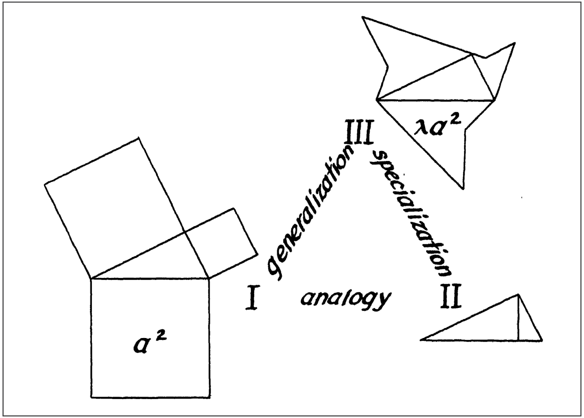
Accounting for the cost of substitution in structured rewriting

Vincent van Oostrom

Part I: Structured Rewriting

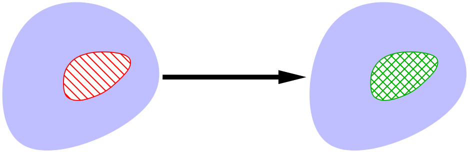
Part II: Implementation of 👀

Pólya’s triangle



Mathematics and Plausible Reasoning, Volume1, 1954, Fig. 2.3

Pólya’s triangle in structured rewriting



rewrite step $C[\ell]{\downarrow} \to C[r]{\downarrow}$ for rewrite rule $\varrho : \ell \to r$ and context C

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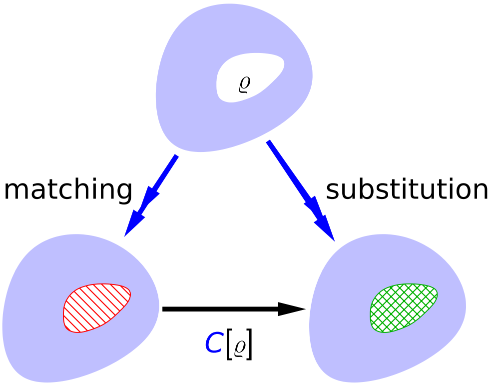

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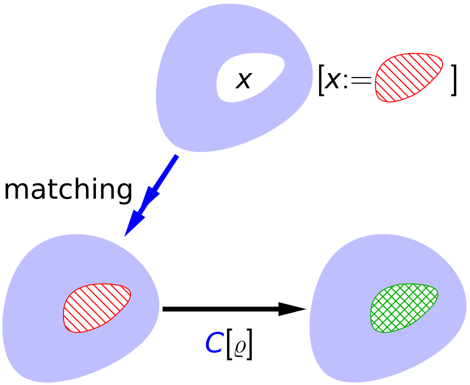

matching for rewrite step $C[\varrho] : C[\ell]{\downarrow} \to C[r]{\downarrow}$ for structure $C[x]$ and rule $\varrho : \ell \to r$

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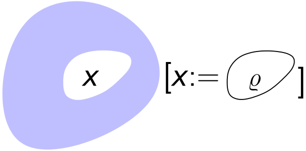

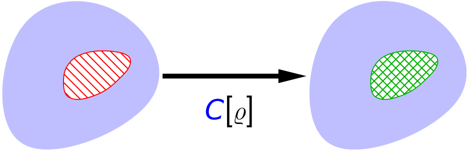

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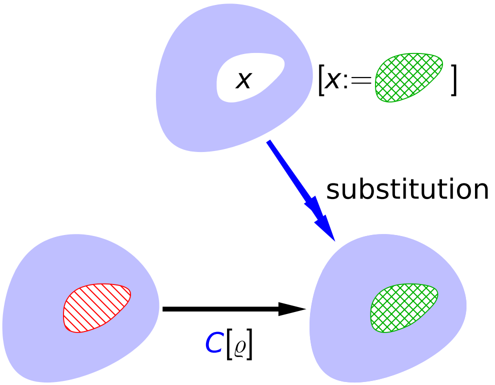
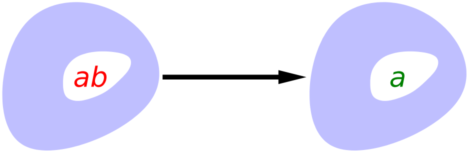


substitution for rewrite step $C[\varrho] : C[\ell]{\downarrow} \rightarrow C[r]{\downarrow}$ for structure $C[x]$ and rule $\varrho : \ell \rightarrow r$

Pólya's triangle in string rewriting



rewrite step $abb \to ab$ for rewrite rule $\varrho : ab \to b$ and context b

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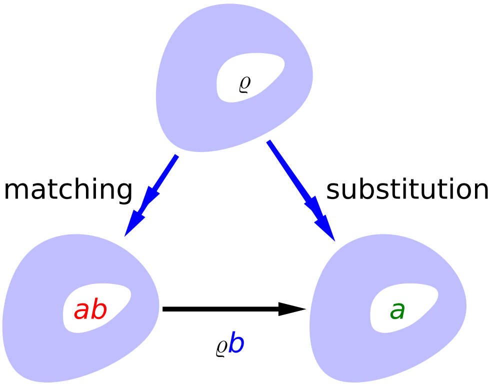

rewrite step $\varrho b : abb \to ab$ for rewrite rule $\varrho : ab \to b$ and context b

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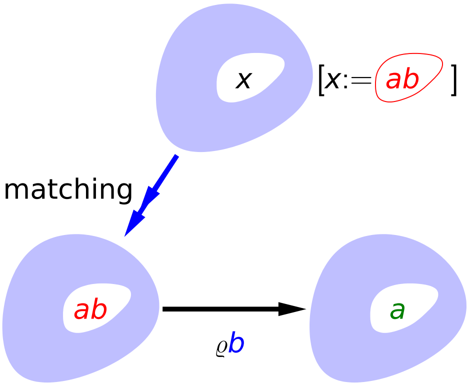

matching for rewrite step $\varrho b : abb \to ab$ for structure xb and rule $\varrho : ab \to b$

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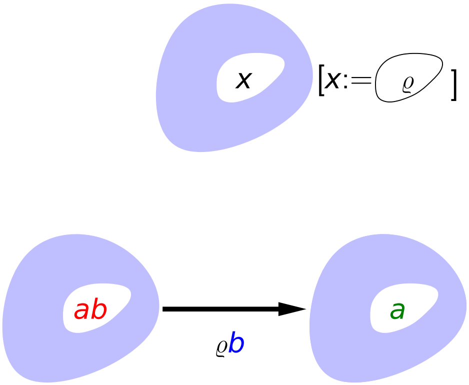

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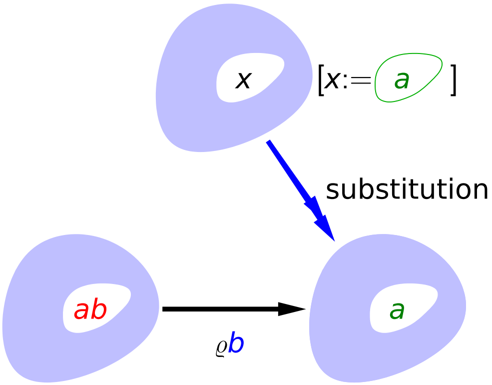

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Matching and substitution (may) come at a cost

Definition (of string)

string is an element of the free monoid A^* over alphabet A

poor: only interface to user à la abstract data types; no data structure

Matching and substitution (may) come at a cost

Definition (of Haskell strings)

string built from empty list Nil by consing (:) characters, *e.g.*, $a : (b : (b : \mathrm{Nil}))$

