Confluence via critical valleys

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Universiteit Utrecht

HOR, Nagoya, June 2, 2012

Critical peak systems

Higher-order pattern rewrite systems

Development closed critical peaks

Critical valley systems

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term rewrite system $\mathcal{T}$ is a critical peak system if

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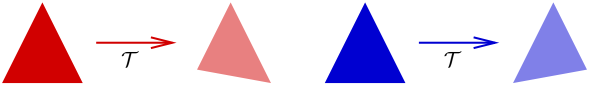
- critical peak rules terminating modulo, $\mathrm{SN}(\mathcal{CPR}(\mathcal{T})/\mathcal{T})$

$$\mathcal{CPR}(\mathcal{T}) = \{t \to t_i \mid t_0 \leftarrow t \to t_1 \text{ is a critical peak of } \mathcal{T}\}$$

$$\mathcal{CPR}(\mathcal{T})/\mathcal{T} = \twoheadrightarrow_{\mathcal{T}} \cdot \to_{\mathcal{CPR}(\mathcal{T})} \cdot \twoheadrightarrow_{\mathcal{T}}$$

Critical peak rules



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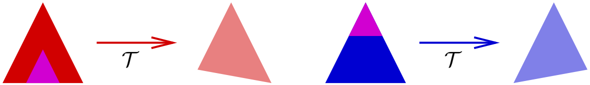

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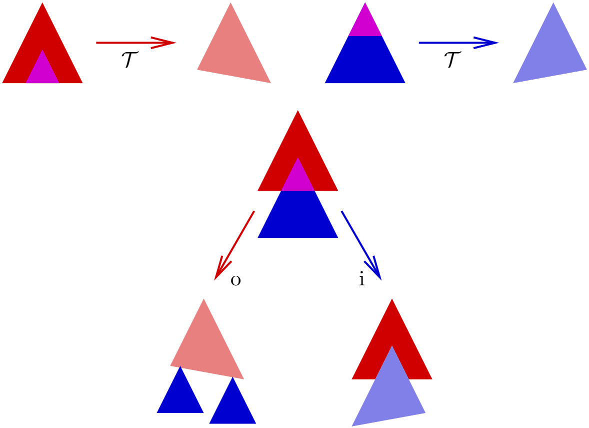

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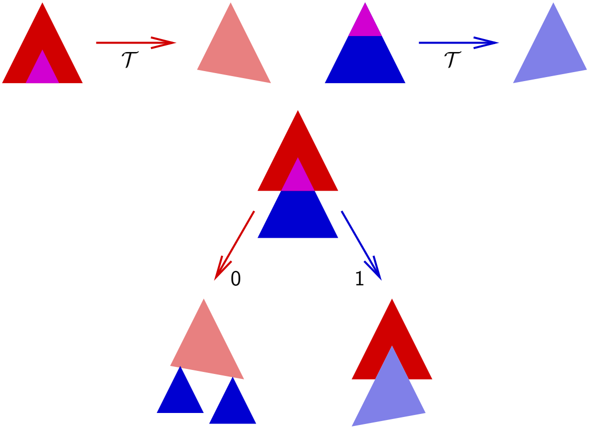

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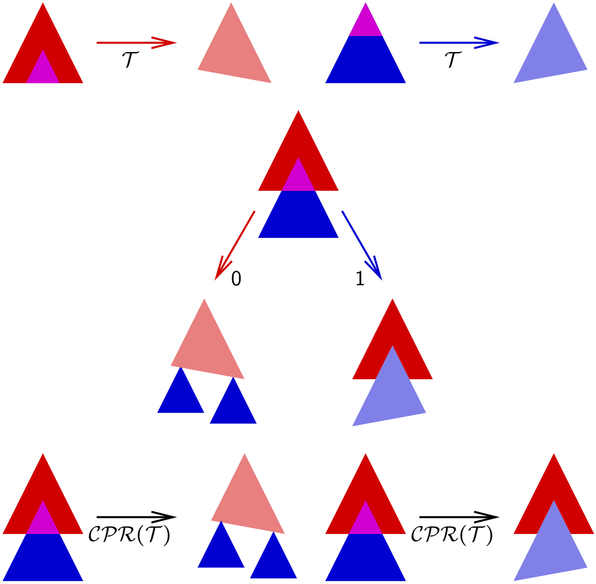

