Closures vs. Freeness

Vincent van Oostrom

Newman's Lemma and Random Descent, I and II

From rewrite relations to rewrite systems

Various 1-algebras in modelling rewrite systems

Newman's Lemma

Definition (Blue Preprint; Barendregt, Bergstra, Klop, Volken)

fork-in-the-road is a peak $b \twoheadleftarrow a \twoheadrightarrow c$ where b, c are not joinable; a is its root

Newman’s Lemma

Definition (Blue Preprint)

fork-in-the-road is a peak $b \twoheadleftarrow a \twoheadrightarrow c$ where b, c are not joinable; a is its root

$\twoheadrightarrow$ denotes finite $\rightarrow$-reduction

note neither reduction in a fork-in-the-road can be empty

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Lemma (fork-in-the-road; Barendregt 70s)

if a is the root of some fork-in-the-road for a locally confluent $\rightarrow$
then $a \rightarrow a'$ for some a' that again is the root of a fork-in-the-road

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Proof.

assuming the fork-in-the-road has shape $b' \twoheadleftarrow b \leftarrow a \to c \twoheadrightarrow c'$ there is valley $b \twoheadrightarrow d \twoheadleftarrow c$ by local confluence.

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Lemma (fork-in-the-road)

if a is the root of some fork-in-the-road for a *locally confluent* $\rightarrow$
then $a \rightarrow a'$ *for some* a' *that again is the root of a fork-in-the-road*

Proof.

assuming the fork-in-the-road has shape $b' \twoheadleftarrow b \leftarrow a \rightarrow c \twoheadrightarrow c'$ there is valley $b \twoheadrightarrow d \twoheadleftarrow c$ by local confluence. then either (i) $b' \twoheadleftarrow b' \twoheadrightarrow d$ is a fork-in-the-road or (ii) $b' \twoheadrightarrow d' \twoheadleftarrow d$ and $d' \twoheadleftarrow d \twoheadleftarrow c \twoheadrightarrow c'$ is a fork-in-the road $\square$

Newman’s Lemma

Lemma (fork-in-the-road)

if a is the root of some fork-in-the-road for a locally confluent $\rightarrow$
then $a \rightarrow a'$ for some a' that again is the root of a fork-in-the-road

Proof.

$$\begin{tikzcd}[column sep=small]
 & & a \arrow[dl] \arrow[dr] & & \\
 & b \arrow[dl, two heads] & & c \arrow[dr, two heads] & \\
b' & & & & c'
\end{tikzcd}$$

$\square$

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Lemma (fork-in-the-road)

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$\square$

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Proof.

$$\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
 & b \arrow[dl, two heads] \arrow[dr, two heads] \arrow[rr, phantom, "\mathrm{WCR}"] & & c \arrow[dr, two heads] \arrow[dl, two heads] & \\
b' & & d & & c'
\end{tikzcd}$$

$\square$

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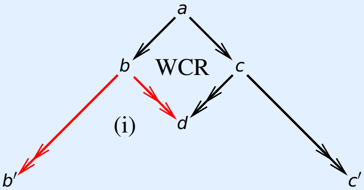

□

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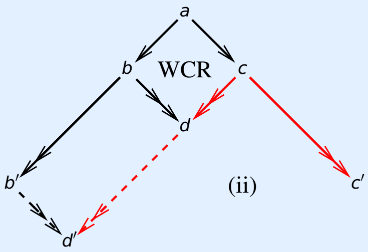

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Lemma (Newman 42)

if rewrite system $\rightarrow$ is locally confluent and terminating then it is confluent

Newman's Lemma

Lemma (fork-in-the-road)

if a is the root of some fork-in-the-road for a locally confluent $\rightarrow$
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Lemma (Newman)

if rewrite system $\rightarrow$ is locally confluent and terminating then it is complete

Proof.

if there were a fork-in-the-road there would be an infinite reduction $\square$

1st simple proof of Newman's Lemma; Barendregt bar induction; Oberwolfach 70

Newman's I Lemma

Lemma (Newman I)

if rewrite system $\to$ is locally confluent and terminating then it is complete

Newman's II Lemma

Lemma (Newman II)

if rewrite system is locally confluent and terminating then conversions between same 2 *objects deformable*

deformable by pasting local confluence diagrams

Newman's II Lemma

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if rewrite system is locally confluent and terminating then conversions between same 2 objects deformable. cyclic conversion on object contractible to loop

deformable by pasting local confluence diagrams

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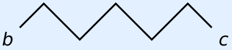
Proof by tiling (there and back again).



b and c convertible

□

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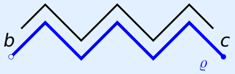

conversion ϱ from b to c (formal; composed of forward / backward steps) $\square$

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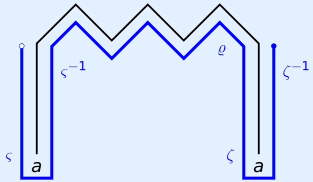

deform ϱ into $\varsigma \cdot \varsigma^{-1} \cdot \varrho \cdot \zeta \cdot \zeta^{-1}$ by reductions ς and ζ to (unique) normal form $\square$

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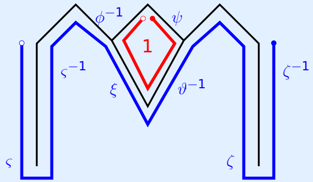

deform $\phi^{-1} \cdot \psi$ into $\xi \cdot \vartheta^{-1}$ by local confluence diagram (cyclic conversion) □

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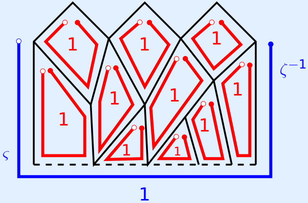

repeating deforming gives unique valley $\varsigma \cdot \zeta^{-1}$ (ς and ζ chosen for b and c) $\square$

Newman's II Lemma

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if rewrite system is locally confluent and terminating then conversions between same 2 objects deformable

Formally?