Matching and substitution (may) come at a cost

Example (of string rewrite step from a structured perspective)

step $\varrho b : abb \to ab$ by rewrite rule $\varrho : ab \to a$

but note $a : (b : \mathrm{Nil})$ does not occur in $a : (b : (b : \mathrm{Nil}))$

Matching and substitution (may) come at a cost

Example (of string rewrite step from a structured perspective)

step $\varrho b : abb \to ab$ by rewrite rule $\varrho : ab \to a$

how to apply $\varrho : a : (b : \mathrm{Nil}) \to a : \mathrm{Nil}$ for Haskell strings?

$$\begin{tikzcd}
 & \varrho : (b : \mathrm{Nil}) \arrow[dl, "\text{matching}"'] \arrow[dr, "\text{substitution}"] \arrow[d, phantom, "?"] & \\
a : (b : (b : \mathrm{Nil})) \arrow[rr] & {} & a : (b : \mathrm{Nil})
\end{tikzcd}$$

Matching and substitution (may) come at a cost

Example (of string rewrite step from a structured perspective)

step $\varrho b : abb \to ab$ by rewrite rule $\varrho : ab \to a$

Idea

all obtained by substituting for string variable x in $x +\!\!+ (b : \mathrm{Nil}) = x +\!\!+ b$

Matching and substitution (may) come at a cost

Example (of string rewrite step from a structured perspective)

step $\varrho b : abb \to ab$ by rewrite rule $\varrho : ab \to a$

- matching-phase decomposes $abb = a : (b : (b : \mathrm{Nil}))$ as substituting $a : (b : \mathrm{Nil})$ for x in $x ++ (b : \mathrm{Nil})$
- step $\varrho b = \varrho : (b : \mathrm{Nil})$ is formed by substituting $\varrho : \mathrm{Nil}$ for x in $x ++ (b : \mathrm{Nil})$
- substitution-phase yields $ab = a : (b : \mathrm{Nil})$ by substituting $a : \mathrm{Nil}$ for x in $x ++ (b : \mathrm{Nil})$

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still easy but shows matching- and substitution-phases (may) come at a cost

Structured rewriting

Definition (of structured rewriting modulo substitution calculus)

- structures over a signature having variables $x, y, \ldots$ over structures
- substitution calculus $\rightarrow_{\mathcal{SC}}$ on structures; $\downarrow$ denotes $\mathcal{SC}$-normal form ($\mathcal{SC}$-nf)
- rules $\varrho : \ell \rightarrow r$ with ϱ in signature and ℓ, r structures
- contexts like $C[x]$, $D[x, y]$ indicating variable occurrences
- $C[s]$ denotes replacement of variable occurrence x by structure s in C

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Definition (of structured rewrite step)

step $C[\varrho] : s \rightarrow t$, for context C and structures s, t in $\mathcal{SC}$-nf and rule $\varrho : \ell \rightarrow r$ if

$$s = C[\ell]{\downarrow}\ {}_{\mathcal{SC}}\!\twoheadleftarrow C[\ell] \rightarrow_{\varrho} C[r] \twoheadrightarrow_{\mathcal{SC}} C[r]{\downarrow} = t$$

Structured rewriting

$$\begin{tikzcd}
C[\ell] \arrow[r, "\text{replacement}"] \arrow[d, "\text{matching}"'] & C[r] \arrow[d, "\text{substitution}"] \\
t \arrow[r, "\text{step}"'] & s
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Structured rewriting

$$
\begin{tikzcd}
C[\ell_1,\dots,\ell_n] \arrow[r, "\text{replacement}"] \arrow[d, "\text{matching}"'] & C[r_1,\dots,r_n] \arrow[d, "\text{substitution}"] \\
t \arrow[r, "\circ", "\text{multistep}"'] & s
\end{tikzcd}
$$

Definition (of structured rewrite multistep)

multistep $C[\vec{\varrho}] : s \mathrel{-\!\!\circ\!\!\to} t$, for context C, structures s, t in $\mathcal{SC}$-nf, rules $\varrho_i : \ell_i \to r_i$ if

$$s = C[\ell_1,\dots,\ell_n]{\downarrow}\ {}_{\mathcal{SC}}{\twoheadleftarrow}\ C[\ell_1,\dots,\ell_n] \mathrel{-\!\!\circ\!\!\to}_{\vec{\varrho}} C[r_1,\dots,r_n] \twoheadrightarrow_{\mathcal{SC}} C[r_1,\dots,r_n]{\downarrow} = t$$

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Definition (of cost of step)

cost of step $C[\varrho] : t \to s$ is sum of costs of matching, replacement, substitution

Higher-order terms as structures

Instance

- structures: simply typed λ-terms over a simply typed signature, in long η-nf
- substitution calculus: $\mathcal{SC}$ is β modulo α

- notation: we write $x.t$ for abstraction in the $\mathcal{SC}$ (no λ)
- 1–1 correspondence between structured rewrite steps and usual HRS-steps

Higher-order terms as structures

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Example (of HRS step and its cost)

- rule $\varrho : x.f(x) \to x.d(x,x)$ over obvious signature (in fact first-order)

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Example (of HRS step and its cost)

- rule $\varrho : x.f(x) \to x.d(x,x)$ over obvious signature
- step $f(\varrho(f(a))) : f(f(f(a))) \to f(d(f(a),f(a)))$ (substituting for X in $f(X(f(a))))$

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$$\begin{tikzcd}
f((x.f(x))\,f(a)) \arrow[r, "\text{replacement}"] \arrow[d, "\text{matching}"'] & f((x.d(x,x))\,f(a)) \arrow[d, "\text{substitution}"] \\
f(f(f(a))) \arrow[r, "f(\varrho(f(a)))"'] & f(d(f(a),f(a)))
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- step $f(\varrho(f(a))) : f(f(f(a))) \to f(d(f(a), f(a)))$
- $\mathcal{SC}$-cost should take replication (in rhs) into account; choose algebra: n-ary function symbol by $(n_1, \ldots, n_n) \mapsto 1 + \sum_i n_i$ and ϱ by $n \mapsto 1 + 2n$

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  cost of step $f(\varrho(f(a)))$ is $1 + (1 + 2(1 + 1)) = 6$

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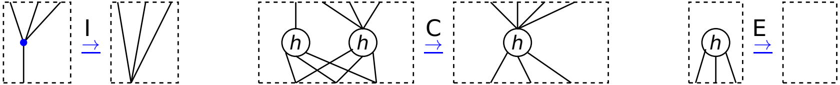
reversing rule $\varrho^{-1} : x.d(x,x) \to x.f(x)$, step $f(\varrho^{-1}(f(a)))$, same cost analysis

Termgraphs as structures

Instance

- structures: rooted dags over a signature extended with indirection •
- substitution calculus: the ж-calculus



Termgraphs as structures

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- ж-calculus has implicit garbage collection
- termgraphs in ж-normal form are maximally shared

Termgraphs as structures

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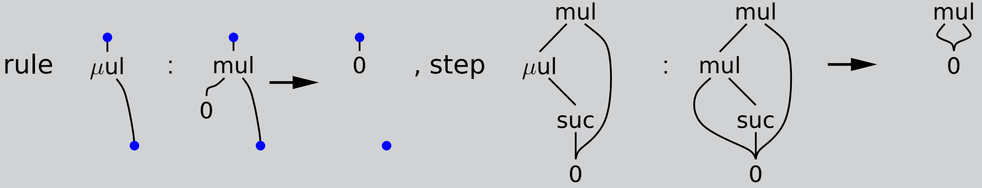
Example (of termgraph step modulo ж)



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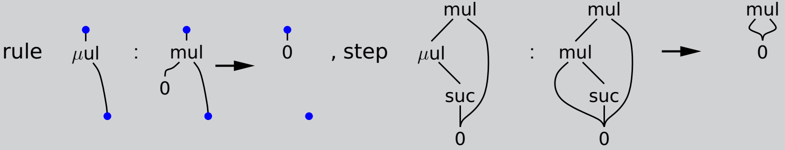

cost: substitution may knock-on erasures and sharing (bounded by graph size)

Substitution Calculi

Example (of some substitution calculi)

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- net rewriting: proofnet-calculus (PN)

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- term rewriting: linear substitution calculus (LSC)?

