Z-property for TRSs via complete CSR

Vincent van Oostrom

http://www.javakade.nl/

First-order left-linear term rewriting (TRS)

Example (TRS $\mathcal{T}$)

$$\begin{aligned} \mathsf{nats} &\to_1 \mathsf{from}(0) & \mathsf{tl}(x:y) &\to_4 y \\ \mathsf{inc}(x:y) &\to_2 \mathsf{s}(x):\mathsf{inc}(y) & \mathsf{from}(x) &\to_5 x:\mathsf{from}(\mathsf{s}(x)) \\ \mathsf{hd}(x:y) &\to_3 x & \mathsf{inc}(\mathsf{tl}(\mathsf{from}(x))) &\to_6 \mathsf{tl}(\mathsf{inc}(\mathsf{from}(x))) \end{aligned}$$

Question

is $\mathcal{T}$ confluent, i.e. is induced rewrite system $\to$ confluent?

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\mathsf{hd}(x : y) &\to_3 x & \mathsf{inc}(\mathsf{tl}(\mathsf{from}(x))) &\to_6 \mathsf{tl}(\mathsf{inc}(\mathsf{from}(x)))
\end{aligned}
$$

Question

is $\mathcal{T}$ confluent, i.e. is induced rewrite system $\to$ confluent?

Methodology this talk (from Gramlich & Lucas, RTA 2006):

transfer confluence of context-sensitive term rewrite system $\mathcal{T}, \mu$ to that of $\mathcal{T}$, for appropriate replacement map μ

Context-sensitive term rewriting (CSR)

Definition (context-sensitive rewriting)

- replacement map μ maps symbol to subset of active argument positions
- rewrite system $\hookrightarrow$ induced by $\mathcal{T}, \mu$: $\rightarrow$ restricted to redexes at active positions

frozen = non-active, indicated by overlining

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Example (CSR $\mathcal{T}, \mu$ with $\mu(\mathsf{inc}) := \mu(\mathsf{tl}) := \{1\}$, $\mu(a) := \emptyset$ otherwise)

$$\begin{aligned} \mathsf{nats} &\rightarrow_1 \mathsf{from}(\overline{0}) & \mathsf{tl}(\overline{x} : \overline{y}) &\rightarrow_4 y \\ \mathsf{inc}(\overline{x} : \overline{y}) &\rightarrow_2 \overline{\mathsf{s}(\overline{x})} : \overline{\mathsf{inc}(y)} & \mathsf{from}(\overline{x}) &\rightarrow_5 \overline{x} : \overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})} \\ \mathsf{hd}(\overline{\overline{x} : \overline{y}}) &\rightarrow_3 x & \mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) &\rightarrow_6 \mathsf{tl}(\mathsf{inc}(\mathsf{from}(\overline{x}))) \end{aligned}$$

$\mathsf{hd}(\mathsf{nats})$ rewrites for $\rightarrow$, but $\mathsf{hd}(\overline{\mathsf{nats}})$ is in $\hookrightarrow$-normal form as nats occurs frozen

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Example (CSR $\mathcal{T}, \mu$ with $\mu(\mathsf{inc}) := \mu(\mathsf{tl}) := \{1\}$, $\mu(a) := \emptyset$ otherwise)

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{nats} & \rightarrow_1 & \mathsf{from}(\overline{0}) & \mathsf{tl}(\overline{x} : \overline{y}) & \rightarrow_4 & y \\
\mathsf{inc}(\overline{x} : \overline{y}) & \rightarrow_2 & \overline{\mathsf{s}(\overline{x})} : \overline{\mathsf{inc}(y)} & \mathsf{from}(\overline{x}) & \rightarrow_5 & \overline{x} : \overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})} \\
\mathsf{hd}(\overline{\overline{x} : \overline{y}}) & \rightarrow_3 & x & \mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) & \rightarrow_6 & \mathsf{tl}(\mathsf{inc}(\mathsf{from}(\overline{x})))
\end{array}
$$

note: $\hookrightarrow = \rightarrow$ if $\mu(f) := \{1, \ldots, n\}$ for every n-ary f; all positions active

Transferring confluence from CSR to TRS

Idea

choose μ such that CSR $\hookrightarrow$ is complete, but without losing critical pairs

otherwise no hope to transfer confluence of $\hookrightarrow$ to that of $\rightarrow$

Transferring confluence from CSR to TRS

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choose μ such that CSR $\hookrightarrow$ is complete, but without losing critical pairs