- what is a rewrite system?
- what are the conversions of a rewrite system?
- what are deformations of them?
- what structure on conversions used in reasoning?

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Applications of deformation?

- every reduction deformable into a standard one (left-normal OCRS; Klop 80)

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Applications of deformation?

- every reduction deformable into a standard one (left-normal OCRS; Klop 80)
- (i) in complete TRS deformation generated from its critical peaks
  (ii) decidable equational theories where deformation not finitely generated
  $\Longrightarrow$ deciding by means of complete finite TRSs incomplete (Squier 80s)

Newman's II Lemma

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Applications of deformation?

- every reduction deformable into a standard one (left-normal OCRS; Klop 80)
- deciding by means of complete finite TRSs incomplete (Squier 80s)
- diagrammatically enriching rewrite theory: residuation (Newman 42), axiomatic standardisation, stability, ... (Melliès 96), Garside theory (Dehornoy & al. 15), improvement (& Toyama; next), higher-categories (Ara & al. 23), homological invariants (Ikebuchi 24)...

Newman's Random Descent I

Definition (ordered diagram)

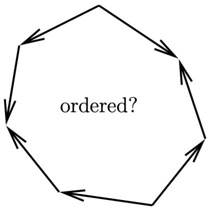
conversion ϱ for rewrite system $\rightarrow$ is ordered if number of forward steps ($\rightarrow$) is greater than or equal to number of backward steps ($\leftarrow$) in ϱ. in symbols: $d(\varrho) := \#_{\rightarrow}(\varrho) - \#_{\leftarrow}(\varrho) \geq 0$

Newman's Random Descent I

Definition (ordered diagram)

conversion ϱ for rewrite system $\to$ is ordered if $d(\varrho) := \#_{\to}(\varrho) - \#_{\leftarrow}(\varrho) \geq 0$

extends to diagrams via cyclic conversion (for object on boundary; widdershins)



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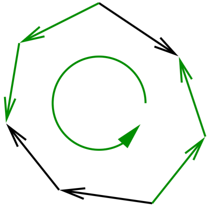

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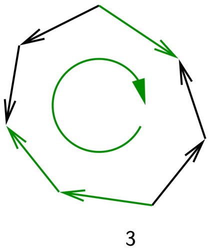

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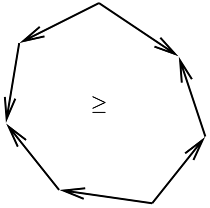

$4 \geq 3$

Newman's Random Descent I

Definition (locally Dyck)

if for every local peak $\phi \cdot \psi^{-1} : b \leftarrow a \rightarrow c$ there is a conversion $\varrho : b \leftrightarrow^* c$ giving Dyck diagram: prefixes ς of $\phi \cdot \varrho \cdot \psi^{-1}$ ordered ($d(\varsigma) > 0$ if ς non-trivial prefix)

(depends on object on boundary; source of local peak)

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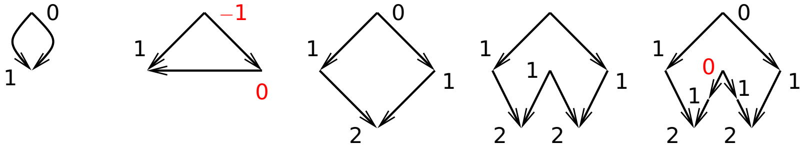

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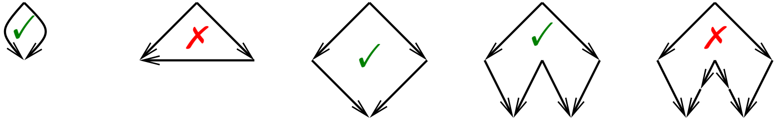

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special cases: Newman $\leftarrow \cdot \rightarrow \subseteq = \cup (\rightarrow \cdot \leftarrow)$, Toyama 92, Eriksson 93, ...
quantitative improvement of 2 systems: 07 & Toyama 16, Hamana & Muroya 24

Newman’s Random Descent I

Theorem (random descent I & Toyama 16)

locally Dyck $\implies$ *uniformly complete: all objects convertible to normal form are complete and all maximal reductions from complete object have same nf and d*

gives distance function d on objects; length of reduction to normal form else ∞

converse holds when allowed to assign weights to steps (22)

Newman's Random Descent I

Theorem (random descent I)

locally Dyck $\implies$ *uniformly complete with distance d on objects*

many heavily used (Lafont, Accattoli, ...) special cases (Newman, Toyama, Eriksson, Gramlich, ...)