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- sharing graph rewriting: deep inference?

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- ...

Axioms on substitution calculi (SC)

Axioms

A1 the SC is complete (confluent and terminating)

A2 the SC is only needed for gluing (rules are closed)

A3 multisteps can be sequentialised / serialised (some development)

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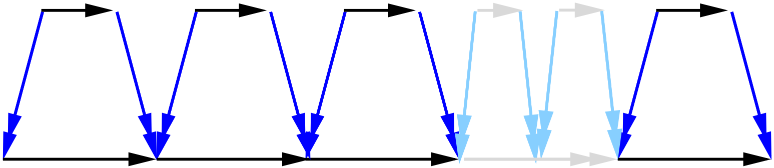
A3 multisteps can be sequentialised / serialised (some development)

$$
\begin{tikzcd}
C[\ell_0,\vec{\ell}] \arrow[rrr, "{C[\varrho_0,\vec{\ell}]}"'] \arrow[dd] \arrow[dr] & & & C[r_0,\vec{\ell}] \arrow[rrr, "{C[r_0,\vec{r}]}"'] \arrow[dl] \arrow[dd] \arrow[dr] & & & C[r_0,\vec{r}] \arrow[dl] \arrow[dd] \\
& D[\ell_0] \arrow[r, "{D[\varrho_0]}"'] \arrow[dl] & D[r_0] \arrow[dr] & & E[\vec{\ell}] \arrow[r, "{E[\vec{\varrho}]}"'] \arrow[dl] \arrow[dr, phantom, "\mathrm{IH}"] & E[\vec{r}] \arrow[dr] & \\
t_0 \arrow[rrr, "\mathcal{T}"'] & & & t_1 \arrow[rrr, two heads, "\mathcal{T}"'] & & & t_n
\end{tikzcd}
$$

Exploiting substitution calculi to redistribute costs

Cost-saving observations

- should avoid substitution and subsequent matching inverse to each other
- matching more lhss simultaneously (multisteps) enables parallelism
- by not going to $\mathcal{SC}$-normal forms we may sometimes eliminate matching



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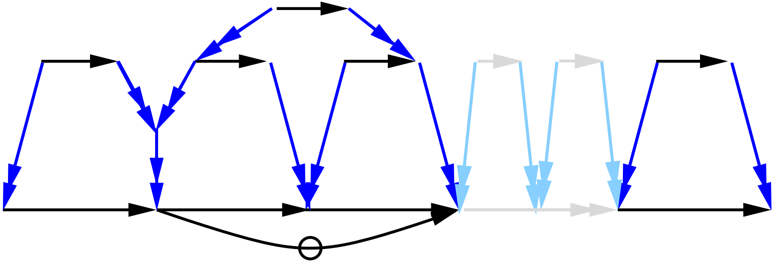

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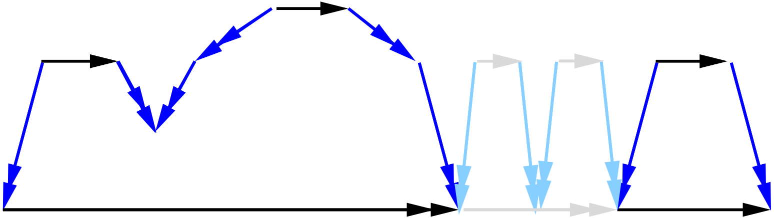

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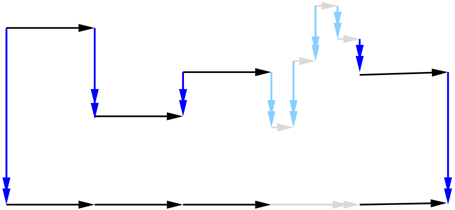

deterministic rewrite system $\Longrightarrow$ dipper SC

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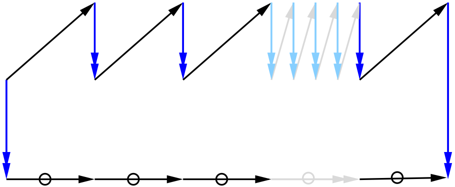

matching trivial $\Longrightarrow$ cascading SC

Exploiting substitution calculi to redistribute costs

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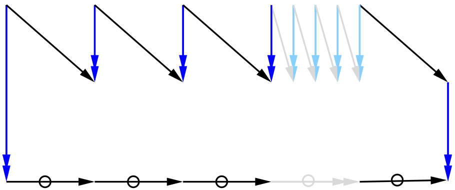

substitution trivial $\Longrightarrow$ stepping SC

Dipper SC Example: leftmost string rewriting

Dipper idea

record current location in string (state) by splitting it into three strings: to the left, matched, to the right

since costly to scan from the start after each step

Dipper SC Example: leftmost string rewriting

Example

for rule $\varrho : ab \to a$ and string *aaba*

$$(\varepsilon, \varepsilon, babb) \leftarrow (b, \varepsilon, abb) \leftarrow (b, a, bb) \leftarrow (b, ab, b) \leftarrow (b, \varrho, b) \to$$

$$(b, a, b) \to (b, ab, \varepsilon) \to (b, \varrho, \varepsilon) \to (b, a, \varepsilon) \to (\varepsilon, \varepsilon, ba)$$

- state (split) components combine to current string
- state represents how far we have currently dipped into the string
- substitution calculus scans string from left to right
- back up (at most) 1 after $\to$-step

Implementation of 👀

Motivation for 👀

- TRSs interesting as target when compiling functional programming
- matching is simple (lhss linear and exactly two function symbols; cascading)
- substitution can be made to avoid replication by termgraph rewriting
- cost (time and space) linear by combining the above two items

Applicative Inductive Interaction System ()

Definition (of an)

TRS with signature $\{@/2, C_1/n_1, C_2/n_2, \ldots\}$ and for each i, rule $\varrho_{C_i}(x_0, x_1, \ldots, x_{n_i})$:

$$C_i(x_1, \ldots, x_{n_i})\, x_0 \quad \to \quad r$$

right-hand side r constructed from variables, @, and constructors C_j, for $j < i$

notational conventions:

- application @ infix, implicit as in Combinatory Logic (CL)
- usually leave arguments of rule symbols implicit (derivable from lhs of rule)

Applicative Inductive Interaction System (👁)

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right-hand side r constructed from variables, @, and constructors C_j, for $j < i$

Example (of an 👁)

$$\begin{array}{rcrcl} \varrho_C(x_0, x_1, x_2) & : & C(x_1, x_2)\, x_0 & \to & x_1\, (x_2\, x_0) \\ \varrho_D(x_0) & : & D\, x_0 & \to & C(x_0, x_0) \end{array}$$

Applicative Inductive Interaction System (👁)

Definition (of an 👁)

TRS with signature $\{@/2, C_1/n_1, C_2/n_2, \ldots\}$ and for each i, rule $\varrho_{C_i}(x_0, x_1, \ldots, x_{n_i})$:

$$C_i(x_1, \ldots, x_{n_i})\, x_0 \quad \rightarrow \quad r$$

right-hand side r constructed from variables, @, and constructors C_j, for $j < i$

