Uniform Completeness

Vincent van Oostrom[1]

[1] Supported by EPSRC Project EP/R029121/1 Typed lambda-calculi with sharing and unsharing.

Completeness

Definition

rewrite system $\rightarrow := \langle A, \Phi, \mathrm{src}, \mathrm{tgt} \rangle$ with objects A and steps Φ

$\phi : a \rightarrow b$ or $a \rightarrow_{\phi} b$ denotes step ϕ with source $\mathrm{src}(\phi) = a$, target $\mathrm{tgt}(\phi) = b$

Completeness

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rewrite systems have same data as multigraphs, quivers, pre-categories

Completeness

Definition

rewrite system is complete if confluent (CR) and terminating (SN)

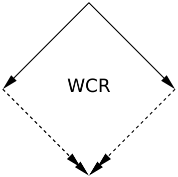
Completeness

Definition

rewrite system is complete if confluent and terminating

Lemma (Complete iff)

- *locally confluent (WCR) and terminating (SN) (Newman 1942)*



Completeness

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Lemma (Complete iff)

- *locally confluent and terminating*
- *ordered locally confluent (OWCR) and normalising (WN) (this talk)*

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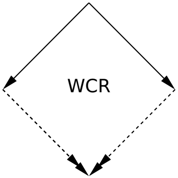

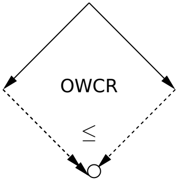

Completeness

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Lemma (Complete iff)

- *locally confluent and terminating*
- *ordered locally confluent and normalising*

Theorem (Newman 1942, 2007)

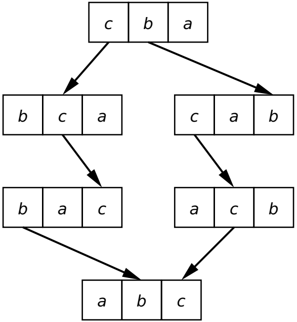
ordered local confluence $\iff$ *random descent (RD):*

if convertible to nf max reductions same length: $\mathrm{NF} \ni a \; {}^{n}\!\leftrightarrow^{m} b \implies a \; {}^{n \dot{-} m}\!\leftarrow b$

Example 1: Sorting by swapping adjacent inversions

Example (RTA 2007)

$\rightarrow$ swaps adjacent out-of-order letters in finite strings of letters

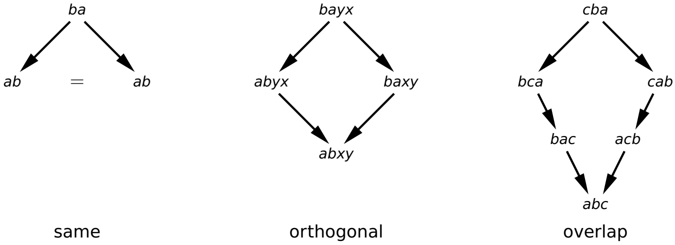


Example 1: Sorting by swapping

Example (RTA 2007)

$\to$ swaps adjacent out-of-order letters in finite strings of letters

- $\to$ is ordered weak Church–Rosser:



Example 1: Sorting by swapping

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- $\rightarrow$ is normalising by termination of some sorting algorithm, e.g. bubble sort

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hence $\rightarrow$ is complete because it has random descent

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hence $\rightarrow$ is complete

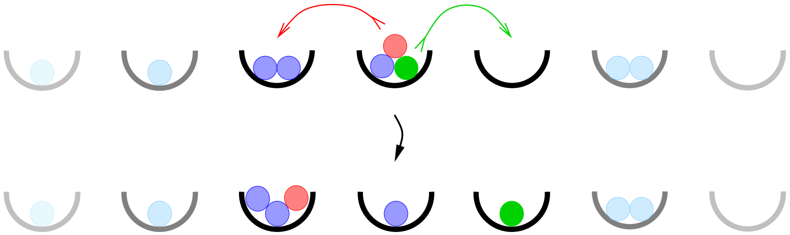
and all ways of sorting a string by swapping have same length; $O(n^2)$