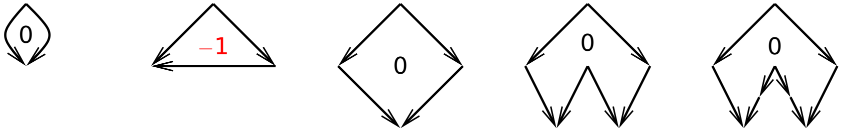
Newman's Random Descent I

Theorem (random descent I)

locally Dyck $\implies$ *uniformly complete with distance d on objects*

Corollary

object of a locally Dyck $\rightarrow$ *is non-normalising (*∞*) if it is on a non-0-cycle*



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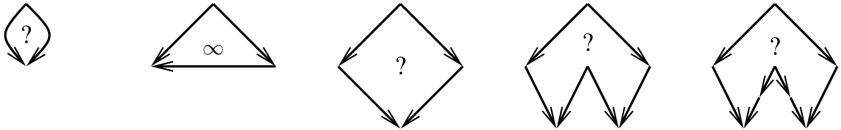


Newman's Random Descent I

Theorem (random descent I)

locally Dyck $\implies$ *uniformly complete with distance d on objects*

Proof.

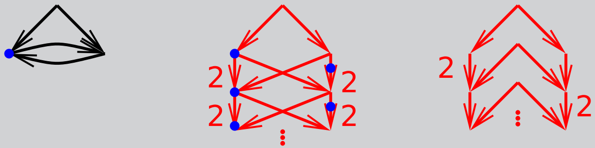
if non-0-cycle then distance *d* of objects on cycle cannot be finite $\square$

Newman's Random Descent I

Theorem (random descent I)

locally Dyck $\Longrightarrow$ *uniformly complete with distance d on objects*

Example (of locally Dyck ∞)



each locally out-of-sync; no search for ∞ reduction; no local confluence needed

Newman's Random Descent II

Theorem (random descent II)

locally Dyck $\Longrightarrow$ *uniformly complete with distance d on objects and conversions between same 2 complete objects deformable*

Newman's Random Descent II

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locally Dyck $\Longrightarrow$ uniformly complete with distance d on objects and conversions between same 2 complete objects deformable

Proof by tiling (there and back again).

as for Newman's Lemma II (non-0-diagrams impossible by corollary) $\square$

Newman's Random Descent II

Theorem (random descent II)

locally Dyck $\Longrightarrow$ uniformly complete with distance d on objects and conversions between same 2 complete objects deformable

Formally?

- what is a rewrite system?
- what are the conversions of a rewrite system?
- what are deformations of them?
- what structure on conversions used in reasoning?

Rewrite relations vs. rewrite systems

Definition

rewrite relation is a binary endorelation on a set
rewrite system sets of objects, steps, with src, tgt maps from steps to objects

various variations on core distinction, *e.g.*, families, labeling, in literature
abstract . . . used to stress no further structure assumed

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Conflated

Newman, Hindley, Stark, Melliès, Khasidashvili, . . . use rewrite systems
Dershowitz & Jouannaud, Klop, Baader & Nipkow, Ohlebusch . . . rewrite relations

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Dershowitz & Jouannaud, Klop, Baader & Nipkow, Ohlebusch . . . rewrite relations
results for one transferred to / used in the other, without mention
Newman's Lemma invariably stated for rewrite relations but attributed to
Newman 42 where it is stated for rewrite systems (amount to the same)

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Rewrite theory methodology (Newman, Huet)

factorise theory for structured (string, term, graph, ...) system $\mathcal{T}$, through theory for its underlying abstract rewrite relation / system $\to_{\mathcal{T}}$

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factorise theory for structured (string, term, graph, ...) system $\mathcal{T}$, through theory for its underlying abstract rewrite relation / system $\to_{\mathcal{T}}$
properties depending on objects like confluence, termination, reachability, convertibility (FORT) naturally stated for (constructions on) rewrite relations

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Rewrite theory methodology (Newman, Huet)

factorise theory for structured (string, term, graph, ...) system $\mathcal{T}$, through theory for its underlying abstract rewrite relation / system $\to_{\mathcal{T}}$
properties depending on steps naturally expressed for (constructions on) systems, *e.g.*, standardisation, implementation, strategies, complexity, ...

Rewrite relations vs. rewrite systems

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Rewrite theory methodology (Newman, Huet)

factorise theory for structured (string, term, graph, ...) system $\mathcal{T}$, through theory for its underlying abstract rewrite relation / system $\to_{\mathcal{T}}$

how to lift constructions from rewrite relations to rewrite systems?

Lifting constructions from rewrite relations to systems?

Rewrite relation specifications in rewrite theory

- $\rightarrow^{=}$ is the reflexive closure of $\rightarrow$

Lifting constructions from rewrite relations to systems?

Rewrite relation specifications in rewrite theory

- $\rightarrow^=$ is the reflexive closure of $\rightarrow$
- $\leftrightarrow$ is the symmetric closure of $\rightarrow$

Lifting constructions from rewrite relations to systems?

