Normalisation by Random Descent

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Strategy

- Normalisation
- Spine
- Needed

Ordering strategies

- Ordered Church–Rosser
- Local Dyck

Normalisation of strategies

- Random Descent
- Compatibility

Strategy

Definition

Rewrite system is set of objects and steps between them

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Example

- λ-terms with β-steps

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Rewrite strategy is subsystem with same set of normal forms

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- Leftmost–outermost/spine strategy

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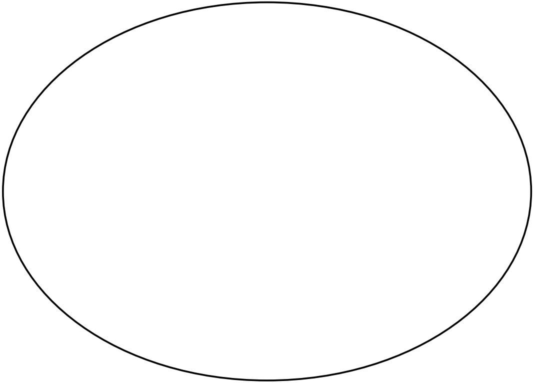
- Leftmost–outermost/spine strategy
- Needed strategy
- Not call by value

Normalisation of strategy

Definition

Strategy S is normalising for R if $\mathrm{WN}(R) = \mathrm{SN}(S)$



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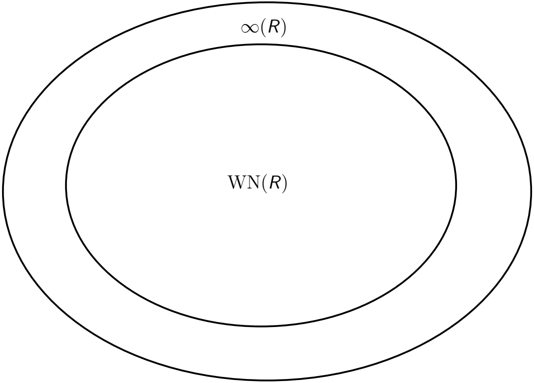

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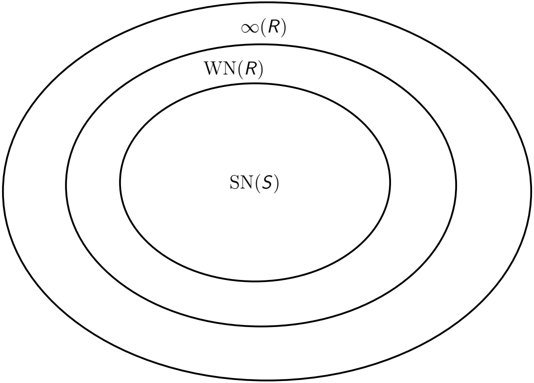

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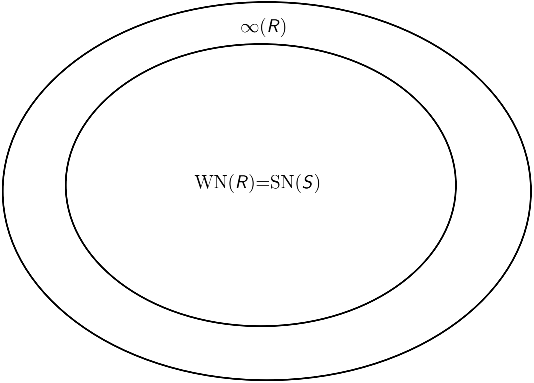

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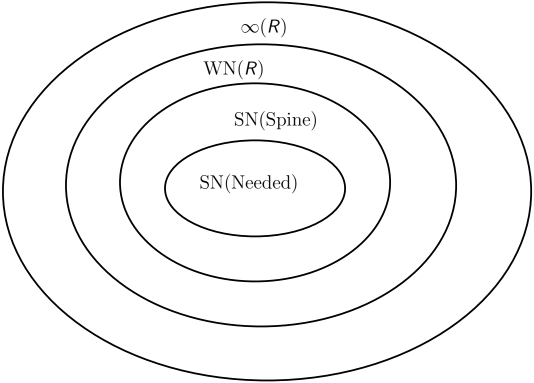