Assumptions

- (i) $\mathcal{T}$ critical peaks are $\mathcal{T}, \mu$ critical peaks
- (ii) $\mathcal{T}, \mu$ is a left-linear and complete (confluent and terminating) CSR

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Definition

μ is convective if inner redexes in critical peaks are active

Example (non-convectivity loses critical peak)

$$\ldots {}_5\!\leftarrow \mathsf{inc}(\mathsf{tl}(\mathsf{from}(x))) \rightarrow_6 \ldots$$

$\mathsf{from}(x)$ inner redex of critical peak, so must have $1 \in \mu(\mathsf{inc}), \mu(\mathsf{tl})$

Transferring confluence from CSR to TRS

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2. $\mathcal{T}, \mu$ is a left-linear and complete (confluent and terminating) CSR

Definition

μ is convective if inner redexes in critical peaks are active

Lemma

if μ is convective, then assumption (i) holds

for example $\mathcal{T}$, $\mu(\text{inc}) := \mu(\text{tl}) := \{1\}$ is convective

Z-property

Definition (Z)

rewrite system $\to$ has Z-property for map $\bullet$ on objects, if $a \to b$ entails $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$

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$b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$

Theorem (Loader,Dehornoy, ,...)

if $\to$ has Z-property for some $\bullet$, then

- *$\to$ is confluent*
- *bullet strategy $\stackrel{\bullet}{\to}$, repeatedly rewriting a to $a^\bullet$, is hyper-normalising*

more properties entailed by Z; see FSCD 2021

Z-property of CSR $\mathcal{T}, \mu$

Definition (of bullet map $\bullet$ for CSR $\mathcal{T}, \mu$)

Let $\bullet$ map a term to its $\hookrightarrow$-normal form (FSCD 2021)

$\bullet$ is extensive, i.e. $t \twoheadrightarrow t^\bullet$, by completeness assumption (ii)

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Lemma (Z of $\hookrightarrow$)

- *$\hookrightarrow$ has Z-property for $\bullet$*
- *if $t \;\twoheadrightarrow\!\!\!\!\!\!\!\!+\; s$ then $t^{\bullet} \twoheadrightarrow s^{\bullet}$*

Z-property of CSR $\mathcal{T}, \mu$

Lemma (Z of $\hookrightarrow$)

- *$\hookrightarrow$ has Z-property for $\bullet$*
- *if $t \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; s$ then $t^\bullet \twoheadrightarrow s^\bullet$*

Proof.

- since $\hookrightarrow$-normal forms exist uniquely by completeness assumption (ii)
- by induction on t ordered by $\hookleftarrow$, well-founded by assumption (ii)
  - if $t \hookrightarrow t' \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; s$, then by IH for $t' \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; s$;
  - elseif $t \hookrightarrow t'$, then by $t' \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; s' \hookleftarrow s$ using assumption (i), and IH for $t' \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; s'$ □

Z-property of TRS $\mathcal{T}$

Definition (of bullet map $\circledbullet$ for TRS $\mathcal{T}$, based on map $\bullet$ for CSR $\mathcal{T}, \mu$)

- write $C\langle\vec{t}\rangle$ to denote decomposition based on maximal active context C
- layering $\circledbullet$ of $\bullet$ inductively defined by $C\langle\vec{t}\rangle^{\circledbullet} := C\langle\vec{t}^{\circledbullet}\rangle^{\bullet}$.

maximal active context of a term is unique

Z-property of TRS $\mathcal{T}$

Definition (of bullet map $\odot$ for TRS $\mathcal{T}$, based on map $\bullet$ for CSR $\mathcal{T}, \mu$)

- write $C\langle\vec{t}\rangle$ to denote decomposition based on maximal active context C
- layering $\odot$ of $\bullet$ inductively defined by $C\langle\vec{t}\rangle^{\odot} := C\langle\vec{t}^{\odot}\rangle^{\bullet}$.

Lemma (Z of $\rightarrow$)

- *$f(\vec{t}^{\odot}) \twoheadrightarrow f(\vec{t})^{\odot}$; extends from symbols f to contexts C*
- *$\rightarrow$ has the Z-property for $\odot$*

Z-property of TRS $\mathcal{T}$

Lemma (Z of $\to$)

- *$f(\vec{t}^{\circledcirc}) \twoheadrightarrow f(\vec{t})^{\circledcirc}$; extends from symbols f to contexts C*
- *$\to$ has the Z-property for $\circledcirc$*

first item exploits inside–out nature of $\circledcirc$

Z-property of TRS $\mathcal{T}$

Lemma (Z of $\to$)

- $f(\vec{t}^{\circledbullet}) \twoheadrightarrow f(\vec{t})^{\circledbullet}$; *extends from symbols f to contexts C*
- $\to$ *has the Z-property for* $\circledbullet$

Proof.