Rewrite relation specifications in rewrite theory

- $\to^{=}$ is the reflexive closure of $\to$
- $\leftrightarrow$ is the symmetric closure of $\to$
- $\twoheadrightarrow$ is the reflexive–transitive closure of $\to$ (reachability)

Lifting constructions from rewrite relations to systems?

Rewrite relation specifications in rewrite theory

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- $\twoheadrightarrow$ is the reflexive–transitive closure of $\rightarrow$
- $\leftrightarrow^{*}$ is the equivalence closure of $\rightarrow$ (convertibility)

Lifting constructions from rewrite relations to systems?

Rewrite relation specifications in rewrite theory

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new relations specified as closures

Lifting constructions from rewrite relations to systems?

Definition / construction by closure?

closure is a specification, neither a definition nor a construction (*cf.* ADT)

- specification: least relation extending rewrite relation having property
- not a definition, need not exist (uniquely; irreflexive closure?)
- different (complexity) constructions may exist (*e.g.*, for transitive closure)

Typically $\twoheadrightarrow := \bigcup_i \rightarrow^i$ for some definition of i-fold iteration

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Rewrite system constructions by analogy with closure?

rewrite system induces conversions analogously to how
rewrite relation induces convertibility

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conversion appears already in the title of Church & Rosser 36.

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maybe (let me know!), but not done / used in standard rewrite literature

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  - construction, not a specification; not analogous
  - sequences unsatisfactory: misses out on involution
    e.g., reversing twice is the identity on conversions (*cf.* involutive monoid)

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  - sequences unsatisfactory: misses out on involution
  - empty conversion is not empty sequence
  - good basis for deformation / rewriting by diagrams? (*desiderata*)

Lifting constructions from rewrite relations to systems?

Rewrite system constructions by analogy with closure?

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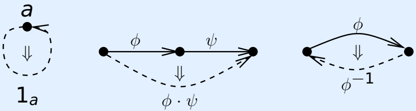
rewrite relation induces convertibility

- conversion typically defined as sequence of forward and backward steps
- conversion missing from / treated as an aside in categorical rewriting
  sometimes specificed as free groupoid (wrong; cancellation is not desired)
  it is a free dagger category (giving it away; but what is $\leftrightarrow$?)

Rewrite system constructions by freeness

Operations 1, $\cdot$, $^{-1}$ to steps subject to 1-equations:



$$
\begin{aligned}
1 \cdot \varrho &= \varrho \\
\varrho \cdot 1 &= \varrho \\
(\varrho \cdot \varsigma) \cdot \zeta &= \varrho \cdot (\varsigma \cdot \zeta)
\end{aligned}
\qquad\qquad
\begin{aligned}
1^{-1} &= 1 \\
(\varrho \cdot \varsigma)^{-1} &= \varsigma^{-1} \cdot \varrho^{-1} \\
(\varrho^{-1})^{-1} &= \varrho
\end{aligned}
$$

Rewrite system constructions by freeness

Operations 1, $\cdot$, $^{-1}$ to steps subject to 1-equations:

$$
\begin{array}{rclcrcl}
1 \cdot \varrho & = & \varrho & \qquad & 1^{-1} & = & 1 \\
\varrho \cdot 1 & = & \varrho & & (\varrho \cdot \varsigma)^{-1} & = & \varsigma^{-1} \cdot \varrho^{-1} \\
(\varrho \cdot \varsigma) \cdot \zeta & = & \varrho \cdot (\varsigma \cdot \zeta) & & (\varrho^{-1})^{-1} & = & \varrho
\end{array}
$$

Rewrite system constructions in rewrite theory

- $\rightarrow^{=}$ is carrier of free 1-reflectoid $\langle \rightarrow^{=}, 1 \rangle$ for $\rightarrow$
  unary 1 maps object of $\rightarrow$ to loop on it; no equations apply

Rewrite system constructions by freeness

Operations 1, $\cdot$, ${}^{-1}$ to steps subject to 1-equations:

$$\begin{aligned} 1 \cdot \varrho &= \varrho & 1^{-1} &= 1 \\ \varrho \cdot 1 &= \varrho & (\varrho \cdot \varsigma)^{-1} &= \varsigma^{-1} \cdot \varrho^{-1} \\ (\varrho \cdot \varsigma) \cdot \zeta &= \varrho \cdot (\varsigma \cdot \zeta) & (\varrho^{-1})^{-1} &= \varrho \end{aligned}$$

Rewrite system constructions in rewrite theory

- $\to^{=}$ is carrier of free 1-reflectoid $\langle \to^{=}, 1 \rangle$ for $\to$
- $\leftrightarrow$ is carrier of free 1-involutoid $\langle \leftrightarrow, {}^{-1} \rangle$ for $\to$
  unary ${}^{-1}$ maps step of $\to$ to reverse swapping src,tgt; involution
  $(\varrho^{-1})^{-1} = \varrho$

Rewrite system constructions by freeness

Operations 1, $\cdot$, $^{-1}$ to steps subject to 1-equations:

$$
\begin{array}{rcl@{\qquad}rcl}
1 \cdot \varrho & = & \varrho & 1^{-1} & = & 1 \\
\varrho \cdot 1 & = & \varrho & (\varrho \cdot \varsigma)^{-1} & = & \varsigma^{-1} \cdot \varrho^{-1} \\
(\varrho \cdot \varsigma) \cdot \zeta & = & \varrho \cdot (\varsigma \cdot \zeta) & (\varrho^{-1})^{-1} & = & \varrho
\end{array}
$$