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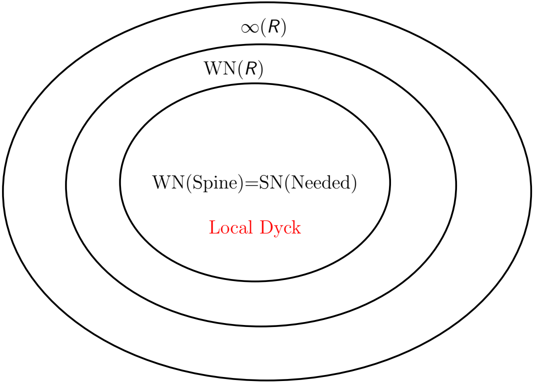

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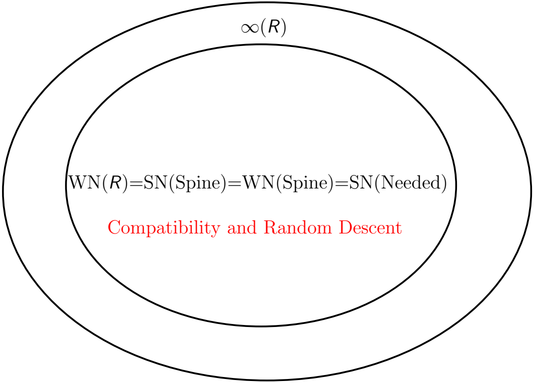
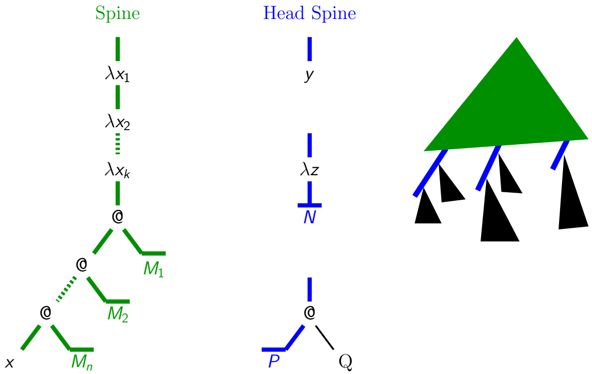


Spine strategy

Definition

Spine: if head normal form recur, else Head Spine.

Head Spine: recur on left.



Example

$x((\lambda x.(\lambda z.zz))y)(xx)(II)$

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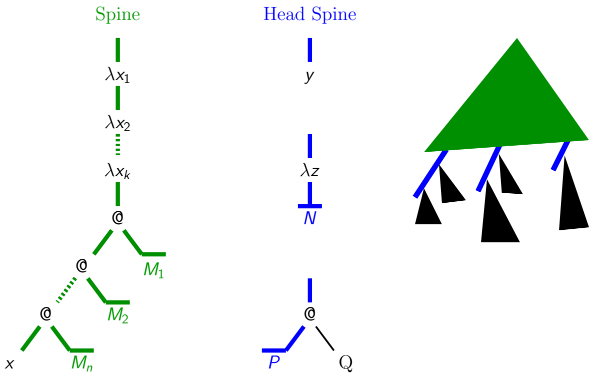

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Every term not in normal form has Spine redex

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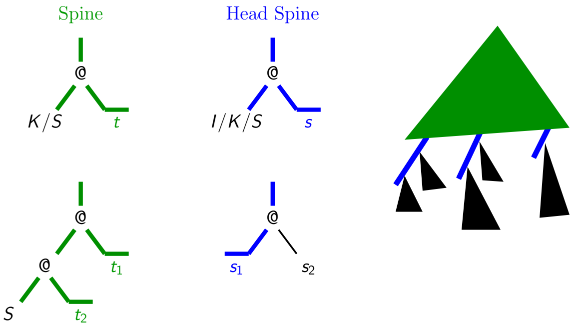

Example

S(SISI)(K(IK))

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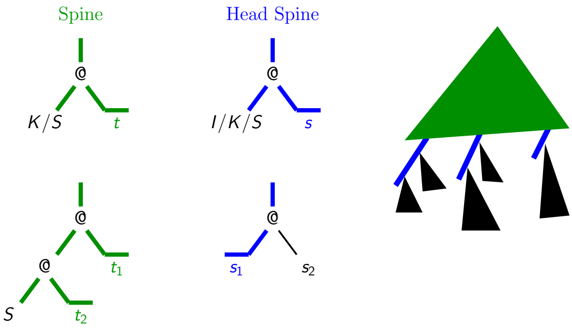

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$K(II)(II)$

Undecidable in general: R Needed in tR?