Example (👁 confluent (via orthogonality), Turing complete (via CL))

$$\begin{array}{rrcl}
\varrho_{S_2}: & S_2(x_1, x_2)\, x_0 & \rightarrow & (x_1\, x_0)\,(x_2\, x_0) \\
\varrho_{S_1}: & S_1(x_1)\, x_0 & \rightarrow & S_2(x_1, x_0) \\
\varrho_{S}: & S\, x_0 & \rightarrow & S_1(x_0)
\end{array}
\qquad
\begin{array}{rrcl}
\varrho_{K_1}: & K_1(x_1)\, x_0 & \rightarrow & x_1 \\
\varrho_{K}: & K\, x_0 & \rightarrow & K_1(x_0)
\end{array}$$

Applicative Inductive Interaction System (👁)

Example (of an 👁)

$$\begin{array}{rcrcl} \varrho_C(x_0, x_1, x_2) & : & C(x_1, x_2)\,x_0 & \to & x_1\,(x_2\,x_0) \\ \varrho_D(x_0) & : & D\,x_0 & \to & C(x_0, x_0) \end{array}$$

Example (two-step reduction $(\varrho_C(D, D)\,z_1) \cdot (\varrho_D(D\,z_1))$)

$$C(D, D)\,z_1 \to_{\varrho_C(z_1, D, D)} D\,(D\,z_1) \to_{\varrho_D(D\,z_1)} C(D\,z_1, D\,z_1)$$

duplicates $D\,z_1$ redex; ends in (constructor C-)head normal form

Implementing 👀

Question (on implementation of 👁)

do 👀 have an efficient (hyper-(head-))normalising reduction strategy?

efficient in time / space

Implementing 👁👁

Question (on implementation of 👁)

do 👁👁 have an efficient (hyper-(head-))normalising reduction strategy?

efficient in time / space

Observations (explored further below)

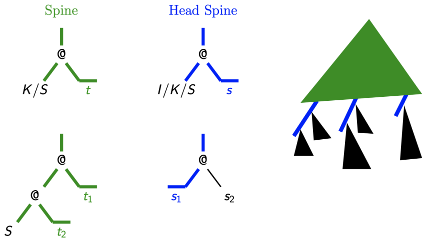
- spine strategy is (hyper-(head-))normalising
  since every 👁 is left-normal orthogonal TRSs
- matching-phase is trivial (since lhss left-linear, comprise two symbols)
  substitution-phase not trivial (rhss may replicate arguments)

Spine strategy

Definition

Spine: if head normal form recur, else Head Spine.

Head Spine: recur on left.



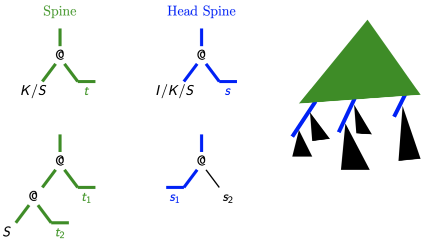
Example

S(SISI)(K(IK))

Spine strategy

Definition

Spine: if head normal form recur, else Head Spine.
Head Spine: recur on left.



Lemma

Every term not in normal form has Spine redex

Spine strategy for 👁

Definition (of spine for 👁-terms)

- spine: t or $x\, t_1\,,\ldots,\, t_n$
- head spine: x or $C(t_1,\ldots,t_n)$ or $t\, s$

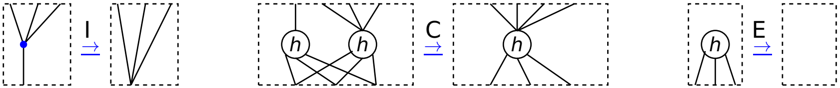
Lemma (normalising strategy)

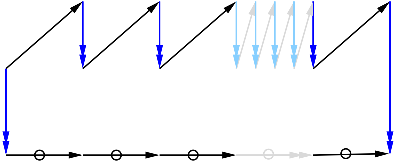
- *every term not in normal form has redex-pattern on spine, so a strategy*
- *spine strategy is a normalising strategy having random descent*

- random descent: reductions to normal form have same length / measure
- leftmost–outermost strategy is a spine-strategy

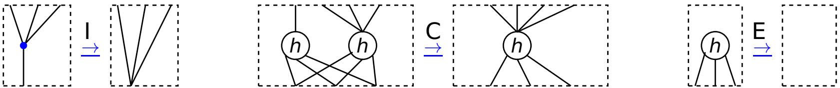
Implementing 👁 in termgraphs by cascading ж

Recall termgraph rewriting with ж-calculus as SC, and cascading:



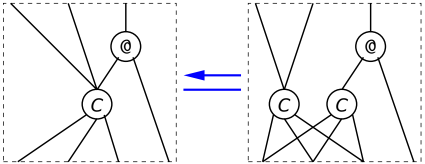


Implementing 👁 in termgraphs by cascading ж



Idea (minimal unsharing; Wadsworth's admissibility)

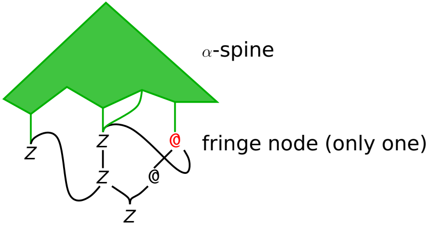
- instead of maximal sharing, unshare only constructors in redex-patterns
- goal: amortise cost of ж-steps by charging 👁-steps



Termgraph α-spine strategy

Definition (of (head / α-)spine nodes)

- spine: head spine, or such in normal form (hsnf) with spine vertebrae
- head spine: path from root through bodies of @,● to variable or constructor
- α-spine: spine prefix; fringe nodes: nodes covered by α-spine



Termgraph α-spine strategy

Definition (of (head / α-)spine nodes)

- spine: head spine, or such in normal form (hsnf) with spine vertebrae
- head spine: path from root through bodies of @,● to variable or constructor
- α-spine: spine prefix; fringe nodes: nodes covered by α-spine

Lemma

every termgraph not in normal form has a spine redex-pattern, and any (proper) α-spine prefix of it has a non-empty fringe

Proof.

by minimality using acyclicity of termgraphs □

Termgraph α-spine strategy

Definition (of (head / α-)spine nodes)

- spine: head spine, or such in normal form (hsnf) with spine vertebrae
- head spine: path from root through bodies of @,• to variable or constructor
- α-spine: spine prefix; fringe nodes: nodes covered by α-spine

Definition (of α-spine strategy)

reduce head spines from fringe nodes to hsnf and recurse on spine vertebrae

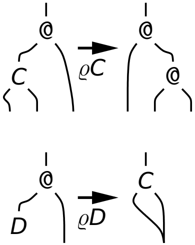
by lemma always some step possible until whole termgraph is α-spine (in nf)

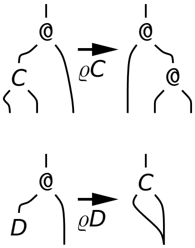
Example α-spine reduction (Java code ⇒ dot ⇒ graphs)

recall 👁-rules:

$\varrho_C : C(x_1, x_2)\, x_0 \to x_1\,(x_2\, x_0)$
$\varrho_D : D\, x_0 \to C(x_0, x_0)$

as termgraph rules:





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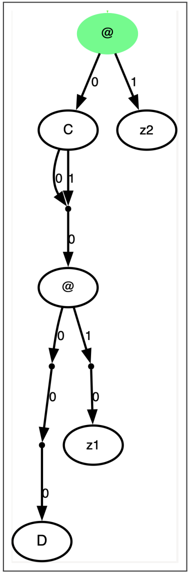

