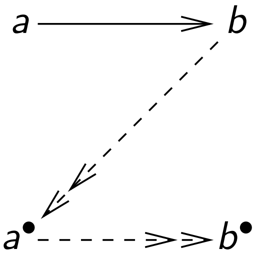
Z

Patrick Dehornoy
Vincent van Oostrom

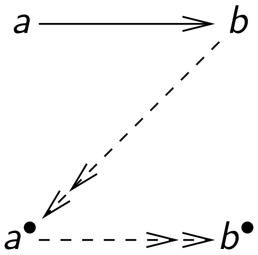
Theoretische Filosofie
Universiteit Utrecht
Nederland

May 6, 2008

Z

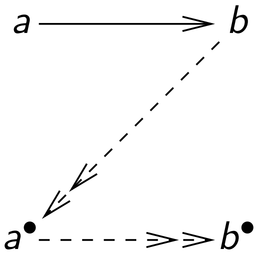


Z



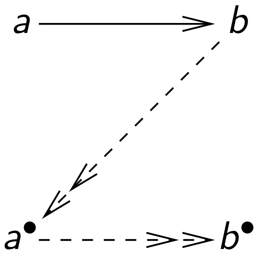
A rewrite relation $\rightarrow$ has the Z-property

Z



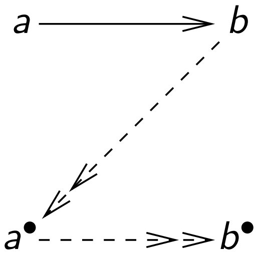
A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects

Z



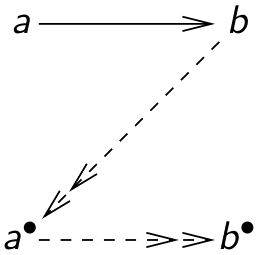
A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step $a \to b$ from a to b

Z



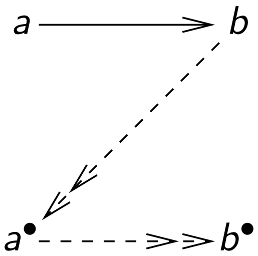
A rewrite relation $\rightarrow$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step $a \rightarrow b$ from a to b
there exists a many-step reduction $b \twoheadrightarrow a^\bullet$ from b to $a^\bullet$

Z



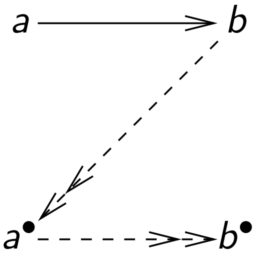
A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step $a \to b$ from a to b
there exists a many-step reduction $b \twoheadrightarrow a^\bullet$ from b to $a^\bullet$
and there exists a many-step reduction $a^\bullet \twoheadrightarrow b^\bullet$ from $a^\bullet$ to $b^\bullet$

Z

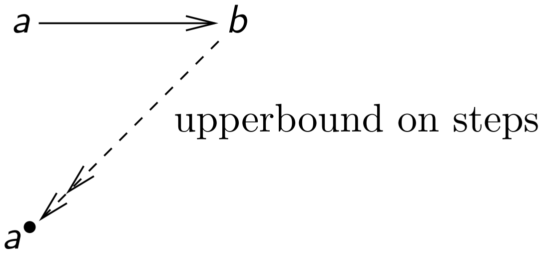


$$\exists \bullet : A \to A, \forall a, b \in A : a \to b \;\Rightarrow\; b \twoheadrightarrow a^{\bullet}, a^{\bullet} \twoheadrightarrow b^{\bullet}$$

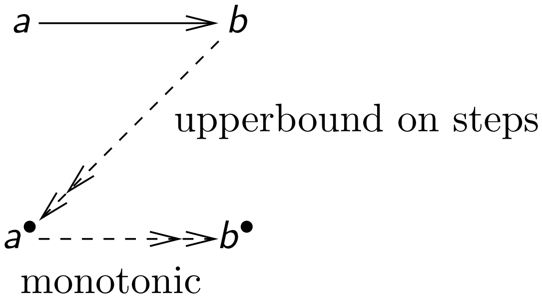
Z intuitions



Z intuitions



Z intuitions



Z $\Rightarrow$ confluence

Definition
$\to$ confluent, if $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

Z $\Rightarrow$ confluence

confluence $\Rightarrow$

- uniqueness of normal forms
- consistent, if some objects not joinable (distinct normal forms)
- decidable, if $\rightarrow$ is terminating

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.