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$K(II)(II)$

Undecidable in general: R Needed in tR?

Lemma

Every term not in normal form has Needed redex

Proof.

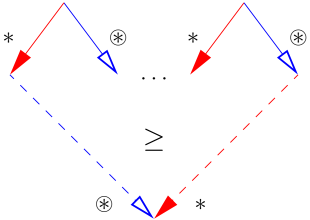
Every Spine redex is needed since each spine symbol has unique descendant until overlapped by contracted redex □

False in general: $f(f(a,a), f(a,a))$ for $f(x,a) \to a$ and $f(a,x) \to a$

Ordering Spine above Needed

Lemma

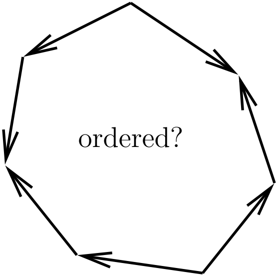
WN(→) ⊆ *SN*(⇾) *if* →, ⇾ *are ordered Church–Rosser*



Corollary

WN(Spine)=SN(Needed) if Spine,Needed ordered Church–Rosser

Ordered diagrams ($\geq$)



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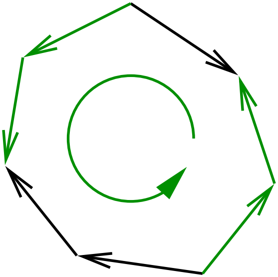
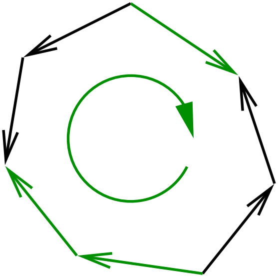


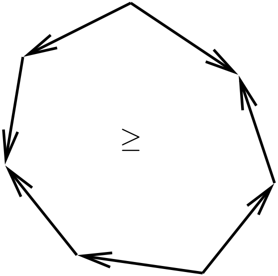
4

Ordered diagrams ($\geq$)



3

Ordered diagrams ($\geq$)

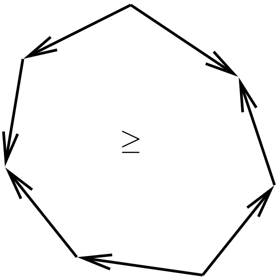


$4 \quad \geq \quad 3$

Definition

Diagram ordered if $\sum$ counter clockwise $\geq \sum$ clockwise

Ordered diagrams ($\geq$)



$4 \quad \geq \quad 3$

Definition

Diagram ordered if $\sum$ counter clockwise $\geq \sum$ clockwise

Lemma

Ordered diagrams preserved by pasting (along segment)

Finitely representing infinite reductions (⊛)

Definition

$a \rightarrow^{\infty} b$ if $a \rightarrow a' \rightarrow a'' \ldots$.

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Order extended with ∞ (top) to measure such steps (μ,ν,...)

Finitely representing infinite reductions ($\circledast$)

Definition

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Order extended with ∞ (top) to measure such steps (μ,ν,...)

Allows to finitely represent infinite reductions

Definition

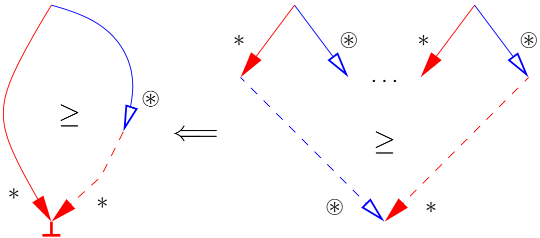
$\rightarrow^{\circledast} = (\rightarrow \cup \rightarrow^{\infty})^*$

Ordering Spine above Needed

Lemma

WN($\rightarrow$) $\subseteq$ *SN*($\rightarrow$) *if* $\rightarrow$, $\rightarrow$ *are ordered Church–Rosser*

Proof.