Critical peak system examples

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  (LL and non-overlapping, $\mathcal{CPR}(\mathcal{T}) = \varnothing$)
- LL and terminating $\mathcal{T}$ with JCP.
  $(\to_{\mathcal{CPR}(\mathcal{T})} \subseteq \to_{\mathcal{T}})$

Critical peak systems are confluent

Theorem (Hirokawa & Middeldorp)

critical peak systems are confluent

Proof.

will show multisteps $\multimap\!\!\to$ are confluent

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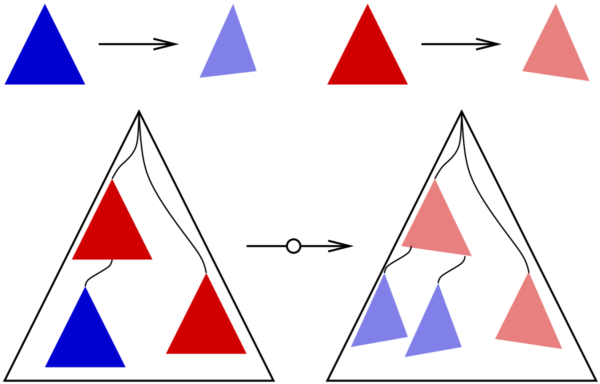
multistep contracts non-overlapping redex-patterns



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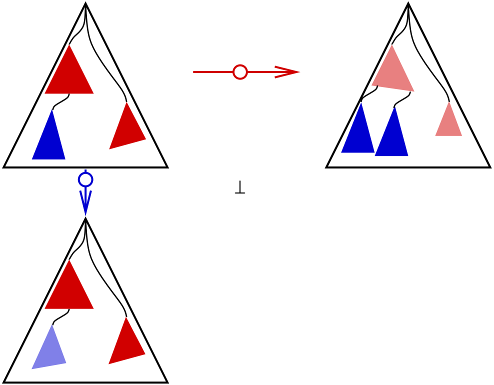

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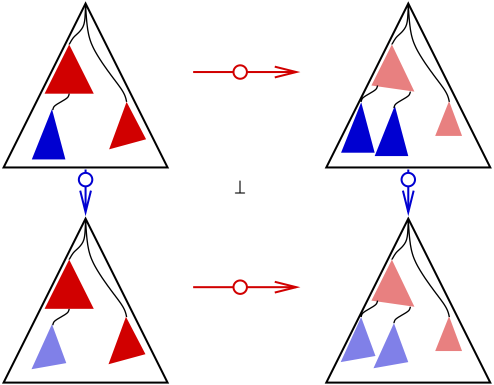

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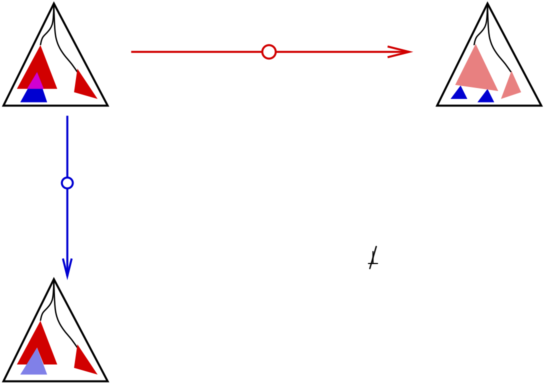

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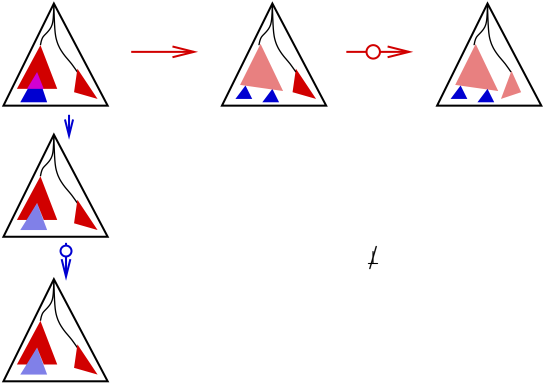

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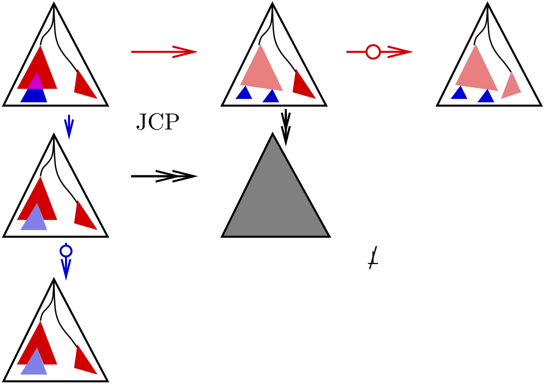