- $f(\vec{t}^{\circledbullet}) \twoheadrightarrow f(\vec{t}^{\circledbullet})^{\bullet} = f(\vec{t})^{\circledbullet}$, with the 1st holding by extensivity and the 2nd since computing $f(\vec{t}^{\circledbullet})^{\bullet}$ proceeds by computing the $t_i^{\circledbullet}$, which each ends in applying $\bullet$, followed by another application of $\bullet$; can be combined
- by induction on the decomposition of t for a step $t \to s$, distinguishing cases on whether redex pattern ℓ of step is in context, a $\hookrightarrow$-step, or not $\square$

Z-property of TRS $\mathcal{T}$

Lemma (Z of $\to$)

- $f(\vec{t}^{\circledast}) \twoheadrightarrow f(\vec{t})^{\circledast}$; *extends from symbols f to contexts C*
- $\to$ *has the Z-property for* $\circledast$

Proof.

- $f(\vec{t}^{\circledast}) \twoheadrightarrow f(\vec{t}^{\circledast})^{\bullet} = f(\vec{t})^{\circledast}$, with the 1st holding by extensivity and the 2nd since computing $f(\vec{t}^{\circledast})^{\bullet}$ proceeds by computing the $t_i^{\circledast}$, which each ends in applying $\bullet$, followed by another application of $\bullet$; can be combined
- if $s \hookleftarrow t = C\langle\vec{t}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledast}\rangle \hookrightarrow\!\!\!\!\rightarrow C\langle\vec{t}^{\circledast}\rangle^{\bullet} = t^{\circledast}$ then $s = E[\vec{u}] \twoheadrightarrow E[\vec{u}^{\circledast}] = u \hookleftarrow C\langle\vec{t}^{\circledast}\rangle$ and $s \twoheadrightarrow t^{\circledast} = u^{\bullet} = E[\vec{u}^{\circledast}]^{\bullet} \twoheadrightarrow (E[\vec{u}]^{\circledast})^{\bullet} = (s^{\circledast})^{\bullet} = s^{\circledast}$ for some $u, E, \vec{u}$ $\square$

Sufficient conditions

Definition

$\mathcal{T}, \mu$ is active-preserving, if, whenever a variable occurs active in lhs of a rule then all occurrences in rhs of the rule are active

vacuously true for $\mathcal{T}, \mu$; not for $f(x) \to \mathsf{from}(\overline{x})$

Lemma (Lucas)

if $\mathcal{T}$ left-linear, assumption (i) holds, and critical peaks $\hookrightarrow$-joinable, then $\hookrightarrow$ locally confluent

Sufficient conditions

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Theorem

If $\mathcal{T}, \mu$ is left-linear active-preserving CSR such that μ is convective, critical peaks are $\hookrightarrow$-joinable, and $\hookrightarrow$ is terminating, then $\to$ has Z-property for $\circledcirc$

example TRS $\mathcal{T}$ is confluent, because CSR $\mathcal{T}, \mu$ satisfies conditions

Conclusion

- OSR: we envision a friendly atmosphere during the meeting, which enables fruitful exchanges leading to joint research and subsequent publications
  this research inspired by IWC invited talk of, and discussion with, Nao Hirokawa last year; also feedback of Salvador Lucas; thanks!

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- partially solves open problem 1 of (Lucas & Gramlich 2006) asking can level-decreasingness be dropped?

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- long draft The Z-property and ω-confluence by context-sensitive termination solves open problem 2 (ω-confluence?) in the affirmative

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- long draft The Z-property and ω-confluence by context-sensitive termination solves open problem 2 (ω-confluence?) in the affirmative
- CSR stratification related to study of modularity (non-height-increasingness) but a priori number of layers may increase by rewriting

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- long draft The Z-property and ω-confluence by context-sensitive termination solves open problem 2 (ω-confluence?) in the affirmative
- CSR stratification related to study of modularity
- CSR related to the study of mixed inductive / co-inductive systems but no a priori conditions on shapes of terms

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- generalisation from canonical to convective replacement maps
- transfer of Z? (instead of of confluence)
  sufficient conditions such that Z-property of CSR $\rightarrow$ entails that of TRS $\hookrightarrow$, other than completeness of $\rightarrow$, assumption (ii)?

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- generalisation from canonical to convective replacement maps
- transfer of Z?
- automation / implementation?
  incorporation in Valencia tools

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