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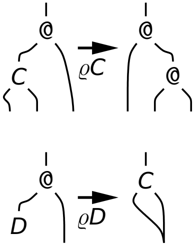

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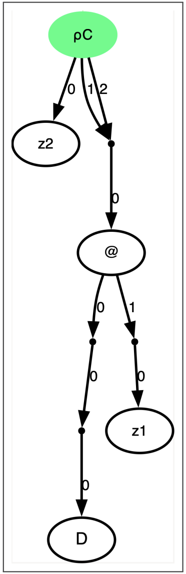

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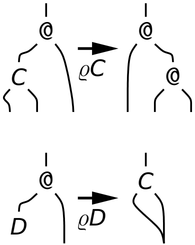
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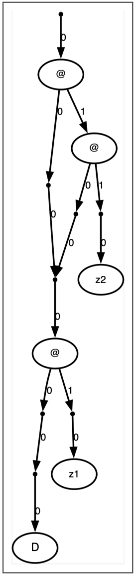
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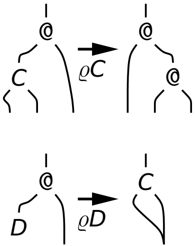
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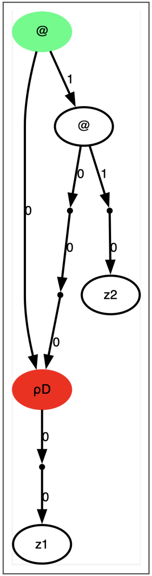
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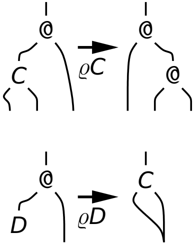
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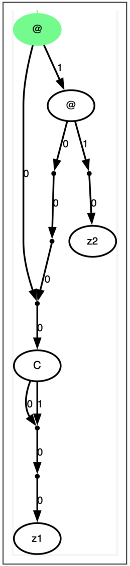
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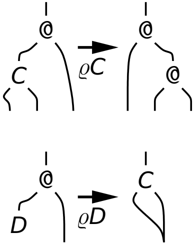
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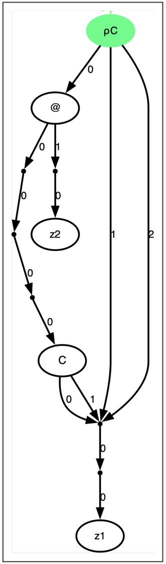
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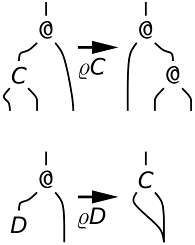

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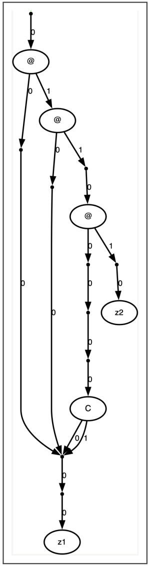

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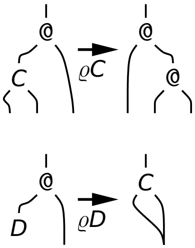

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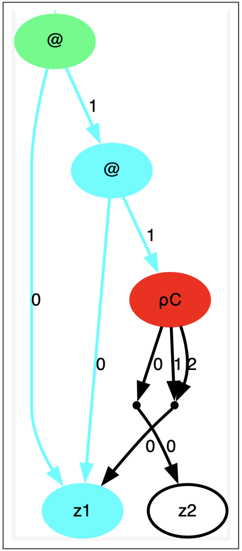

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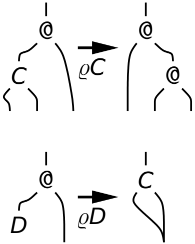
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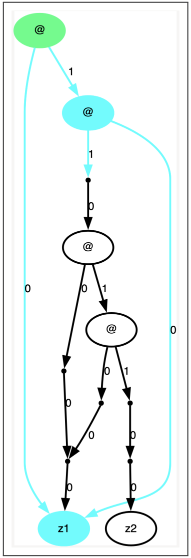
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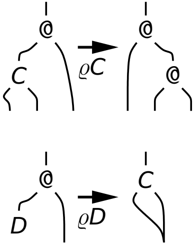
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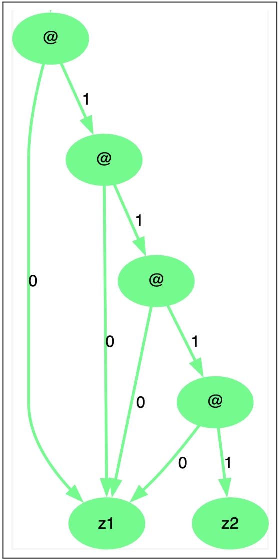
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Correspondence between termgraphs and terms

Theorem

- *α-spine step maps to multistep having at least one spine redex ((hyper-)(head) normalising strategy)*

Correspondence between termgraphs and terms

Theorem

- *α-spine step maps to multistep having at least one spine redex*
- *multistep comprises redex-patterns having same creation history (family-step, so optimal strategy (qua horizontal sharing))*

Correspondence between termgraphs and terms

Theorem

- *α-spine step maps to multistep having at least one spine redex*
- *multistep comprises redex-patterns having same creation history*
- *cost and size linear in number of termgraph steps*
  (graph grows linearly; strategy visits links only few times à la DFS)

Correspondence between termgraphs and terms

Theorem

- *α-spine step maps to multistep having at least one spine redex*
- *multistep comprises redex-patterns having same creation history*
- *cost and size linear in number of termgraph steps*

- α-spine reduction length not longer than spine
  (👀 are orthogonal for which doing more in parallel is better)
- number of spine steps always the same (random descent property)
- reduction length not longer than that of leftmost–outermost stategy

Decompiling 👁 to the λ-calculus

Definition (of tree homomorphism $(\ )_\lambda$ into λ-terms)

$C_i(t_1,\ldots,t_n) \mapsto (\lambda x_0.(r)_\lambda)[x_1,\ldots,x_n{:=}t_1,\ldots,t_n]$

- capture avoiding substitution (avoid capture of free variables of the t_k)
- $(t[\vec{x}{:=}\vec{t}])_\lambda = (t)_\lambda[\vec{x}{:=}\overrightarrow{(t)_\lambda}]$ (substitution lemma)
- well-defined by 👁 being inductive (in r only C_j for $j < i$ may occur)

Decompiling to the λ-calculus

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$C_i(t_1, \ldots, t_n) \mapsto (\lambda x_0.(r)_\lambda)[x_1, \ldots, x_n{:=}t_1, \ldots, t_n]$

Example (of tree homomorphism for example)

$$
\begin{array}{rcl|rcl}
 & \text{rule} & & & \text{tree homomorphism} & \\
\varrho_C : C(x_1, x_2)\, x_0 & \to & x_1\,(x_2\, x_0) & C(t_1, t_2) & \mapsto & \lambda x_0.t_1\,(t_2\, x_0) \\
\varrho_D : \qquad D\, x_0 & \to & C(x_0, x_0) & D & \mapsto & \lambda x_0 x_0'.x_0\,(x_0\, x_0')
\end{array}
$$

as $D \mapsto \lambda x_0.(C(x_0, x_0))_\lambda = \lambda x_0.(\lambda x_0.x_1\,(x_2\, x_0))[x_1, x_2{:=}x_0, x_0] =_\alpha \lambda x_0 x_0'.x_0\,(x_0\, x_0')$