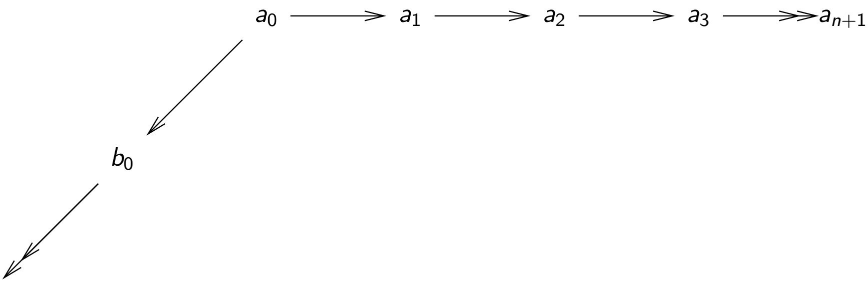
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



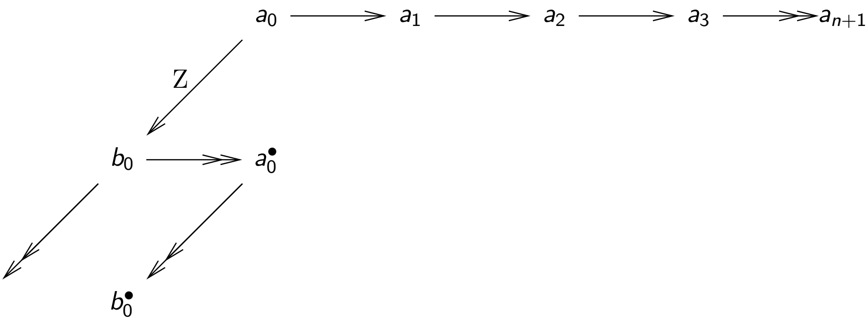
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



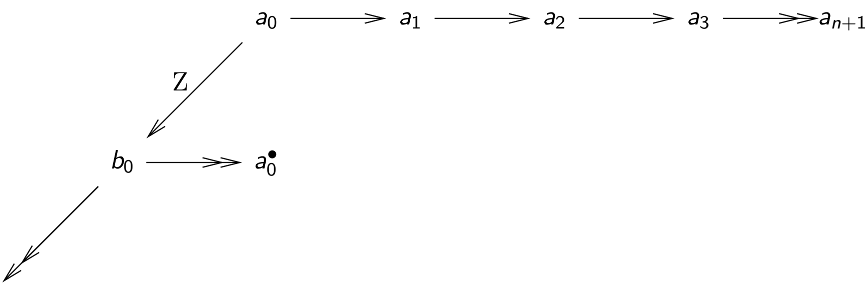
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



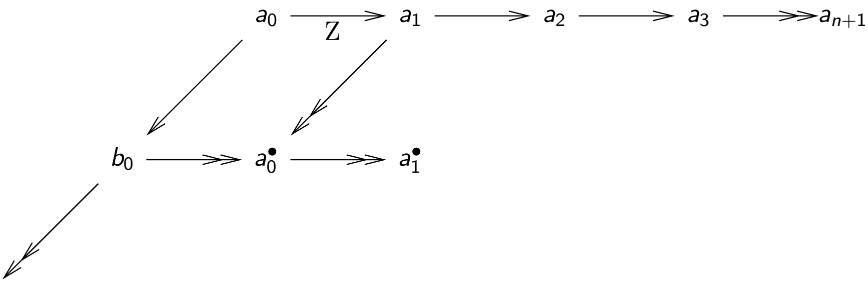
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



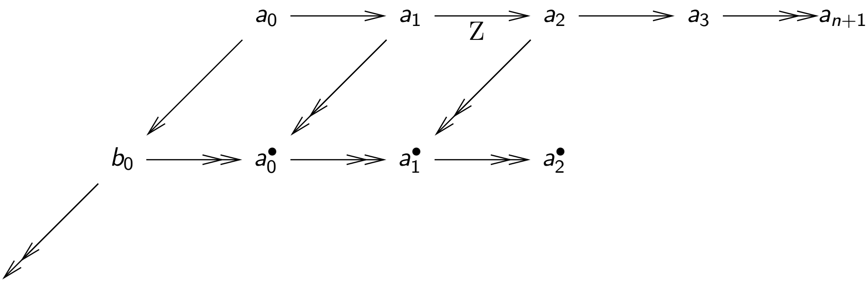
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



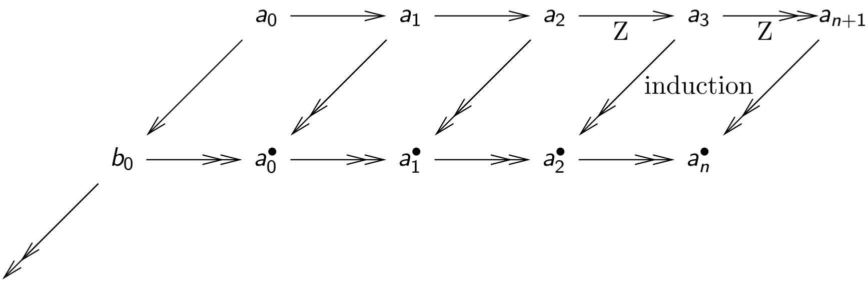
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



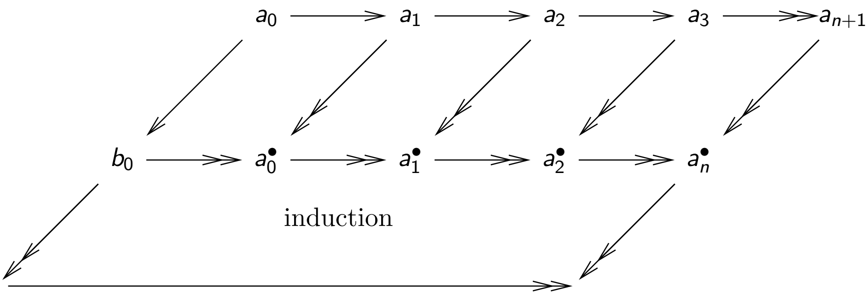
□

Z $\Rightarrow$ confluence

Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.