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labeled multistep $t \multimap\!\!\to_{t'} s$ iff $t' \twoheadrightarrow t \multimap\!\!\to s$

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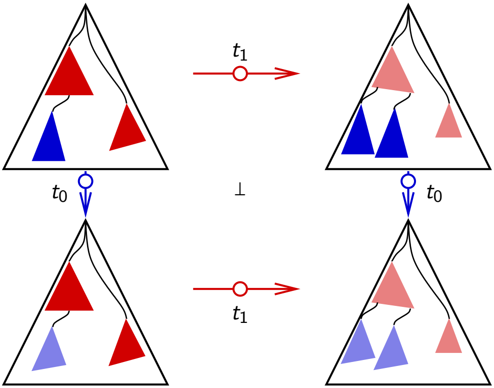
labeling orthogonal peak $s \leftarrow\!\circ_{t_0} t \circ\!\!\rightarrow_{t_1} r$



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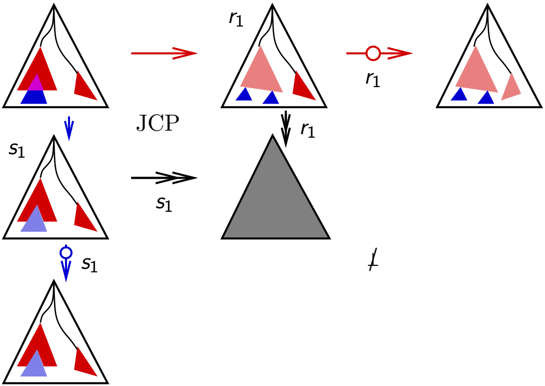
labeling non-orthogonal peak $s \leftarrow\!\circ\!-_{t_0} t \multimap\!\to_{t_1} r$



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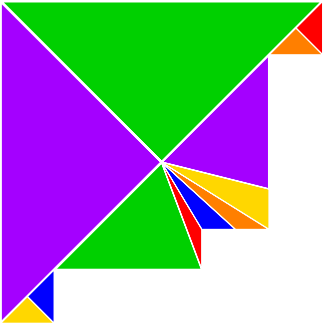
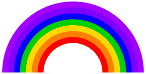
ordering labels by $\mathcal{CPR}(\mathcal{T})/\mathcal{T}$ makes both cases decreasing





rainbow order on colours

Generalising critical peak systems

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1. from first-order to higher-order pattern rewrite systems;
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1. from first-order to higher-order pattern rewrite systems; (. . . extension of Theorems 2 and 3 to higher-order pattern rewrite systems (PRSs) as defined by Mayr and Nipkow [19] . . .)
2. from omitting trivial to development closed critical peaks; (. . . decreasing the set $\mathcal{CPS}(\mathcal{T})$ of critical pair steps that need to be relatively terminating with respect to $\mathcal{T}$. We anticipate that some of the many critical pair criteria for confluence that have been proposed in the literature (e.g. [15, 24, 26]) can be used . . .) (omitting trivial critical pairs due to Middeldorp and Hirokawa)

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   (omitting trivial critical pairs due to Middeldorp and Hirokawa)
3. from critical peak to valley steps.

Higher-order pattern rewrite systems

Definition

- higher-order term rewrite system (HOTRS) is rewrite system on $\alpha\beta\eta$-equivalence classes of terms over simply typed signature with lhss and rhss of rules of same type; (Wolfram)

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- $t \multimap\!\!\!\rightarrow s$ iff $t = C\ell_{i_1}\ldots\ell_{i_n}$, $s = Cr_{i_1}\ldots r_{i_n}$ for rules $\ell_i \to r_i$. (the redex-pattern occurrences in t are orthogonal)

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Fact

PRS multisteps behave as for first-order term rewriting ($\to \subseteq \multimap \subseteq \twoheadrightarrow$, orthogonal multisteps commute, multistep factors through each of its elements)

Pattern rewrite systems are non-borromean

case split between $\perp$ and $\not\perp$ needs

Lemma (patterns non-borromean)

set of redex-patterns is orthogonal iff pairwise orthogonal

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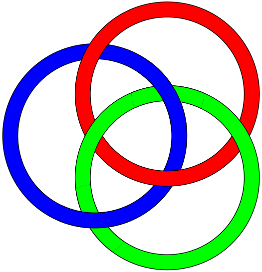