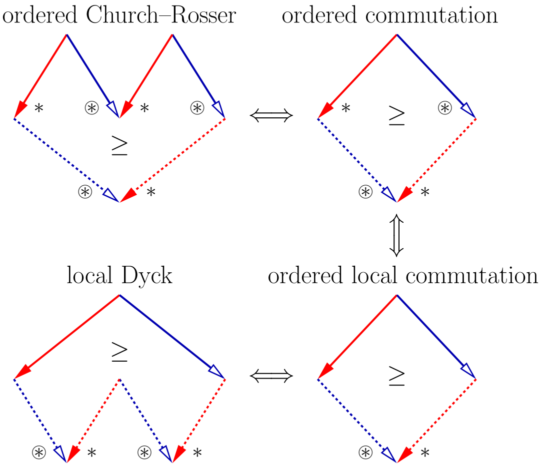


□

Corollary

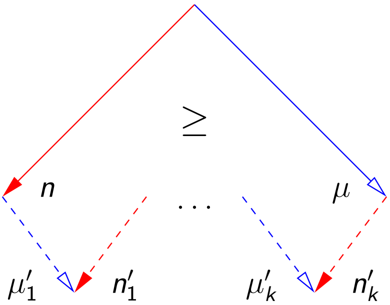
WN(Spine)=SN(Needed) if Spine, Needed are ordered CR

Localising Ordered Church–Rosser



restricting ∀, widening ∃
(cf. Winkler–Buchberger, Decreasing Diagrams converted)

Local Dyck



satisfying $n + \sum \mu'_i > \sum n'_i$ (Dyck, matching parentheses)

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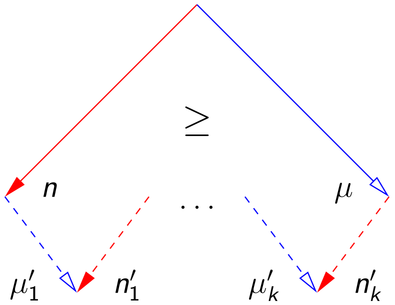

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Theorem

Ordered Church–Rosser iff Local Dyck

Proof.

Only–if trivial. If via ordered commutation, see paper □

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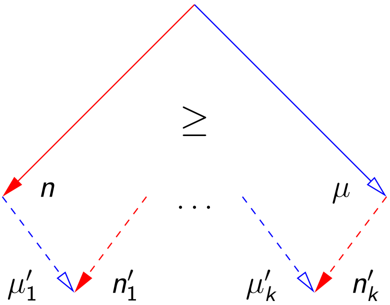

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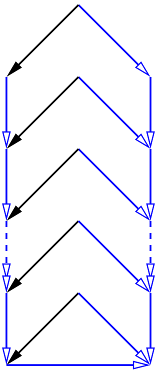
Incomparable to commutation

Local Dyck Example

Let $\rightharpoonup$ be a strategy for $\rightarrow$



Local Dyck Example

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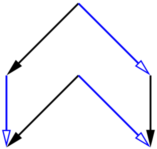
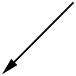


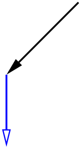
Local Dyck Example

Let $\rightarrowtriangle$ be a strategy for $\rightarrow$



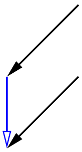
Local Dyck Example

Let $\rightarrow$ (blue, open arrowhead) be a strategy for $\rightarrow$



Local Dyck Example

Let $\textcolor{blue}{\rightharpoonup}$ be a strategy for $\rightarrow$



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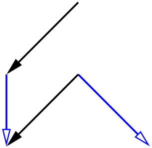

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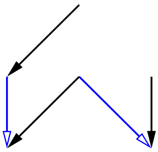

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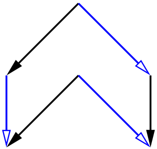
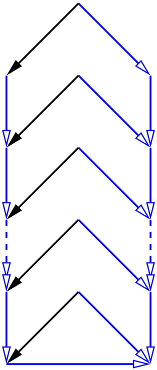


Local Dyck Example

Let $\rightarrowtriangle$ be a strategy for $\to$



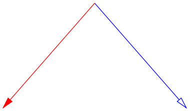
Corollary

$\rightarrowtriangle$ *is optimal strategy for* $\to$

Spine,Needed are Locally Dyck

Let → denote Spine step, ⇾ denote Needed step.

Proof by cases on relative positions of →,⇾-redexes.