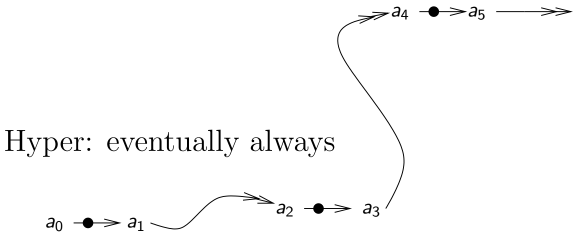
□

Z $\Rightarrow$ $\mathrel{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;}$ strategy is hyper-cofinal

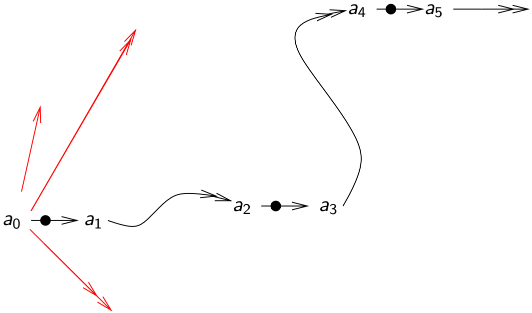
Definition ($\bullet$-strategy)

$a \mathrel{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} b$ if a is not a normal form and $b = a^{\bullet}$

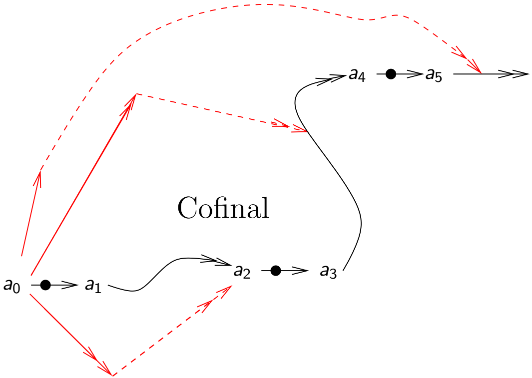
Z $\Rightarrow$ $\multimap\!\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal



Z $\Rightarrow$ $\multimap\!\!\bullet\!\!\to$ strategy is hyper-cofinal



Z $\Rightarrow$ $\multimap\!\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal



Z $\Rightarrow$ $\multimap\!\!\bullet\!\!\to$ strategy is hyper-cofinal

Definition

$\multimap\!\!\bullet\!\!\to$ hyper-cofinal, if for any reduction which eventually always contains a $\multimap\!\!\bullet\!\!\to$-step, any co-initial reduction can be extended to reach the first

Z $\Rightarrow$ $-\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

hyper-cofinal $\Rightarrow$

- confluent
- (hyper-)normalising
- bullet-fast ...

Z $\Rightarrow$ $\multimap\!\!\!\bullet\!\!\!\rightarrow$ strategy is hyper-cofinal

Theorem

$\multimap\!\!\!\bullet\!\!\!\rightarrow$ *is hyper-cofinal*

Proof.

$\bullet$

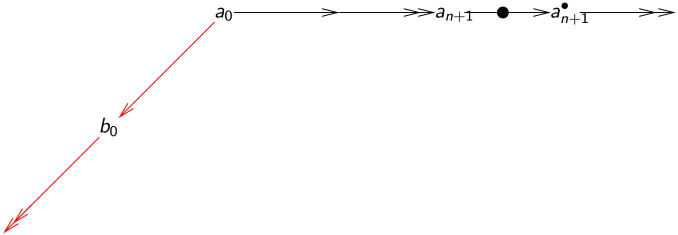
$\square$

Z $\Rightarrow$ $\multimap\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

Theorem

$\multimap\!\!\!\bullet\!\!\rightarrow$ *is hyper-cofinal*

Proof.



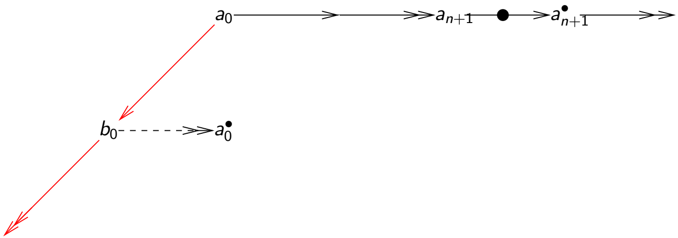
□

Z $\Rightarrow$ $\multimap\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

Theorem

$\multimap\!\!\!\bullet\!\!\rightarrow$ is hyper-cofinal

Proof.