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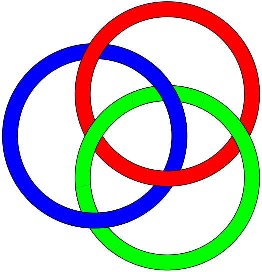

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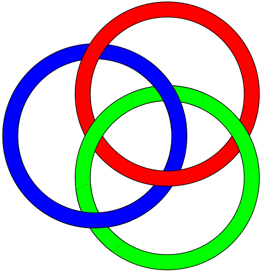

Theorem

higher-order critical peak systems are confluent

generalises confluence by orthogonality and by LL,JCP,SN

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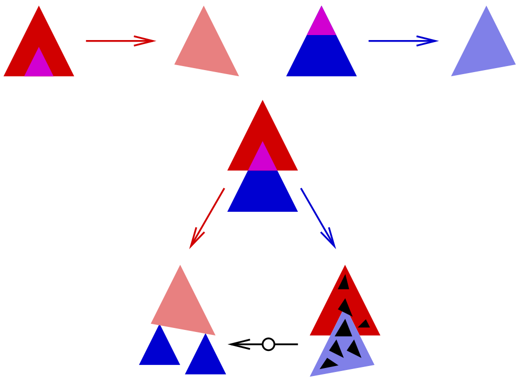
Development closed critical peaks

Definition

critical peak is development closed if



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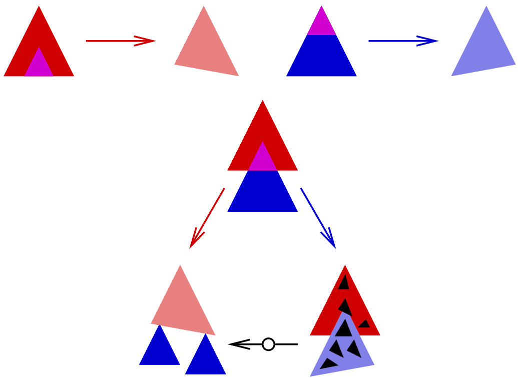

Example

$\{f(g(x)) \rightarrow h(c), g(a) \rightarrow i(b), f(i(x)) \rightarrow h(x), b \rightarrow c\}$

Development closed critical peaks

Theorem

higher-order critical peak systems, with only critical peak rules for non-development closed rules, are confluent

Proof.

will show confluence $\multimap\!\!\rightarrow$ by reduction to earlier cases ($\perp$ and $\not\perp$)

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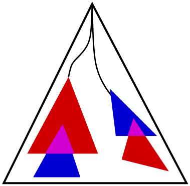
by induction on amount of overlap # between multisteps



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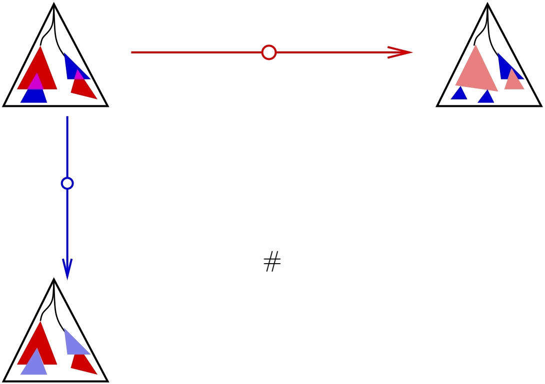

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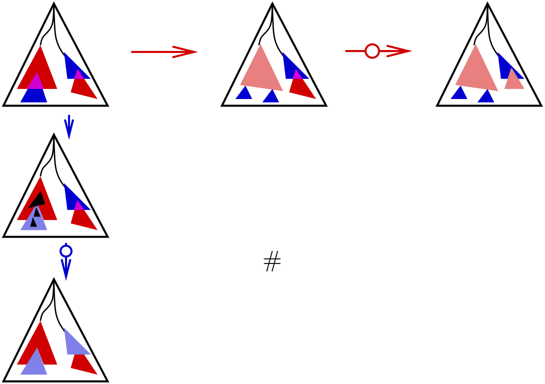