Rewrite system constructions in rewrite theory

- $\to^{=}$ is carrier of free 1-reflectoid $\langle \to^{=}, 1 \rangle$ for $\to$
- $\leftrightarrow$ is carrier of free 1-involutoid $\langle \leftrightarrow, {}^{-1} \rangle$ for $\to$
- $\twoheadrightarrow$ is carrier of free 1-monoid $\langle \twoheadrightarrow, 1, \cdot \rangle$ for $\to$
  binary $\cdot$ maps 2 steps of $\to$ to their composition; free category

Rewrite system constructions by freeness

Operations 1, $\cdot$, $^{-1}$ to steps subject to 1-equations:

$$\begin{array}{rcl} 1 \cdot \varrho & = & \varrho \\ \varrho \cdot 1 & = & \varrho \\ (\varrho \cdot \varsigma) \cdot \zeta & = & \varrho \cdot (\varsigma \cdot \zeta) \end{array} \qquad \begin{array}{rcl} 1^{-1} & = & 1 \\ (\varrho \cdot \varsigma)^{-1} & = & \varsigma^{-1} \cdot \varrho^{-1} \\ (\varrho^{-1})^{-1} & = & \varrho \end{array}$$

Rewrite system constructions in rewrite theory

- $\rightarrow^{=}$ is carrier of free 1-reflectoid $\langle \rightarrow^{=}, 1 \rangle$ for $\rightarrow$
- $\leftrightarrow$ is carrier of free 1-involutoid $\langle \leftrightarrow, {}^{-1} \rangle$ for $\rightarrow$
- $\twoheadrightarrow$ is carrier of free 1-monoid $\langle \twoheadrightarrow, 1, \cdot \rangle$ for $\rightarrow$
- $\leftrightarrow^{*}$ is carrier of free 1-involutive 1-monoid $\langle \twoheadrightarrow, 1, \cdot, {}^{-1} \rangle$ for $\rightarrow$ subject to all above equations; free dagger category

Rewrite system constructions by freeness

Operations 1, $\cdot$, ${}^{-1}$ to steps subject to 1-equations:

$$\begin{aligned} 1 \cdot \varrho &= \varrho & 1^{-1} &= 1 \\ \varrho \cdot 1 &= \varrho & (\varrho \cdot \varsigma)^{-1} &= \varsigma^{-1} \cdot \varrho^{-1} \\ (\varrho \cdot \varsigma) \cdot \zeta &= \varrho \cdot (\varsigma \cdot \zeta) & (\varrho^{-1})^{-1} &= \varrho \end{aligned}$$

Rewrite system constructions in rewrite theory

- $\to^{=}$ is carrier of free 1-algebra $\langle \to^{=}, 1 \rangle$ for $\to$
- $\leftrightarrow$ is carrier of free 1-algebra $\langle \leftrightarrow, {}^{-1} \rangle$ for $\to$
- $\twoheadrightarrow$ is carrier of free 1-algebra $\langle \twoheadrightarrow, 1, \cdot \rangle$ for $\to$
- $\leftrightarrow^{*}$ is carrier of free 1-algebra $\langle \twoheadrightarrow, 1, \cdot, {}^{-1} \rangle$ for $\to$
- ...

Rewrite system constructions by freeness

Rewrite system constructions in rewrite theory

- $\to^{=}$ is carrier of free 1-algebra $\langle \to^{=}, 1 \rangle$ for $\to$
- $\leftrightarrow$ is carrier of free 1-algebra $\langle \leftrightarrow, {}^{-1} \rangle$ for $\to$
- $\twoheadrightarrow$ is carrier of free 1-algebra $\langle \twoheadrightarrow, 1, \cdot \rangle$ for $\to$
- $\leftrightarrow^{*}$ is carrier of free 1-algebra $\langle \twoheadrightarrow, 1, \cdot, {}^{-1} \rangle$ for $\to$
- $\ldots$

specified by freeness for rewrite systems (by closures for rewrite relations)

Rewrite system constructions by freeness

1-algebra?

not universal algebra because operations not total; 1-equations not equations
e.g., composition $\phi \cdot \psi$ only defined if $\mathrm{tgt}(\phi) = \mathrm{src}(\psi)$

Rewrite system constructions by freeness

1-algebras by essentially algebraic theories

particular essentially algebraic (ea-)theories (here as in Palmgren & Vickers 07, but old *e.g.*, Burroni) with partial Horn-logic having only equality as predicate

Rewrite system constructions by freeness

1-algebras by essentially algebraic theories

particular ea-theories with partial Horn-logic having only equality as predicate

- ea-theory on two sorts, objects and steps
  total symbols src and tgt from the latter to former
  (intuition: algebra having rewrite system as carrier)