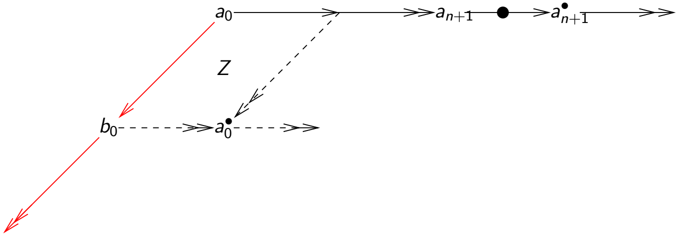
□

Z ⇒ $\twoheadrightarrow\!\!\!\!\bullet$ strategy is hyper-cofinal

Theorem

$\multimap$ *is hyper-cofinal*

Proof.



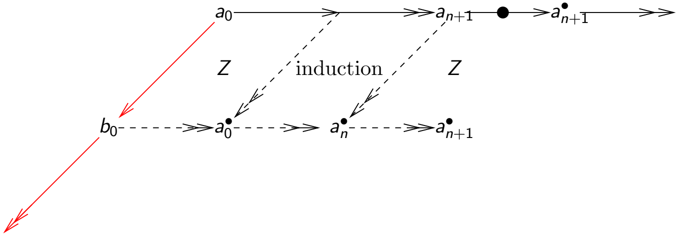
□

Z ⇒ $\multimap\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

Theorem

$\multimap\!\!\bullet\!\!\rightarrow$ is hyper-cofinal

Proof.



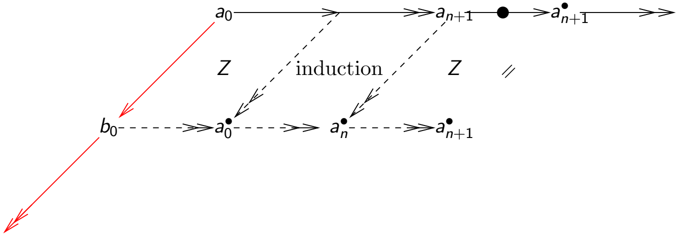
□

Z $\Rightarrow$ $\multimap\!\!\bullet\!\!\to$ strategy is hyper-cofinal

Theorem

$\to\!\!\bullet\!\!\to$ *is hyper-cofinal*

Proof.



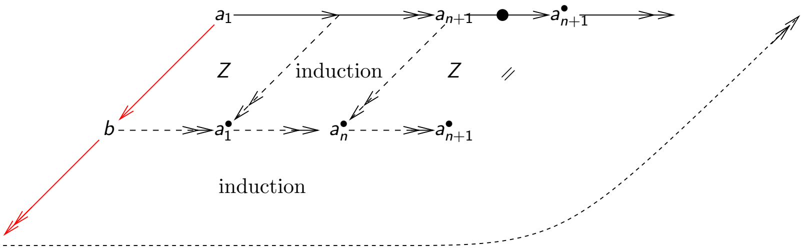
□

Z $\Rightarrow$ $\multimap\!\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

Theorem

$\multimap\!\!\!\!\bullet\!\!\rightarrow$ is hyper-cofinal

Proof.



□

Examples

Example: braids

Definition

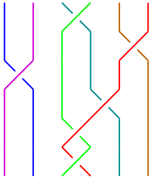
Braid rewriting: cross adjacent strands, right over left.

Example: braids

Definition

Braid rewriting: cross adjacent strands, right over left.

Example:

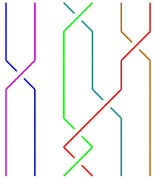


Example: braids

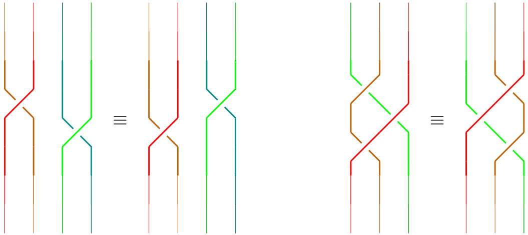
Definition

Braid rewriting: cross adjacent strands, right over left.

Example:



Up to topological equivalence:



Example: braids

Theorem

Braid rewriting has the Z-property, for • full crossing

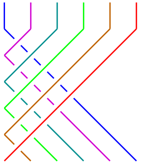
Example

Example: braids

Theorem

Braid rewriting has the Z-property, for • full crossing

Proof.



Example: braids

Theorem

Braid rewriting has the Z-property, for • full crossing

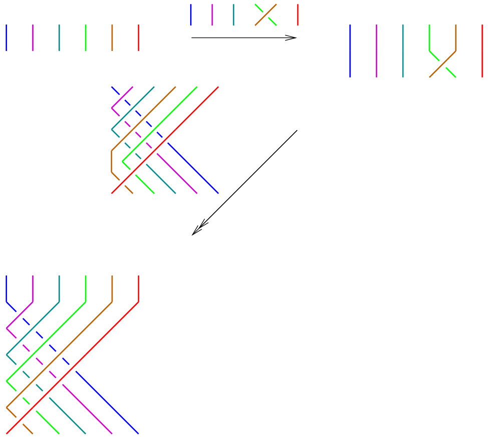
Proof.