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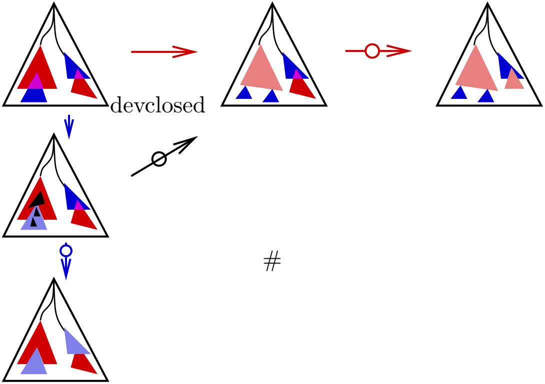

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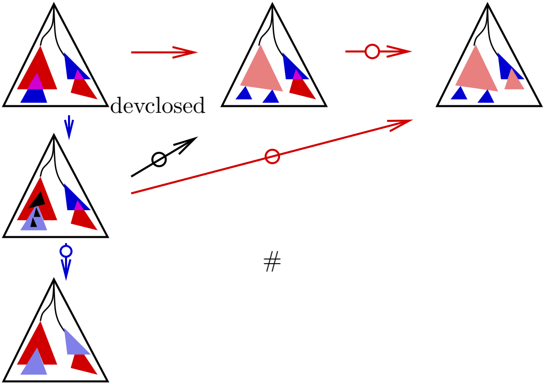

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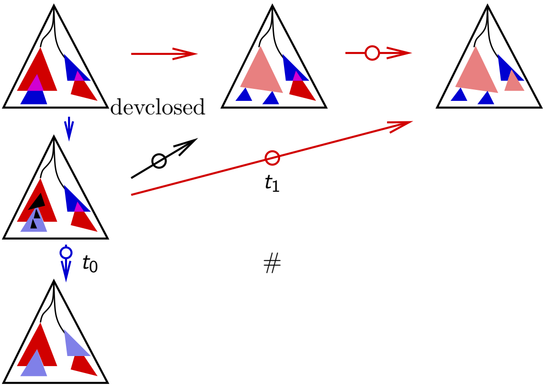

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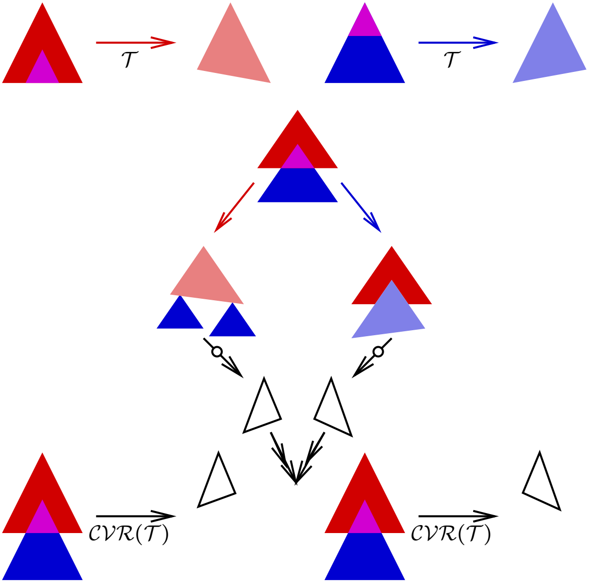
- development closed peaks do not contribute to $\mathcal{CVR}(\mathcal{T})$.

Critical valley rules



Critical valley systems are confluent

Theorem

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Proof.

show decreasingness invariant ${}_{t_0}\!\leftarrow\!\!\circ\!\!- \cdot \multimap\!\!\rightarrow_{t_1} \subseteq \multimap\!\!\rightarrow_{t_1} \cdot \leftrightarrow^*_< \cdot \leftarrow\!\!\circ\!\!-_{t_0}$

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Proof.

as before, by induction on amount of overlap $\#$, and cases $\perp$, $\not\perp$

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Theorem

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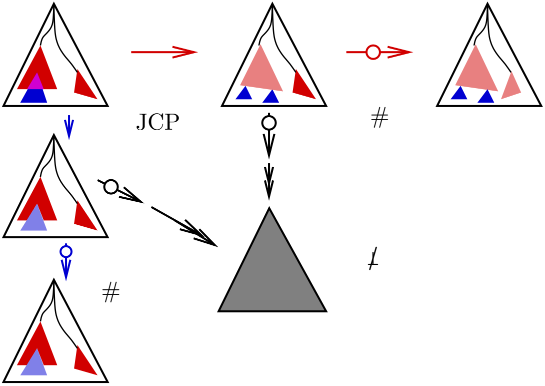
Proof.

new $\not\perp$-case: decrease of amount of overlap # in both corners



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Conclusion

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