Decompiling 👁 to the λ-calculus

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$$C_i(t_1,\ldots,t_n) \mapsto (\lambda x_0.(r)_\lambda)[x_1,\ldots,x_n{:=}t_1,\ldots,t_n]$$

Example (of tree homomorphism for example 👁)

	rule			tree homomorphism		
$\varrho_C :$	$C(x_1,x_2)\,x_0$	$\to$	$x_1\,(x_2\,x_0)$	$C(t_1,t_2)$	$\mapsto$	$\lambda x_0.t_1\,(t_2\,x_0)$
$\varrho_D :$	$D\,x_0$	$\to$	$C(x_0,x_0)$	D	$\mapsto$	$\lambda x_0 x_0'.x_0\,(x_0\,x_0')$

- D maps to the Church numeral $\underline{2}$ for $\underline{n} := \lambda sz.s^n\,z$
- S maps to $\lambda xyz.x\,z\,(y\,z)$ and K to $\lambda xy.x$ as expected / hoped for

Decompiling 👁 to the λ-calculus

Definition (of tree homomorphism $(\ )_\lambda$ into λ-terms)

$C_i(t_1, \ldots, t_n) \mapsto (\lambda x_0.(r)_\lambda)[x_1, \ldots, x_n := t_1, \ldots, t_n]$

Lemma (implementation of 👁 by $\lambda\beta$)

if $t \rightarrow_{\text{👁}} s$ *then* $(t)_\lambda \rightarrow_\beta (s)_\lambda$

Decompiling 👁 to the λ-calculus

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Lemma (implementation of 👁 by $\lambda\beta$)

if $t \to_{\text{👁}} s$ *then* $(t)_\lambda \to_\beta (s)_\lambda$

Example (of implementing $D\,(D\,z_1) \to_{\text{👁}} C(D\,z_1, D\,z_1)$)

$$(D\,(D\,z_1))_\lambda = (\lambda xy.x\,(x\,y))\,(\underline{2}\,z_1) \to_\beta \lambda y.\underline{2}\,z_1\,(\underline{2}\,z_1\,y) =_\alpha (C(D\,z_1, D\,z_1))_\lambda$$

Compiling the λ-calculus to 👁

Lemma (??)

if $M \rightarrow_{\beta} N$ *then* $(M)_{👁} \rightarrow_{\mathcal{I}} (N)_{👁}$ *for* $\mathcal{I}$ *an* 👁

Compiling the λ-calculus to 👁

Lemma (??)

if $M \to_\beta N$ *then* $(M)_{\text{👁}} \to_{\mathcal{I}} (N)_{\text{👁}}$ *for* $\mathcal{I}$ *an* 👁

- no implementation $(\;)_{\text{👁}}$ can achieve that, for full β
- for weak β (wβ; contract redex if has no variable bound outside) it can:
- weak β is first-order (α-conversion never needed), and
- weak β basis of Haskell (no contraction under λ, but that's not confluent)

Compiling the λ-calculus to 👁

Lemma (??)

if $M \to_\beta N$ then $(M)_{👁} \to_{\mathcal{I}} (N)_{👁}$ for $\mathcal{I}$ an 👁

Definition (of $(\ )_{👁}$ mapping a λ-term to a pair of an 👁 and term in it)

- $(x)_{👁} := (\emptyset, x)$
- $(M_1\, M_2)_{👁} := (\mathcal{I}_1 \cup \mathcal{I}_2, t_1\, t_2)$, where $(\mathcal{I}_i, t_i) := (M_i)_{👁}$ for $i \in \{1, 2\}$
- $(\lambda x.M)_{👁} := (\{\varrho_C : C(z_1, \ldots, z_n)\, x \to r[z_1, \ldots, z_n]\} \cup \mathcal{I}, C(t_1, \ldots, t_n))$, where $(\mathcal{I}, r[t_1, \ldots, t_n]) := (M)_{👁}$, r skeleton, t_i maximal x-free subterm occurrences

do allow components to share constructors when these have the same rules
compilation known variation on the abstraction algorithm (custom combinators)

Compiling the λ-calculus to 👁

Definition (of 👁-lifting)

- $(x)_{👁} := (\emptyset, x)$
- $(M_1\, M_2)_{👁} := (\mathcal{I}_1 \cup \mathcal{I}_2, t_1\, t_2)$, where $(\mathcal{I}_i, t_i) := (M_i)_{👁}$ for $i \in \{1, 2\}$
- $(\lambda x.M)_{👁} := (\{\varrho_C : C(z_1, \ldots, z_n)\, x \to r[z_1, \ldots, z_n]\} \cup \mathcal{I}, C(t_1, \ldots, t_n))$, where $(\mathcal{I}, r[t_1, \ldots, t_n]) := (M)_{👁}$, r skeleton, t_i maximal x-free subterm occurrences

Example (of $(\underline{2})_{👁}$; recall $\underline{2} := \lambda xy.x\,(x\,y)$)

Compiling the λ-calculus to 👁

Definition (of 👁-lifting)

- $(x)_{\text{👁}} := (\emptyset, x)$
- $(M_1\, M_2)_{\text{👁}} := (\mathcal{I}_1 \cup \mathcal{I}_2, t_1\, t_2)$, where $(\mathcal{I}_i, t_i) := (M_i)_{\text{👁}}$ for $i \in \{1, 2\}$
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Example (of $(\underline{2})_{\text{👁}}$; recall $\underline{2} := \lambda xy.x\,(x\,y)$)

- $(x\,(x\,y))_{\text{👁}} := (\emptyset, x\,(x\,y))$ using only first two items of the definition

Compiling the λ-calculus to 👁

Definition (of 👁-lifting)

- $(x)_{\text{👁}} := (\emptyset, x)$
- $(M_1\, M_2)_{\text{👁}} := (\mathcal{I}_1 \cup \mathcal{I}_2, t_1\, t_2)$, where $(\mathcal{I}_i, t_i) := (M_i)_{\text{👁}}$ for $i \in \{1, 2\}$
- $(\lambda x.M)_{\text{👁}} := (\{\varrho_C : C(z_1, \ldots, z_n)\, x \to r[z_1, \ldots, z_n]\} \cup \mathcal{I}, C(t_1, \ldots, t_n))$, where $(\mathcal{I}, r[t_1, \ldots, t_n]) := (M)_{\text{👁}}$, r skeleton, t_i maximal x-free subterm occurrences

Example (of $(\underline{2})_{\text{👁}}$; recall $\underline{2} := \lambda xy.x\,(x\,y)$)

- $(x\,(x\,y))_{\text{👁}} := (\emptyset, x\,(x\,y))$, so
- $(\lambda y.x\,(x\,y))_{\text{👁}} := (\{\varrho_C : C(z_1, z_2)\, y \to z_1\,(z_2\, y)\}, C(x, x))$
  since x and x are maximal y-free subterm occurrences in $x\,(x\,y)$

Compiling the λ-calculus to 👁

Definition (of 👁-lifting)

- $(x)_{\text{👁}} := (\emptyset, x)$
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- $(\lambda xy.x\,(x\,y))_{\text{👁}} := (\{\varrho_C : C(z_1, z_2)\, y \to z_1\,(z_2\,y), \varrho_D : D\, x \to C(x, x)\}, D)$ since no x-free subterm occurrence in $C(x, x)$

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Lemma (👁-lifting)

if $M \to_{\mathrm{w}\beta} N$ then $(M)_{\text{👁}} \to_{\mathcal{I}} (N)_{\text{👁}}$ for some 👁-lifting $\mathcal{I}$.