Rewrite system constructions by freeness

1-algebras by essentially algebraic theories

particular ea-theories with partial Horn-logic having only equality as predicate

- ea-theory on two sorts, objects and steps
  total symbols src and tgt from the latter to former
- axioms expressed in logic, *e.g.* for the operation 1:

$$\vdash^a 1_a^\checkmark \qquad \vdash^a \mathrm{src}(1_a) = a \qquad \vdash^a \mathrm{tgt}(1_a) = a$$

  expressing that 1 is defined for every object a, which is both its src and tgt

$(\phi \cdot \psi)^\checkmark \dashv\vdash^{\phi,\psi} \mathrm{tgt}(\phi) = \mathrm{src}(\psi)$ captures partiality of composition

Rewrite system constructions by freeness

Universality of free 1-algebra

$$\begin{tikzcd} \to \arrow[r, mapsto] \arrow[dr, mapsto, "f"'] & \langle \to^F, \ldots \rangle \arrow[d, dashed, "\text{unique homomorphism}"] & \text{free } \mathbf{1}\text{-algebra} \\ & \langle \to', \ldots \rangle & \mathbf{1}\text{-algebra} \end{tikzcd}$$

Rewrite system constructions by freeness

Example (Universality of free category; *cf.* reflexive–transitive closure)

$$\begin{tikzcd}
\to \arrow[r, mapsto] \arrow[dr, mapsto, "\text{constant-}\bullet"'] & \langle \twoheadrightarrow, \mathbf{1}, \cdot \rangle \arrow[d, dashed, red, "\color{red}\text{unique homomorphism}"] & \text{path } \mathbf{1}\text{-algebra} \\
& \langle \circlearrowright, \mathbf{1}_{\circlearrowright}, \cdot_{\circlearrowright} \rangle & \text{black hole } \mathbf{1}\text{-algebra}
\end{tikzcd}$$

Rewrite system constructions by freeness

Universality of free 1-algebra

$$\begin{tikzcd} \to \arrow[r, mapsto] \arrow[dr, mapsto, "f"'] & \langle \to^F, \ldots \rangle \arrow[d, dashed, "\text{unique homomorphism}"] & \text{free } \mathbf{1}\text{-algebra} \\ & \langle \to', \ldots \rangle & \mathbf{1}\text{-algebra} \end{tikzcd}$$

do free 1-algebras exist? are they effective? efficient?

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
  (as in universal algebra: form formal expressions, quotient out equivalence)

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
- make effective by orienting / completing 1-equations into rewrite rules

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
- make effective by orienting / completing 1-equations into rewrite rules
- make efficient through appropriate data-structure
  (show elements are in bijection with normal forms of rewrite system)

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
- make effective by orienting / completing 1-equations into rewrite rules
- make efficient through appropriate data-structure

Example

given a rewrite system $\rightarrow$, the free category $\langle \twoheadrightarrow, 1, \cdot \rangle$ exists (by ea)

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
- make effective by orienting / completing 1-equations into rewrite rules
- make efficient through appropriate data-structure

Example

given a rewrite system $\rightarrow$, the free category $\langle \twoheadrightarrow, 1, \cdot \rangle$ exists
objects / steps of $\twoheadrightarrow$ are equivalence classes of formal expressions built from objects and steps of $\rightarrow$, 1 and $\cdot$, then quotienting-out 1-equations

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
- make effective by orienting / completing 1-equations into rewrite rules
- make efficient through appropriate data-structure

Example

given a rewrite system $\rightarrow$, the free category $\langle \twoheadrightarrow, 1, \cdot \rangle$ exists
objects / steps of $\twoheadrightarrow$ are equivalence classes of formal expressions
orienting 1-equations gives complete rewrite system having Haskell lists (right combs of composable steps) as normal forms making operations effective

Rewrite system constructions by freeness

Existence, effectivity, efficiency of free algebras

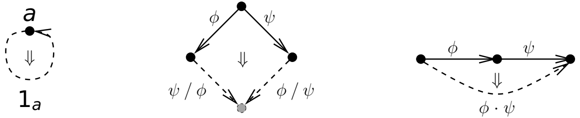
- free 1-algebras exist because exist for ea-algebras (Palmgren & Vickers)
- make effective by orienting / completing 1-equations into rewrite rules
- make efficient through appropriate data-structure

Example

given a rewrite system $\rightarrow$, the free category $\langle \twoheadrightarrow, 1, \cdot \rangle$ exists
objects / steps of $\twoheadrightarrow$ are equivalence classes of formal expressions
orienting 1-equations gives complete system having lists as normal forms
implementing normal forms as linked-lists makes them efficient (NbE)

1-Residual algebras a.k.a. residual systems (Terese 03)

pictures of 1-algebra operations at play: unit 1, residuation $/$, composition $\cdot$



novel feature of residuation: specifies (potentially new; grey) object to exist

1-Residual algebras

Example (Terese 03)

1-ra is $\langle \rightarrow, 1, / \rangle$ satisfying 1-equations:

$$\begin{array}{rcl} \phi/\phi & = & 1 \\ \phi/1 & = & \phi \\ 1/\phi & = & 1 \\ (\phi/\psi)/(\chi/\psi) & = & (\phi/\chi)/(\psi/\chi) \end{array}$$

1-Residual algebras

Example (Terese 03)

1-cra is $\langle \rightarrow, 1, /, \cdot \rangle$ satisfying 1-equations:

$$
\begin{array}{rcl}
\phi/\phi & = & 1 \\
\phi/1 & = & \phi \\
1/\phi & = & 1 \\
(\phi/\psi)/(\chi/\psi) & = & (\phi/\chi)/(\psi/\chi) \\
1 \cdot 1 & = & 1 \\
\chi/(\phi \cdot \psi) & = & (\chi/\phi)/\psi \\
(\phi \cdot \psi)/\chi & = & (\phi/\chi) \cdot (\psi/(\chi/\phi))
\end{array}
$$

1-Residual algebras

Example (Terese 03)

1-cra is $\langle \to, 1, /, \cdot \rangle$ satisfying 1-equations:

1-ra's induce 1-cra's (Plotkin, Stark, Melliès, . . .)