Example 2: Bowls and beans

Example (RTA 2007)

$\to$ moves a bean to both adjacent bowls in two-sided infinite sequence of bowls

sequence s may be modelled as $s : \mathbb{Z} \to \mathbb{N}$ with $\sum s < \infty$ (finite number of beans)



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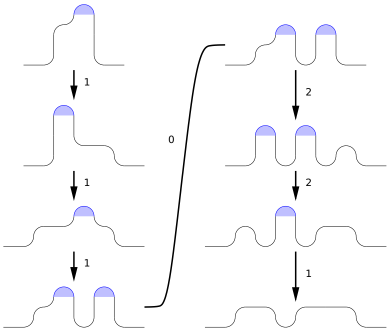

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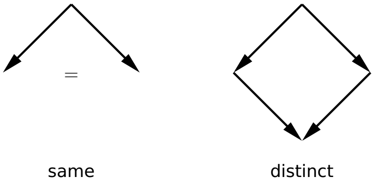

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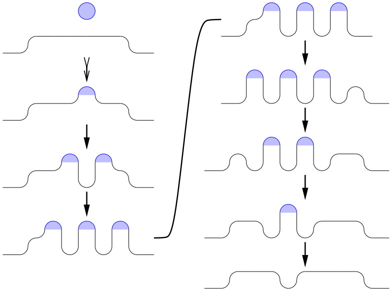

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hence $\to$ is complete

and all bean runs have same length

Incompleteness

Example

$\to$ with steps $a \to b$, $b \to c$ and $a \to c$ trivially complete

Incompleteness

Example

$\to$ with steps $a \to b$, $b \to c$ and $a \to c$ trivially complete

but reductions from a to c do not have same length (1 or 2)

Incompleteness for length

Example

$\rightarrow$ with steps $a \rightarrow b$, $b \rightarrow c$ and $a \rightarrow c$ trivially complete

$\rightarrow$ cannot be proven complete by OWCR & WN; method of (2007) incomplete

Completeness

Example

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Idea

allow to measure steps by appropriate weights (Toyama, 2016)

Completeness

Example

$\to$ with steps $a \to b$, $b \to c$ and $a \to c$ trivially complete

Definition (Toyama, 2016; with minor refinements in paper)

$\langle M, \bot, +, \leq \rangle$ derivation monoid if

- $\langle M, \bot, + \rangle$ a monoid;

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main example: ordinals with zero, addition, less–than–or–equal

Completeness

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Definition (Toyama, 2016; with minor refinements in paper)

$\langle M, \bot, +, \leq \rangle$ derivation monoid

- measure on $\to$ maps steps to $M - \{\bot\}$;

Completeness

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- measure on $\to$ maps steps to $M - \{\bot\}$;
- measure of finite reduction is sum ($+$; tail to head) of steps (starting with $\bot$);

Completeness

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Theorem (Toyama, 2016 & paper)

ordered local confluence (OWCR; wrt measure) $\iff$ *peak random descent (PR):*
peak to nf reductions same weight: $\mathrm{NF} \ni a \;{}_n^{*}\!\!\leftarrow \cdot \to_{\mu}^{\circ} b \implies \exists k.a \;{}_k^{*}\!\!\leftarrow b \;\&\; k + \mu = n$

Completeness

Example

$\to$ has PR, since $a \to_1 b$, $b \to_1 c$ and $a \to_2 c$ is OWCR

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Uniform completeness

Definition (for property Π of objects)

$\to$ is uniformly Π if all objects convertible to nf are Π

Uniform completeness

Definition ($\sqcap$:= CR & SN)

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Theorem

uniformly complete iff has peak random descent wrt some measure

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Theorem

uniformly complete iff has peak random descent wrt some measure

Proof of if-direction.