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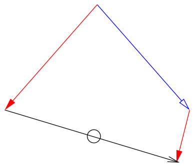

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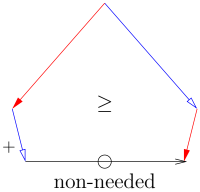

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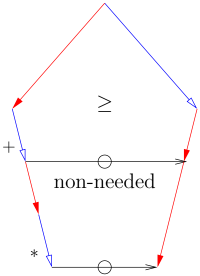

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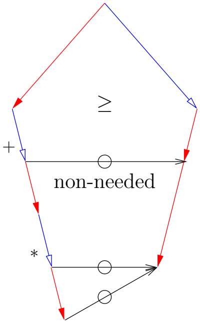

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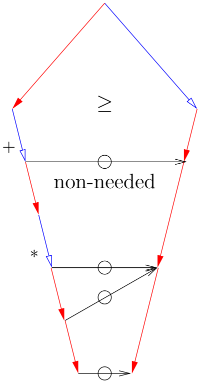

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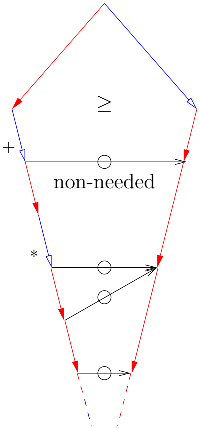
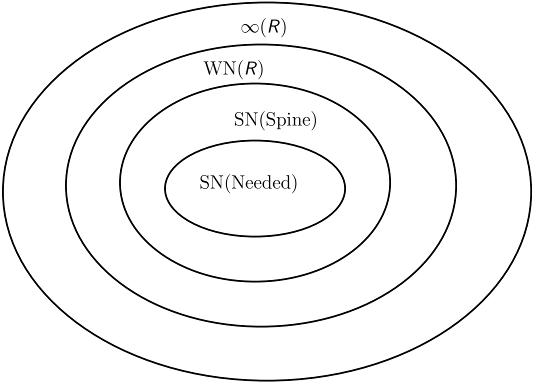


Corollary

WN(Spine)=SN(Needed)

Normalisation of Needed via Spine



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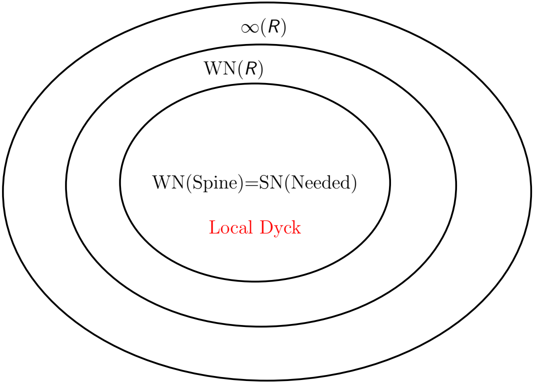

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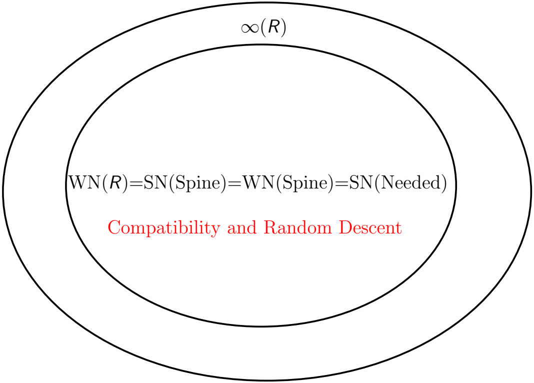

Random Descent

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$\rightarrow$ has random descent if $a \; {}_{n}{\leftrightarrow}^{\circledast}_{\mu} \; b$ with a in normal form implies $a \; {}_{n'}^{\;*}{\leftarrow} \; b$ with $n = \mu + n'$

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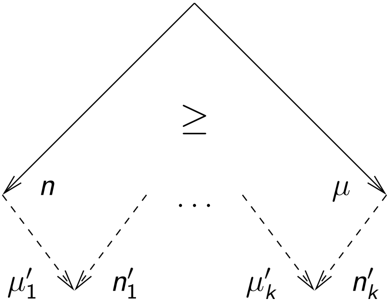
Corollary

Random Descent iff Local Dyck



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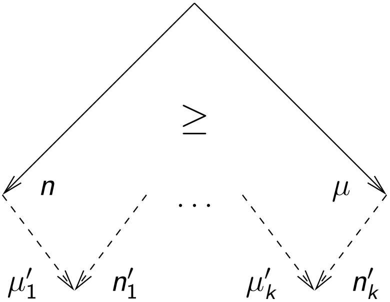

satisfying $n + \sum \mu'_i > \sum n'_i$ (Dyck, matching parentheses)

Incomparable to confluence. Implies uniqueness of normal forms.