Proof.

if $M \to_{\mathrm{w}\beta} N$ and $(\mathcal{I}, t) := (M)_{\text{👁}}$ then $t \to_{\mathcal{I}} s$ for some $(\mathcal{I}', s) := (N)_{\text{👁}}$ with $\mathcal{I} \supseteq \mathcal{I}'$ $\square$

Implementing wβ-reduction via

Observations

- wβ never needs α-conversion, so essentially first-order (that's why it was chosen for Haskell)
- indeed, any λ-term *M* compiles to an and term *t* in it, such that rewriting from *M* respectively *t* is isomorphic
- compilation (finding mfss) can be done efficiently in time and space

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Perspective

Haskell is based on orthogonal 1^{st}-order term rewriting (👁👁), not λ-calculus

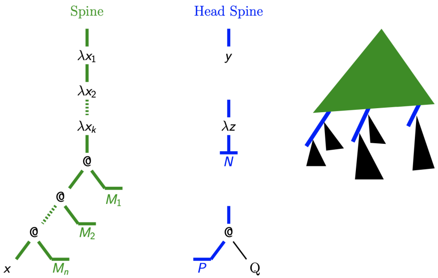
What about Spine strategies for full β?

Spine strategy

Definition

Spine: if head normal form recur, else Head Spine.

Head Spine: recur on left.



Example

$x((\lambda x.(\lambda z.zz))y)(xx)(II)$

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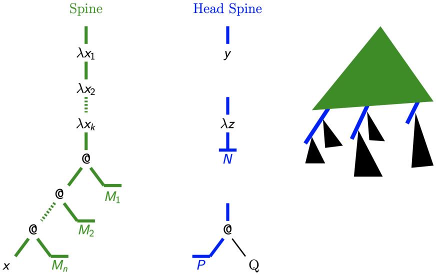


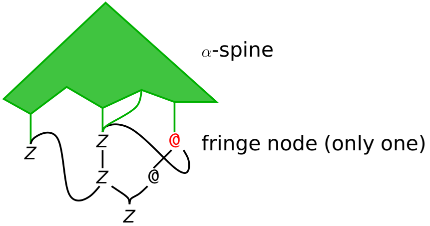
Lemma

Every term not in normal form has Spine redex

Termgraph α-spine strategy **adapted** to spine-β

Definition (of (head / α-)spine nodes)

- spine: head spine, or such in normal form (hsnf) with spine vertebrae
- head spine: path from root through bodies of @,• to variable or constructor
- α-spine: spine prefix; fringe nodes: nodes covered by α-spine



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Definition (of α-spine strategy)

reduce head spines from fringe nodes to hsnf and recurse on spine vertebrae
rewrite fringe constructor $C(t_1, \ldots, t_n)$ to $\lambda x.C(t_1, \ldots, t_n)\, x$ for x fresh

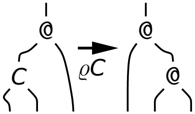
idea: a combinator on fringe / α-spine is a λ-abstraction (in the β-nf), so may iterate on its body, effectuated in 👁 by suppling a fresh variable

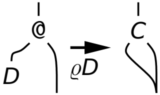
Example α-spine reduction (Java code $\Rightarrow$ dot $\Rightarrow$ graphs)

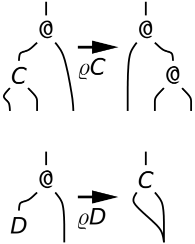
recall -rules:

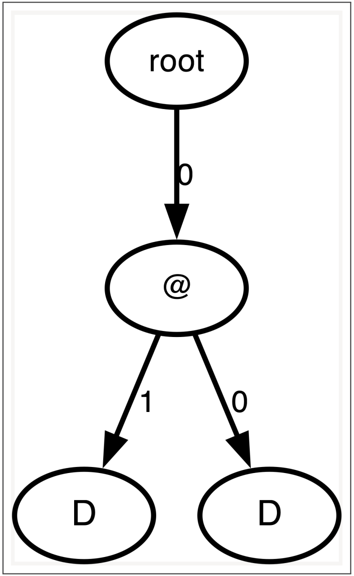
$$\varrho_C : C(x_1, x_2)\, x_0 \to x_1\,(x_2\, x_0)$$
$$\varrho_D : D\, x_0 \to C(x_0, x_0)$$

and termgraph rules:






@
C
ϱC
@
@
@
D
ϱD
C


root
0
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1
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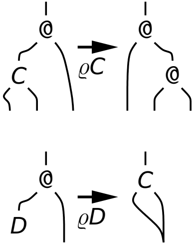
D



@
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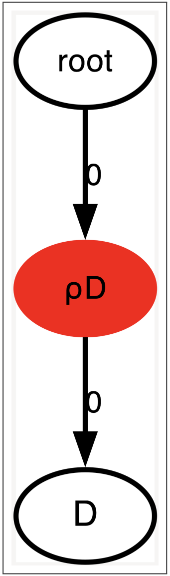

root
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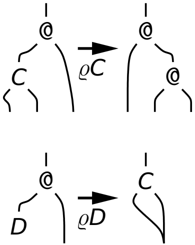
@
C
@
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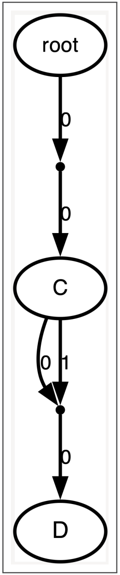
root
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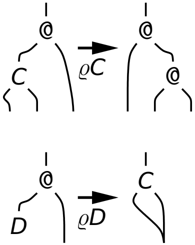
@
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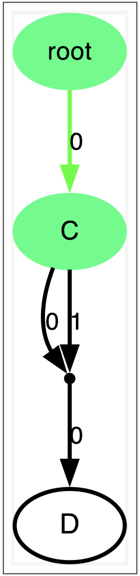
root
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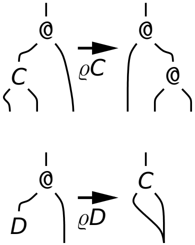
@
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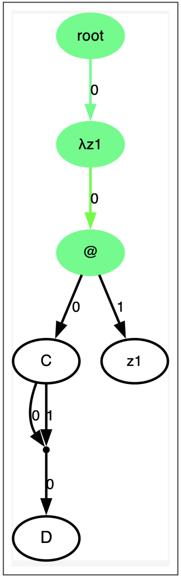
root
0
λz1
0
@
0
1
C
z1
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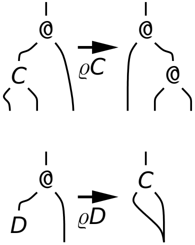
D


@
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ϱC
C
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C
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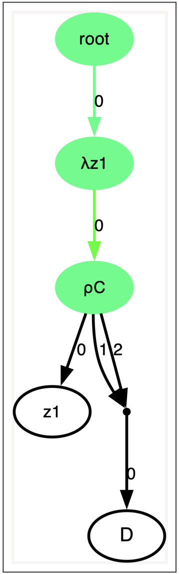
root
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λz1
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ρC
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z1
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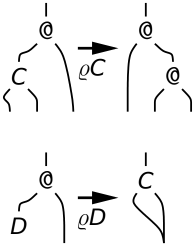
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ϱC
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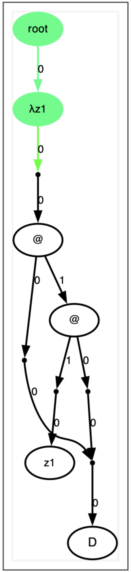
root
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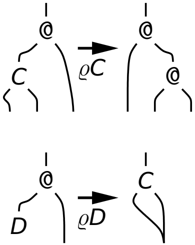
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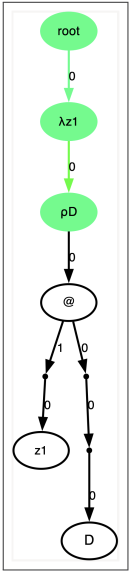
ϱD
C


root
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λz1
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ρD
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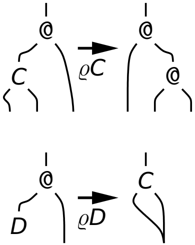
D