1-Residual algebras

Example (Terese 03)

1-cra is $\langle \rightarrow, 1, /, \cdot \rangle$ satisfying 1-equations:

1-ra's induce 1-cra's (Plotkin, Stark, Melliès, . . .)
examples: λ-calculus, orthogonal (C)TRSs / PRSs, braids, self-distributivity, . . .

1-Residual algebras

Example (Terese 03)

1-cra is $\langle \rightarrow, 1, /, \cdot \rangle$ satisfying 1-equations:

1-ra's induce 1-cra's (Plotkin, Stark, Melliès, . . .)
examples: λ-calculus, orthogonal (C)TRSs / PRSs, braids, self-distributivity, . . .

Question

why are 1-(c)ras dealt with outside extant algebraic approaches?
heavily slanted toward categories but neither appropriate nor needed (Burroni)

1-algebra of proofterms of Combinatory Logic (CL)

Steps as structures (term rewrite steps as proofterms in Terese 03)

TRS step as (first-order) term over TRS alphabet + rules ($\#$free vars as arity)
β-step as (higher-order) λ-let-term (let is β-rule reified into symbol)
termgraph (TGRS) rewrite step as termgraph over TGRS alphabet + rules
...

reuse object (term / λ-term / graph) structure for steps (extend with $\mathsf{src}, \mathsf{tgt}$)

1-algebra of proofterms of CL

Example (CL proofterms)

- @ and combinators S, K, I as binary resp. nullary symbols on objects

1-algebra of proofterms of CL

Example (CL proofterms)

- $@$ and combinators S, K, I as symbols on objects
- symbols double as symbols on steps with homomorphism axioms for $\mathrm{src}, \mathrm{tgt}$

1-algebra of proofterms of CL

Example (CL proofterms)

- @ and combinators S, K, I as symbols on objects
- symbols double as symbols on steps with homomorphism axioms for $\mathrm{src}, \mathrm{tgt}$
- symbols on steps for S, K, I-rules with axioms making lhs / rhs their src / tgt

1-algebra of proofterms of CL

Example (CL proofterms)

- $@$ and combinators S, K, I as symbols on objects
- symbols double as symbols on steps with homomorphism axioms for $\mathrm{src}, \mathrm{tgt}$
- symbols on steps for S, K, I-rules with axioms making lhs / rhs their src / tgt

can be naturally equipped with residuation giving rise to 1-(c)ra

1-algebra of proofterms of CL

Example (CL proofterms)

- @ and combinators S, K, I as symbols on objects
- symbols double as symbols on steps with homomorphism axioms for $\mathrm{src}, \mathrm{tgt}$
- symbols on steps for S, K, I-rules with axioms making lhs / rhs their src / tgt

can be naturally equipped with residuation giving rise to 1-(c)ra
tracing obtained by homomorphism mapping a term to its set of positions and a proofterm (step) to relation between the sets of positions of its src / tgt-term

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
(reductions steps of carrier $\twoheadrightarrow$ of free 1-monoid $\langle \twoheadrightarrow, 1, \cdot \rangle$ over $\rightarrow$)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions

deformation by II-rewrite rules based on (cyclic conversion $\phi \cdot \varrho \cdot \varsigma^{-1} \cdot \psi^{-1}$ of) D

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

- vertical tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
  (confluence Newman 42, proof orders Bachmair & Dershowitz 94, etc.)

Deformations of conversions

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deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- horizontal filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions
  (standardisation Klop 80, Melliès 96, Terese 03, etc.)

Deformations of conversions

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- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (Newman's II Lemma)

tiling deforms conversion $\varrho : a \leftrightarrow^* b$ in valley $\varrho_a \cdot \varrho_b^{-1} : a \leftrightarrow^* b$

Deformations of conversions

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deformation by II-rewrite rules based on D

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- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (Newman's II Lemma)

tiling deforms ($\varrho = \varrho_a \cdot \varrho_b^{-1}$ in G) conversion $\varrho : a \leftrightarrow^* b$ in valley $\varrho_a \cdot \varrho_b^{-1} : a \leftrightarrow^* b$
G free 1-group generated by Ds