PR entails:

- uniform termination: if $c \rightarrow_n b \in$ SN and, say, $b \twoheadrightarrow_m a \in$ NF, then $m + n$ is an upperbound on measures of reductions from c;

$\square$

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Definition ($\sqcap$:= CR & SN)

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so if b convertible to nf a, SN(b) by uniform termination, ending in a by NF $\square$

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Proof of only–if-direction.

idea: measure SN objects and steps by wf topological sorting □

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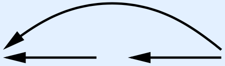
measure a by supremum$\{(\text{measure of } b) + 1 \mid a \rightarrow b\}$; step $a \rightarrow b$ by dif a and b



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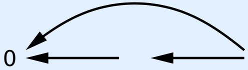

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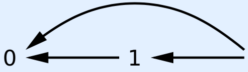

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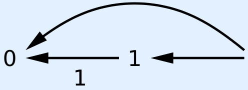

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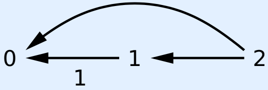

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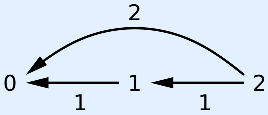

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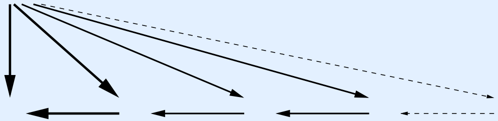

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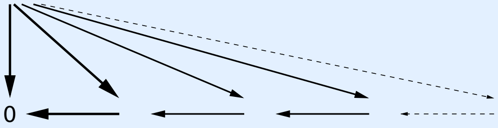

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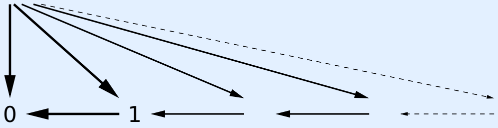

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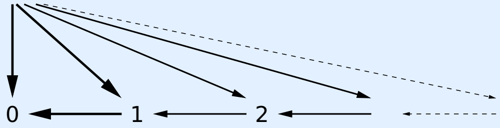

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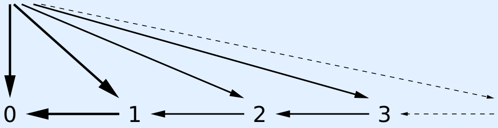

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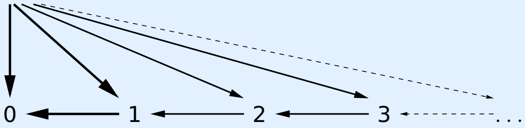

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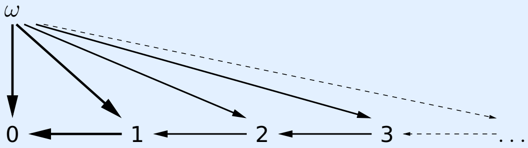

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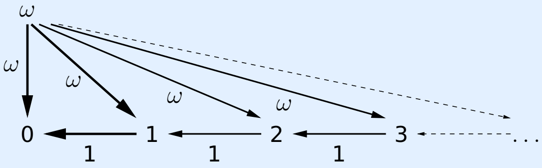

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Theorem

uniformly complete iff has peak random descent wrt some measure

Corollary

uniformly complete iff OWCR for some measure

Example 3: the trivial rewrite system

Example

$\rightarrow$ with steps $a \rightarrow b$, $b \rightarrow c$ and $a \rightarrow c$

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$\to$ with steps $a \to b$, $b \to c$ and $a \to c$

- $\to$ OWCR for measure $a \to_1 b$, $b \to_1 c$ and $a \to_2 c$, hence uniformly complete

Example 3: the trivial rewrite system

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- $\to$ is trivially WN

Example 3: the trivial rewrite system

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- $\to$ OWCR for measure $a \to_1 b$, $b \to_1 c$ and $a \to_2 c$, hence uniformly complete
- $\to$ is trivially WN

hence $\to$ is complete

Finitely branching systems

Observation

for finitely branching (FB) systems, measures in completeness proof in $\mathbb{N}$

Finitely branching systems

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for finitely branching systems, measures in completeness proof in $\mathbb{N}$