Example: braids

Theorem

Braid rewriting has the Z-property, for • full crossing

Proof.

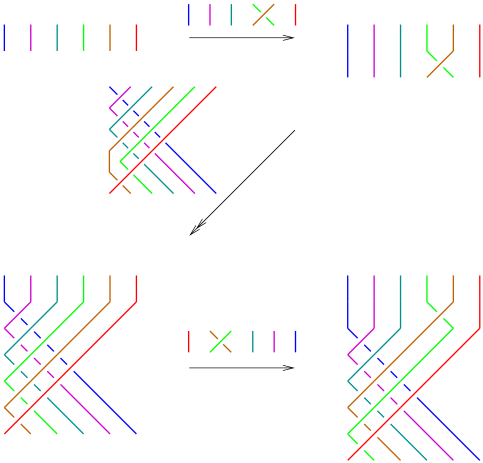


Example: braids

Theorem

Braid rewriting has the Z-property, for • full crossing

Proof.



Example: self-distributivity

Definition

Self-distributivity, rewrite relation generated by $xyz \to xz(yz)$

Example: self-distributivity

Definition

Self-distributivity, rewrite relation generated by $xyz \to xz(yz)$

Some models:

- ACI operations
- take middle of points in space
- substitution

Example: self-distributivity

Definition

Self-distributivity, rewrite relation generated by $xyz \to xz(yz)$

Some models:

- ACI operations
- take middle of points in space
- substitution

In depth: Braids and Self-distributivity (Dehornoy 2000)

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ uniform distribution of s over t:

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ uniform distribution of s over t:

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Example

- $(xy)^\bullet = x[y] = x[x{:=}xy] = xy$;
- $(xyz)^\bullet = (xy)[x{:=}xz, y{:=}yz] = xz(yz)$.

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ uniform distribution of s over t:

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Proof.

By induction on t:

□

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ uniform distribution of s over t:

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Proof.

By induction on t:

- (Sequentialisation) $ts \twoheadrightarrow t[s]$;

□

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ uniform distribution of s over t:

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Proof.

By induction on t:

- (Sequentialisation) $\quad ts \twoheadrightarrow t[s]$;
- (Substitution) $\quad t[s][r] \twoheadrightarrow t[r][s[r]]$

□

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ uniform distribution of s over t:

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Proof.

By induction on t:

- (Sequentialisation) $ts \twoheadrightarrow t[s]$;
- (Substitution) $t[s][r] \twoheadrightarrow t[r][s[r]]$
- (Self) $t \twoheadrightarrow t^\bullet$;

□

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for • full distribution:

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ *uniform distribution of* s *over* t*:*

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Proof.

By induction on t:

- (Sequentialisation) $ts \twoheadrightarrow t[s]$;
- (Substitution) $t[s][r] \twoheadrightarrow t[r][s[r]]$
- (Self) $t \twoheadrightarrow t^\bullet$;
- (Z) $s \twoheadrightarrow t^\bullet \twoheadrightarrow s^\bullet$, if $t \rightarrow s$.

□

Example: normalising and confluent relations

Theorem

Normalising and confluent relations have the Z-property, for $\bullet$ the full reduction map (map to normal form).

Example: normalising and confluent relations

Theorem

Normalising and confluent relations have the Z-property, for $\bullet$ the full reduction map (map to normal form).

Proof.

If $a \rightarrow b$, then $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ since b reduces to its normal form $b^\bullet$ (normalisation) which is the same as the normal form $a^\bullet$ of a (confluence). □

Example: normalising and confluent relations

Theorem

Normalising and confluent relations have the Z-property, for $\bullet$ the full reduction map (map to normal form).

Proof.

If $a \to b$, then $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ since b reduces to its normal form $b^\bullet$ (normalisation) which is the same as the normal form $a^\bullet$ of a (confluence). □

Corollary

Z-property for typed λ-calculi (by confluence and termination)

Example: normalising and confluent relations

Theorem

Normalising and confluent relations have the Z-property, for $\bullet$ the full reduction map (map to normal form).

Proof.

If $a \to b$, then $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ since b reduces to its normal form $b^\bullet$ (normalisation) which is the same as the normal form $a^\bullet$ of a (confluence). □

Corollary

Z-property for typed λ-calculi (by confluence and termination)

Here reverse: use Z-property to establish meta-theory

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \textit{otherwise}
\end{array}
$$

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$\begin{array}{rcll} x^\bullet & = & x & \\ (\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\ (MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x.M' \\ & = & M^\bullet N^\bullet & \text{otherwise} \end{array}$$

Example

- $I^\bullet = I$; $(I = \lambda x.x)$
- $(I(II))^\bullet = I$, $(III)^\bullet = II$;
- $((\lambda xy.x)zw)^\bullet = (\lambda y.z)w$;
- $((\lambda xy.Iyx)zI)^\bullet = (\lambda y.yz)I$;

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \textit{otherwise}
\end{array}
$$

Proof.

By induction on M:

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \textit{otherwise}
\end{array}
$$

Proof.

By induction on M:

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}
$$

Proof.