@
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D
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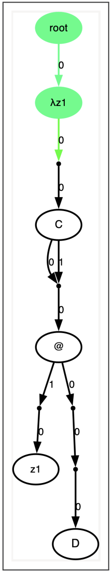
root
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λz1
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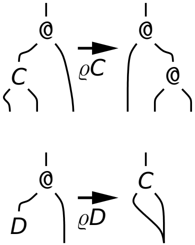
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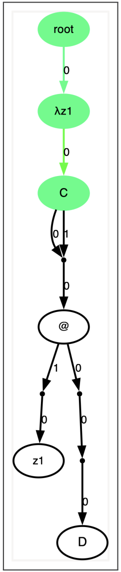
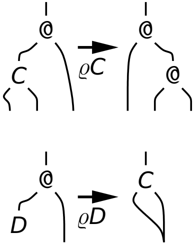

root
λz1
C
@
z1
D



@
C
ϱC
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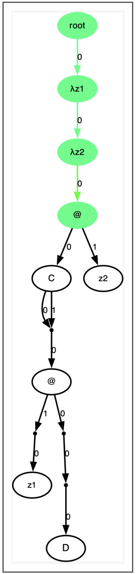
@
D
ϱD
C



root
0
λz1
0
λz2
0
@
0
1
C
z2
0
1
0
@
1
0
0
0
z1
0
D

Implementing some spine-β-strategy via 👁

Observations

- β can be implemented via iterating wβ (for same 👁)

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Implementing some spine-β-strategy via

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- cost of constructor-steps amortised by other steps, for the same reason

Implementing some spine-β-strategy via 👁

Corollary

results for wβ carry over to spine-β, in particular that the cost of reduction to β-normal form is linear in the number of leftmost–outermost β-steps to β-nf

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Perspective

classical 1^{st}-order term(graph) rewrite theory trivialises (extant) cost-analyses

Implementing β-reduction

Complexity unavoidable

convertibility of simply typed λ-calculus is non-elementary. Upshot: whatever way you slice the pie (split into β and substitutions) that can't be overcome.

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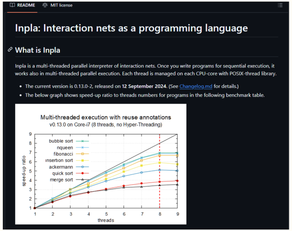
Non-consequence

Optimal reduction for full β is non-interesting. By the same token all implementations shown here would be non-interesting as they are optimal but for wβ.

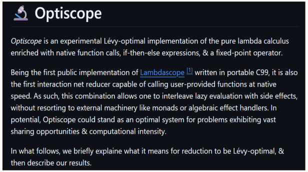
Springtime for interaction nets!



github.com/inpla/inpla

higherorderco.com

vine.dev



github.com/etiams/optiscope

Some conclusions

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- α-spine is 1^{st}-order optimal for 👁, wβ and β
  (only need skeletons present in initial λ-term; no creation of such)

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- amortised analysis: discounting $\bullet$-steps via #nodes, α-steps via β-steps
- higher-order term rewriting useful to bridge λ-calculus and 👁👁

Standing on the shoulders of giants

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10. Blanc, Lévy, Maranget (2005): wβ-family, implemented here (Wadsworth)

Contributions

1. concept of substitution calculus (1994)
2. optimal implementation of lmo-β-family by scope nodes (2004)
3. wβ being isomorphic to orthogonal TRS, given a λ-term (2005)
4. optimality of wβ being an instance of optimality of orthogonal TRSs (2005)
5. the α-spine strategy for 👁👁 (2024)
6. Haskell code implementing wβ into an 👁 and vice versa (2024);
7. linear TGRS implementation of 👁/ wβ / spine-β (2024)
8. Java code for that implementation (2025)
9. naming applicative inductive interaction systems 👁👁 (2025)

Amortised complexity

Idea

measure complexity by averaging over reductions (Tarjan)
(instead of measuring per step)

Amortised complexity

Idea

measure complexity by averaging over reductions

Example

incrementing a counter in binary $011 \rightarrow_{\mathrm{inc}} 111 \rightarrow_{\mathrm{inc}} 0001 \rightarrow_{\mathrm{inc}} 1001 \rightarrow_{\mathrm{inc}} \ldots$
($\rightarrow_{\mathrm{inc}}$-steps not unit-time; $\#$bit-flips unbounded)

Amortised complexity

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measure complexity by averaging over reductions

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Example (inc as term rewrite system; $\to_{\mathrm{inc}} := \to_i \cdot \to_b^!$)

$$s \to_i i(s) \qquad i(0(x)) \to_b 1(x) \qquad i(1(x)) \to_b 0(i(x)) \qquad i(\bullet) \to_b 1(\bullet)$$

Amortised complexity

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$0(0(i(1(\bullet)))) \rightarrow_b 0(0(0(i(\bullet)))) \rightarrow_b 0(0(0(1(\bullet)))) \rightarrow_i \ldots$

Banker's / accounting method in TRSs

Idea

distinguish between charge $\hat{c}$ and cost c of steps. i-steps add charge to pay for cost of subsequent b-steps; labelled ($\mathbb{N}$) symbols as saving-account for charges

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$s \rightarrow_{\hat{3},1} i^{\hat{2}}(s) \qquad i^{\hat{2}}(0(x)) \rightarrow_{\hat{0},1} 1^{\hat{1}}(x) \qquad i^{\hat{2}}(1^{\hat{1}}(x)) \rightarrow_{\hat{0},1} 0(i^{\hat{2}}(x)) \qquad i^{\hat{2}}(\bullet) \rightarrow_{\hat{0},1} 1^{\hat{1}}(\bullet)$

(no need to label 0's or $\bullet$'s)

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  (in general: cost subtracted; charges must remain non-negative, cover costs of steps; $\hat{c} + \sum \ell \geq c + \sum r$ for each (linear) rule $\ell \to_{\hat{c},c} r$)

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- $\hat{\iota}$ initially labels: charge i with $\hat{2}$ and 1 with $\hat{1}$; preserved by steps
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- cost of reduction from t bounded by amortized cost, $\leq 3 \cdot \#i + \sum t^{\hat{\iota}}$

Reduction to (wh)nf in $\lambda\beta$, naïvely, in Haskell

```
data Lam = Lam Head [Lam] deriving (Show)
data Head = Var String | Abs String Lam deriving (Show)
subst x s (Lam h l) = let
  (Lam h' l') = case h of
     (Var y)   | x == y -> s
     (Abs y u) | x /= y -> Lam (Abs y (subst x s u)) []
     _                  -> Lam h [] in (Lam h' (l'++(map (subst x s) l)))
whnf (Lam (Abs x t) (u:l)) = let Lam h s = subst x u t in whnf (Lam h (s++l))
whnf t = t
nf = rnf (\x -> 1)
rnf f t = let
  (Lam h l) = whnf t
  f' x      = \y -> f y + (if (x==y) then 1 else 0)
  v x       = x++"_"++show (f x) in case h of
    (Abs x _) -> Lam (Abs (v x) (rnf (f' x) (Lam h [Lam (Var (v x)) []]))) []
    _         -> Lam h (map (rnf f) l)
```