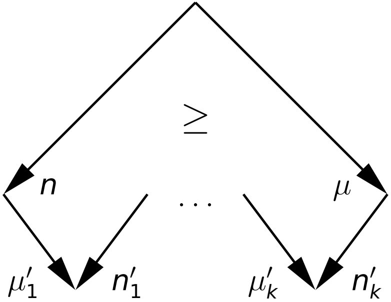
$+$ commutative, cancellative; then OWCR $\iff$ locally Dyck (Toyama, ♥ 2016)

Finitely branching systems

Observation

for finitely branching systems, measures in completeness proof in $\mathbb{N}$

locally Dyck if



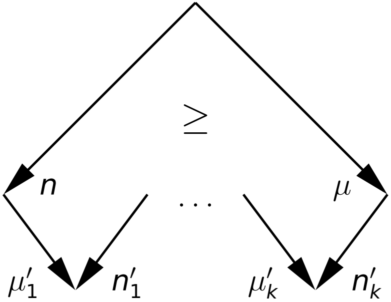
and forward weights $>$ backward weights: $\forall\, i. n + \sum \mu'_i > \sum n'_i$

Finitely branching systems

Corollary

uniformly complete iff locally Dyck for some measure

locally Dyck if



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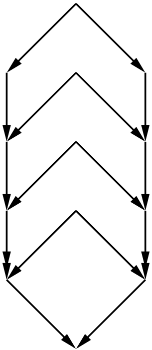
Example 4: deep valleys but shallow conversions

Example (⩔ 2008)

$\to$ with $b_i \leftarrow a_i \to c_i$, $b_i \to b_{i+1}$, and $c_i \to c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$



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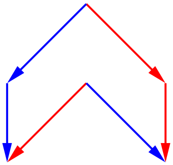
- $\to$ locally Dyck for length measure, hence uniformly complete:



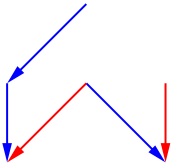
forward $3 \geq 3$ backward

Example 4: deep valleys but shallow conversions

Example (⩔ 2008)

$\to$ with $b_i \leftarrow a_i \to c_i$, $b_i \to b_{i+1}$, and $c_i \to c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$

- $\to$ locally Dyck for length measure, hence uniformly complete:



forward 3 $>$ 2 backward

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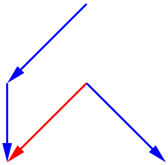

forward 3 $>$ 1 backward

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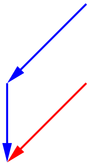


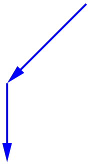
forward 2 $>$ 1 backward

Example 4: deep valleys but shallow conversions

Example (⩔ 2008)

→ with $b_i \leftarrow a_i \rightarrow c_i$, $b_i \rightarrow b_{i+1}$, and $c_i \rightarrow c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$

- → locally Dyck for length measure, hence uniformly complete:



forward $2 > 0$ backward

Example 4: deep valleys but shallow conversions

Example (2008)

$\rightarrow$ with $b_i \leftarrow a_i \rightarrow c_i$, $b_i \rightarrow b_{i+1}$, and $c_i \rightarrow c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$

- $\rightarrow$ locally Dyck for length measure, hence uniformly complete:

forward $1 > 0$ backward

Example 4: deep valleys but shallow conversions

Example (⩔ 2008)

$\to$ with $b_i \leftarrow a_i \to c_i$, $b_i \to b_{i+1}$, and $c_i \to c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$

- $\to$ locally Dyck for length measure, hence uniformly complete
- $\to$ is trivially WN

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Example (⩔ 2008)

$\rightarrow$ with $b_i \leftarrow a_i \rightarrow c_i$, $b_i \rightarrow b_{i+1}$, and $c_i \rightarrow c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$

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- $\rightarrow$ is trivially WN

hence $\rightarrow$ is complete

Conclusions / Directions

1. introduced novel notion uniform completeness (useful?)

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3. finding measures for term rewrite systems?
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5. proof / PL theory fertile hunting ground for WN systems?

thank you

(return to NL tomorrow night; contact me after at `oostrom@javakade.nl`)