By induction on M:

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \textit{otherwise}
\end{array}
$$

Proof.

By induction on M:

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and
- (Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

$$\begin{array}{rcll} x^\bullet & = & x & \\ (\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\ (MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is an abstraction, } M^\bullet = \lambda x.M' \\ & = & M^\bullet N^\bullet & \textit{otherwise} \end{array}$$

Proof.

By induction on M:

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and
- (Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

□

Same method works for all orthogonal first/higher-order TRSs

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is a term, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}
$$

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is a term, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}
$$

Example

- $I^\bullet = I$; $(I = \lambda x.x)$
- $(I(II))^\bullet = I$, $(III)^\bullet = I$;
- $((\lambda xy.x)zw)^\bullet = z$;
- $((\lambda xy.Iyx)zI)^\bullet = Iz$

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$
\begin{aligned}
x^\bullet &= x \\
(\lambda x.M)^\bullet &= \lambda x.M^\bullet \\
(MN)^\bullet &= M'[x{:=}N^\bullet] && \text{if } M \text{ is a term, } M^\bullet = \lambda x.M' \\
&= M^\bullet N^\bullet && \text{otherwise}
\end{aligned}
$$

Proof.

Same ('an abstraction'$\mapsto$ 'a term') proof by induction on M :

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x:=N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$\begin{array}{rcll} x^\bullet & = & x & \\ (\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\ (MN)^\bullet & = & M'[x:=N^\bullet] & \text{if } M \text{ is a term, } M^\bullet = \lambda x.M' \\ & = & M^\bullet N^\bullet & \text{otherwise} \end{array}$$

Proof.

Same ('an abstraction'$\mapsto$ 'a term') proof by induction on M :

- (Substitution) $\quad M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;
- (Self) $\quad M \twoheadrightarrow M^\bullet$;

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is a term, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \textit{otherwise}
\end{array}
$$

Proof.

Same ('an abstraction'$\mapsto$ 'a term') proof by induction on M :

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is a term, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}
$$

Proof.

Same ('an abstraction'$\mapsto$ 'a term') proof by induction on M :

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and
- (Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

□

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \textit{if } M \textit{ is a term}, M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \textit{otherwise}
\end{array}
$$

Proof.

Same ('an abstraction'$\mapsto$ 'a term') proof by induction on M :

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and
- (Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

□

Moral: possibly more than one witnessing map for Z-property

Example: λ-calculus with explicit substitutions

Theorem

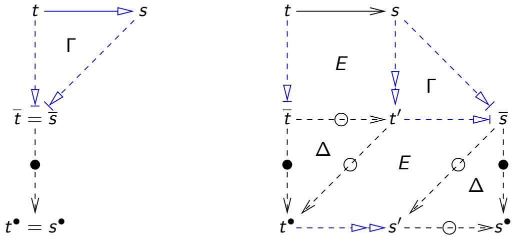
$\lambda\sigma$ has the Z-property, for $\bullet$ the map composed of first σ-normalisation ($\triangleright$), then a Beta-full development ($\multimap\!\!\bullet\!\!\to$)

Example: λ-calculus with explicit substitutions

Theorem

$\lambda\sigma$ has the Z-property, for $\bullet$ the map composed of first σ-normalisation ($\triangleright$), then a Beta-full development ($\multimap\!\!\!\bullet\!\!\to$)

Proof.



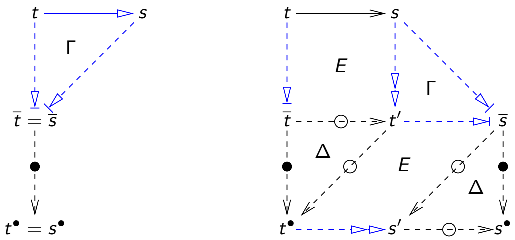
□

Example: λ-calculus with explicit substitutions

Theorem

$\lambda\sigma$ has the Z-property, for $\bullet$ the map composed of first σ-normalisation ($\triangleright$), then a Beta-full development ($\multimap\!\!\!\bullet\!\!\rightarrow$)

Proof.



□

Works for other explicit substitution/proof calculi as well.

Example: weakly orthogonal term rewriting systems

Definition

Rewrite system is weakly orthogonal, if only trivial critical pairs.

Example: weakly orthogonal term rewriting systems

Definition

Rewrite system is weakly orthogonal, if only trivial critical pairs.

Example

- λ-calculus with β and $\eta : \lambda x.Mx \to M$, if $x \notin M$;
- predecessor/successor $\quad S(P(x))) \to x \quad P(S(x)) \to x$;
- parallel-or.