History and consequences of Random Descent

Local peak joinable by

- 0 or 1 steps on both sides (Newman 1942)

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- *n* steps on both sides (Toyama 1992)

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- $n \geq m$ steps (van Oostrom 2007)

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Distance $\mathrm{d}(a)$ of object a:
length of reduction from a to normal form (∞ otherwise)

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Example

- External steps in orthogonal systems (Huet and Lévy 1978)

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Distance $\mathrm{d}(a)$ of object a:
length of reduction from a to normal form (∞ otherwise)

Corollary

Distance well-defined if Random Descent, SN($\to$)=WN($\to$), UN

$\to$ need not be deterministic

Example

- External steps in orthogonal systems (Huet and Lévy 1978)
- Spine steps in $\lambda\beta$ (Barendrecht et.al. 1987)

History and consequences of Random Descent

Local peak joinable by

- 0 or 1 steps on both sides (Newman 1942)
- n steps on both sides (Toyama 1992)
- $n \geq m$ steps (van Oostrom 2007)

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Example

- External steps in orthogonal systems (Huet and Lévy 1978)
- Spine steps in $\lambda\beta$ (Barendrecht et.al. 1987)
- Interaction Net reduction (Lafont 1990)

Spine has Random Descent

Proof.

Two spine steps from the same term either

- have overlap, then the overlap is trivial (0); or
- do not have overlap; then they commute directly (1).

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Corollary

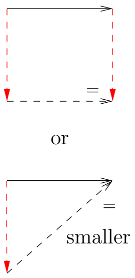
SN(Spine)=WN(Spine)=SN(Needed)

Still to show WN(R)=SN(Spine), i.e. that Spine is normalising.

Normalisation by Compatibility

Lemma

Let $\color{red}{\twoheadrightarrow}$ *be strategy for* $\to$*, having Random Descent and*



Then $\color{red}{\twoheadrightarrow}$ *is* *hyper-normalising and has NF*

NF: convertible to normal form implies reducible to normal form

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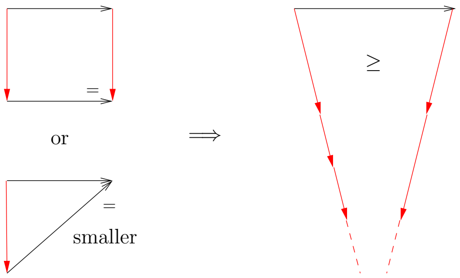

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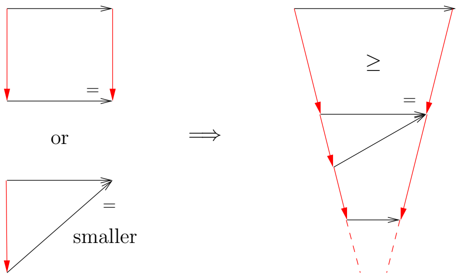

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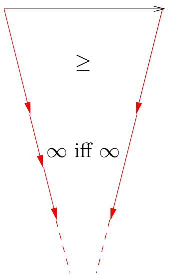

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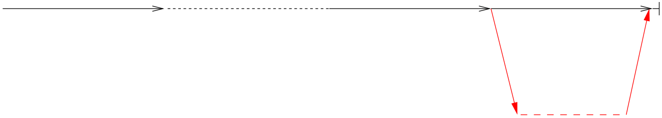

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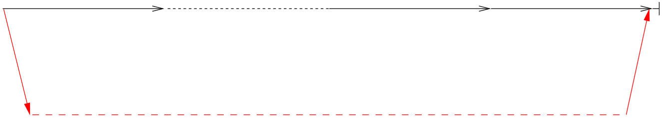

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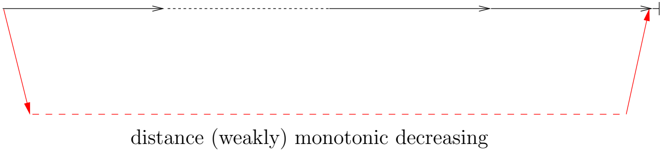