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (Newman's II Lemma)

tiling deforms ($\varrho = \varrho_a \cdot \varrho_b^{-1}$ in G) conversion $\varrho : a \leftrightarrow^* b$ in valley $\varrho_a \cdot \varrho_b^{-1} : a \leftrightarrow^* b$
G free 1-group (groupoid; 1-involutive 1-monoid extended by 1-equations $\phi^{-1} \cdot \phi = 1$ and $\phi^{-1} \cdot \phi = 1$) generated by Ds (1-equations $\phi \cdot \varrho \cdot \varsigma^{-1} \cdot \psi^{-1} = 1$)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (Newman's Random Descent II)

tiling deforms conversion $\varrho : a \leftrightarrow^* b$ into valley $\varrho_a \cdot \varrho_b^{-1}$

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (Newman's Random Descent II)

tiling deforms ($[\![\varrho]\!] = [\![\varrho_a]\!] - [\![\varrho_b]\!]$ in $\mathbb{Z}$) conversion $\varrho : a \leftrightarrow^* b$ into valley $\varrho_a \cdot \varrho_b^{-1}$
$[\![\;]\!]$ homomorphism into $\mathbb{Z}$ group of integers (1-group on singleton set of objects)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
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- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (Klop's II standardisation)

filling deforms reduction $\varrho : a \twoheadrightarrow b$ into standard reduction $\varrho_s : a \twoheadrightarrow b$
for diagrams D with $\psi \cdot \varsigma$ standard and $\phi \cdot \varrho$ anti-standard (inside–out steps)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \to c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \to \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \to \psi \cdot \varsigma$ deformation on reductions

Example (Klop's II standardisation)

filling deforms reduction $\varrho : a \twoheadrightarrow b$ into standard reduction $\varrho_s : a \twoheadrightarrow b$
for diagrams D with $\psi \cdot \varsigma$ standard and $\phi \cdot \varrho$ anti-standard
in free 1-monoid (category; quotiented by Ds)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \to \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \to \psi \cdot \varsigma$ deformation on reductions

Example (Bachmair & Dershowitz's proof orders)

proof-order is morphism to well-founded 1-involutive 1-monoid (dagger cat)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (tiling = filling)

for reductions $\varrho, \varsigma : a \twoheadrightarrow b$, for the same set of diagrams D, tiling deforms $\varrho^{-1} \cdot \varsigma$ in the empty conversion 1 $\iff$ filling deforms ϱ into ς, if undercutting (UC) holds

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
deformation by II-rewrite rules based on D

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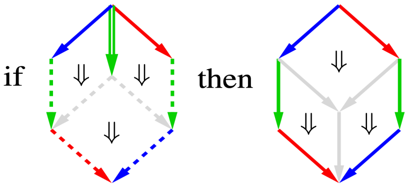
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions



Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions
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Example (tiling = filling)

for reductions $\varrho, \varsigma : a \twoheadrightarrow b$, for the same set of diagrams D, tiling deforms $\varrho^{-1} \cdot \varsigma$ in the empty conversion 1 $\iff$ filling deforms ϱ into ς, if UC holds
UC: trivial diagrams and if tiling deforms $\phi^{-1} \cdot \chi^{-1} \cdot \chi \cdot \psi$ into valley $\varsigma \cdot \zeta^{-1}$ then tiling deforms $\varsigma^{-1} \cdot \phi^{-1} \cdot \psi \cdot \zeta$ in empty conversion in smaller way (steps ϕ, ψ, χ)

Deformations of conversions

Deformation for local confluence diagram D

D peak $\phi^{-1} \cdot \psi : b \leftarrow a \rightarrow c$ of steps and valley $\varrho \cdot \varsigma^{-1} : b \twoheadrightarrow d \twoheadleftarrow c$ of reductions deformation by II-rewrite rules based on D

- tiling $\Downarrow : \phi^{-1} \cdot \psi \rightarrow \varrho \cdot \varsigma^{-1}$ deformation on conversions
- filling $\Rightarrow : \phi \cdot \varrho \rightarrow \psi \cdot \varsigma$ deformation on reductions

Example (tiling = filling)

for reductions $\varrho, \varsigma : a \twoheadrightarrow b$, for the same set of diagrams D, tiling deforms $\varrho^{-1} \cdot \varsigma$ in the empty conversion 1 $\iff$ filling deforms ϱ into ς, if UC holds applications: λ-calculus satisfies UC so permutation equivalent $\iff$ projection equivalent. positive braids satisfy UC so have least common multiples ...

Discussion

Some questions

- essentially algebraic theories implemented (tools) / formalised?

Discussion

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- essentially algebraic theories implemented (tools) / formalised?
- do 1-algebra, 1-completion etc. already exist?
  (no inherent restriction to 1-. . . ; see Ara & al. 23)

Discussion

Some questions

- essentially algebraic theories implemented (tools) / formalised?
- do 1-algebra, 1-completion etc. already exist?
- infinite reductions 1-algebraicaly?

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- ...

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- do 1-algebra, 1-completion etc. already exist?
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- . . .

feedback welcome

Discussion

Some questions

- essentially algebraic theories implemented (tools) / formalised?
- do 1-algebra, 1-completion etc. already exist?
- infinite reductions 1-algebraicaly?
- ...

feedback welcome
(will be in Lisboa until the 28^{th})

Discussion

Some questions

- essentially algebraic theories implemented (tools) / formalised?
- do 1-algebra, 1-completion etc. already exist?
- infinite reductions 1-algebraicaly?
- ...

feedback welcome

Thank you

Conclusion