Example: weakly orthogonal term rewriting systems

Theorem
Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for $\bullet$ full inside-out development

Example: weakly orthogonal term rewriting systems

Theorem

Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for $\bullet$ full inside-out development

Example: weakly orthogonal term rewriting systems

Theorem

Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for $\bullet$ full inside-out *development*

Proof.

$$c(x) \to x$$

$$f(f(x)) \to f(x)$$

$$g(f(f(f(x)))) \to g(f(f(x)))$$

Then $g(f(f(c(f(f(x)))))) \to g(f(f(f(f(x)))))$ gives Z:
$g(f(f(c(f(f(x))))))^\bullet = g(f(f(x))) = g(f(f(f(f(x)))))^\bullet$ □

Example: weakly orthogonal term rewriting systems

Theorem

Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for • full inside-out development

Proof.

$$c(x) \to x$$

$$f(f(x)) \to f(x)$$

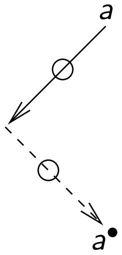
$$g(f(f(f(x)))) \to g(f(f(x)))$$

Then $g(f(f(c(f(f(x)))))) \to g(f(f(f(f(x)))))$ gives Z:
$g(f(f(c(f(f(x))))))^\bullet = g(f(f(x))) = g(f(f(f(f(x)))))^\bullet$ □

Outside-in not monotonic: not $g(f(f(x))) \twoheadrightarrow g(f(f(f(x))))$!

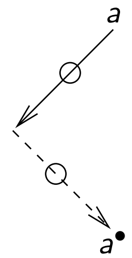
Z vs. angle

- Dehornoy:
  Z-property of $\to$ for $\bullet$;
- Takahashi:
  angle ($\langle$) property of $\to$ for $\bullet$: $\exists \multimap$, $\to \subseteq \multimap \subseteq \twoheadrightarrow$



Z vs. angle

- Dehornoy:
  Z-property of $\to$ for $\bullet$;
- Takahashi:
  angle ($\langle$) property of $\to$ for $\bullet$: $\exists\, -\!\circ\!\!\to$, $\to\ \subseteq\ -\!\circ\!\!\to\ \subseteq\ \twoheadrightarrow$



$-\!\circ\!\!\to$ steps are divisors of $-\!\bullet\!\!\to$

Z $\Leftrightarrow$ angle

Theorem

for any map $\bullet$, $Z \Leftrightarrow \langle$

Proof.

$\square$

Z $\Leftrightarrow$ angle

Theorem

for any map $\bullet$, $Z \Leftrightarrow \langle$

Proof.

(If)

$$a \longrightarrow b$$

□

Z $\Leftrightarrow$ angle

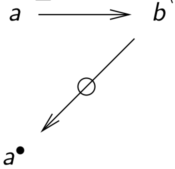
Theorem

for any map $\bullet$, $Z \Leftrightarrow \langle$

Proof.

(If)

$\rightarrow \subseteq \multimap\!\!\rightarrow$ and $\langle$



□

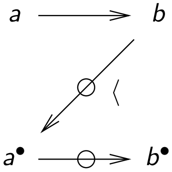
$Z \Leftrightarrow$ angle

Theorem

for any map $\bullet$, $Z \Leftrightarrow \langle$

Proof.

(If)



□

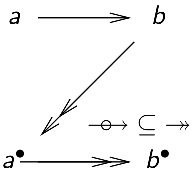
Z $\Leftrightarrow$ angle

Theorem

for any map $\bullet$, $Z \Leftrightarrow \langle$

Proof.

(If)



□

$\mathsf{Z} \Leftrightarrow$ angle

Theorem

for any map $\bullet$*,* $Z \Leftrightarrow \langle$

Proof.

(only if) Def. $a \multimap\!\!\to b$ if b between a and $a^\bullet$, i.e. $a \twoheadrightarrow b \twoheadrightarrow a^\bullet$:

- $a \to b \Rightarrow b \twoheadrightarrow a^\bullet \Rightarrow \to \subseteq \multimap\!\!\to$.
- $a \multimap\!\!\to b \Rightarrow a \twoheadrightarrow b \Rightarrow \multimap\!\!\to \subseteq \twoheadrightarrow$.
- Suppose $a \multimap\!\!\to b$.
  - $a \twoheadrightarrow b \twoheadrightarrow a^\bullet$ by definition of $\multimap\!\!\to$.
  - $a \twoheadrightarrow b \Rightarrow a^\bullet \twoheadrightarrow b^\bullet$ (monotonicity of $\bullet$) by Z
  - $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ so $b \multimap\!\!\to a^\bullet$ by definition of $\multimap\!\!\to$.

$\square$

Non-examples

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has IZ-property;

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has IZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has lZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- *slower* order: $\bullet_1 \leq \bullet_2$, if $\forall a$, $a^{\bullet_1} \twoheadrightarrow a^{\bullet_2}$;

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has IZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- *slower* order: $\bullet_1 \leq \bullet_2$, if $\forall a,\ a^{\bullet_1} \twoheadrightarrow a^{\bullet_2}$;
- $\bullet_i \leq \bullet_1 \circ \bullet_2$;

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\to$ has Z-property iff $\to^=$ has IZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- *slower* order: $\bullet_1 \leq \bullet_2$, if $\forall a$, $a^{\bullet_1} \twoheadrightarrow a^{\bullet_2}$;
- $\bullet_i \leq \bullet_1 \circ \bullet_2$;
- no slowest/minimally slow/fastest/maximally fast;

Some properties of $\bullet$s

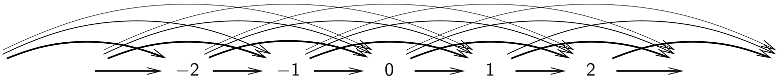
- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has IZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- *slower* order: $\bullet_1 \leq \bullet_2$, if $\forall a$, $a^{\bullet_1} \twoheadrightarrow a^{\bullet_2}$;
- $\bullet_i \leq \bullet_1 \circ \bullet_2$;
- no slowest/minimally slow/fastest/maximally fast;
- for normalising/finite systems: go to 'normal' form fastest.