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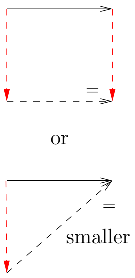

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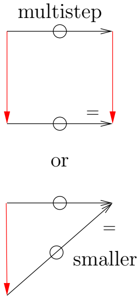
Spine is Compatible in case of multisteps

Let $\rightarrow$ be Needed strategy for $\rightarrow$-multisteps, having Random Descent. Then



Corollary

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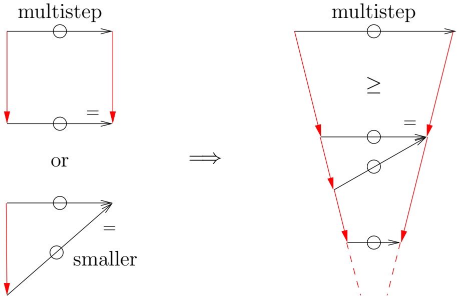

Corollary

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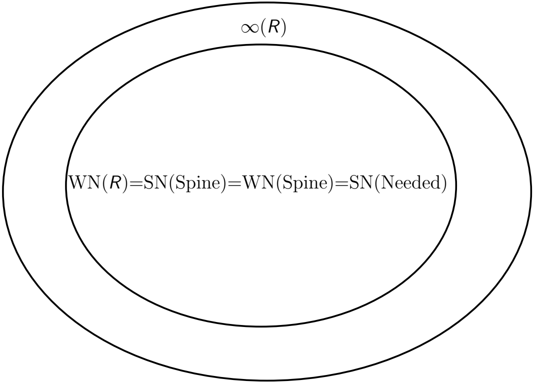
Normalisation of Needed



Summary of Methodology

1. Ordering $\dashrightarrow$ (Needed) below $\rightarrow$ (Spine) by Local-Dyck($\rightarrow$,$\dashrightarrow$)

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Salient features:

- Clean separation abstract vs. concrete (e.g. also graphs, nets)

Summary of Methodology

1. Ordering $\color{blue}{\rightarrowtriangle}$ (Needed) below $\color{red}{\rightarrow}$ (Spine) by Local-Dyck($\color{red}{\rightarrow}$,$\color{blue}{\rightarrowtriangle}$)
2. Conclude $\color{blue}{\rightarrowtriangle}$ is normalising if holds for $\color{red}{\rightarrow}$
3. Show $\color{red}{\rightarrow}$ has via Random Descent by Local-Dyck($\color{red}{\rightarrow}$)
4. Conclude WN($\color{red}{\rightarrow}$)=SN($\color{red}{\rightarrow}$), UN($\color{red}{\rightarrow}$), yields distance-function
5. Show $\color{red}{\rightarrow}$ is Compatible
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- Properties rely on local 'critical peak' analysis (automation)

Summary of Methodology

1. Ordering $\rightarrowtriangle$ (Needed) below $\to$ (Spine) by Local-Dyck($\to$,$\rightarrowtriangle$)
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- Paper: normalisation of left–outer strategy for left-normal h-o

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2. Conclude $-\!\!\triangleright$ is normalising if holds for $\rightarrow$
3. Show $\rightarrow$ has via Random Descent by Local-Dyck($\rightarrow$)
4. Conclude WN($\rightarrow$)=SN($\rightarrow$), UN($\rightarrow$), yields distance-function
5. Show $\rightarrow$ is Compatible
6. Conclude $\rightarrow$ is normalising, so $-\!\!\triangleright$ is.

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- Paper: normalisation of left–outer strategy for left-normal h-o
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- No need for determinism (Hirokawa, Middeldorp, Moser)

Conversion Monoid

- Measure steps in well-foundedly ordered monoid $(M, +, \bot)$.

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Theorem

Ordered Church–Rosser iff Local Dyck, w.r.t. measure (distance)

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Theorem

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Example

$$\mathsf{zeros} \to_1 0 : \mathsf{zeros}$$

$$\mathsf{hd}(x : y) \to_1 x$$

$$\mathsf{hd}(\mathsf{zeros}) \to_2 0$$

Critical peak left–outer Dyck, see paper
Left–outer strategy is normalising.

Future work

- Retrofit known (hyper-)normalisation results in setting (may require slight generalisation of conversion monoid)

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- Extend to other sets of normal forms (head, weak head)