Some properties of $\bullet$s

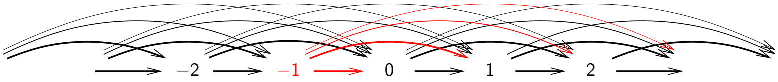
- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has IZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- *slower* order: $\bullet_1 \leq \bullet_2$, if $\forall a,\ a^{\bullet_1} \twoheadrightarrow a^{\bullet_2}$;
- $\bullet_i \leq \bullet_1 \circ \bullet_2$;
- no slowest/minimally slow/fastest/maximally fast;
- for normalising/finite systems: go to 'normal' form fastest.

Used to get ideas about (confluent) systems which do not have Z

$\mathbb{Z}$ does not have Z



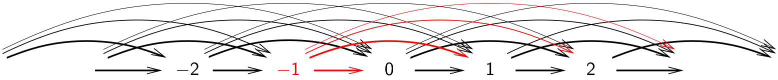
$\mathbb{Z}$ does not have Z



for given integer, no upperbound on steps from it

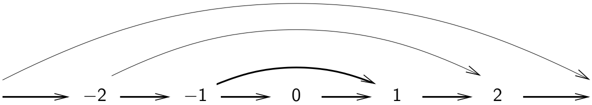
ℤ does not have Z

not finitely branching, no finite TRS

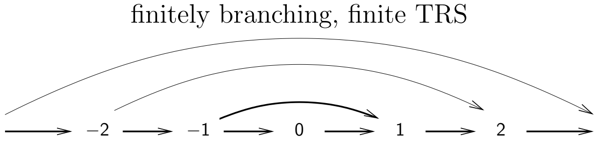


for given integer, no upperbound on steps from it

$\hat{\mathbb{Z}}$ does not imply Z



$\hat{\mathbb{Z}}$ does not imply Z

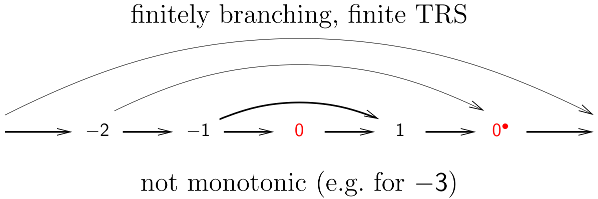


$$n(x) \to p(x) \qquad n(1) \to 0 \qquad 0 \to p(1)$$

$$n(s(x)) \to n(x)$$

$$p(x) \to p(s(x))$$

$\hat{\mathbb{Z}}$ does not imply Z



$n(x) \to p(x) \qquad n(1) \to 0 \qquad 0 \to p(1)$

$n(s(x)) \to n(x)$

$p(x) \to p(s(x))$

$\mathbb{Z}^\flat$ does have Z

$$\longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow$$

$\mathbb{Z}^\flat$ does have Z

finitely branching, finite TRS, no transitivity

$$\longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow$$

$\mathbb{Z}^\flat$ does have Z

finitely branching, finite TRS, no transitivity

$$\longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow$$

Z trivial ($i^\bullet = i + 1$)

$\mathbb{Z}^\flat$ does have Z

finitely branching, finite TRS, no transitivity

$$\longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow$$

Z trivial ($i^\bullet = i + 1$)

Examples show:

- confluent $\not\Rightarrow$ Z
- transitivity might be harmful

Conclusions

- Surprise: Z $\Leftrightarrow$ angle;

Conclusions

- Surprise: Z $\Leftrightarrow$ angle;
- Claim: gives simplest confluence proofs;

Conclusions

- Surprise: Z $\Leftrightarrow$ angle;
- Claim: gives simplest confluence proofs;
- Conjecture: β with restricted η-expansion does not have Z;

Conclusions

- Surprise: Z $\Leftrightarrow$ angle;
- Claim: gives simplest confluence proofs;
- Conjecture: β with restricted η-expansion does not have Z;
- Problem: characterise systems having Z-property;

Conclusions

- Surprise: Z $\Leftrightarrow$ angle;
- Claim: gives simplest confluence proofs;
- Conjecture: β with restricted η-expansion does not have Z;
- Problem: characterise systems having Z-property;
- Puzzle: is Z a modular property of TRSs?;

Conclusions

- Surprise: Z $\Leftrightarrow$ angle;
- Claim: gives simplest confluence proofs;
- Conjecture: β with restricted η-expansion does not have Z;
- Problem: characterise systems having Z-property;
- Puzzle: is Z a modular property of TRSs?;
- Further work: Garside categories $\Leftrightarrow$ residual systems.