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What can string rewriting decide for you?

Vincent van Oostrom

4–12–2019

http://cl-informatik.uibk.ac.at

Overview

1. word problem
2. deciding word problem by complete rewriting system
3. a decidable word problem that cannot be completed
4. changing (finite) presentation such that can be completed
5. finite derivation type (fdt) as invariant wrt change of (finite) presentation
6. a decidable word problem with finite presentation not having fdt
7. each finite complete rewrite system has fdt

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String rewriting

Definition (SRS)

string rewrite system is a system $\mathcal{S} = (\Sigma, R)$, with Σ the alphabet, and R a set of rules (l, r) with $l, r \in \Sigma^*$, i.e. pairs of strings over Σ.

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Example

for $\mathcal{B}_3$, have $1211 \to 2121 \to 2212, 212 \not\to 121$, but $2212 \sim 1211$ and $212 \sim 121$.

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word problem asks to decide $\sim$ for given SRS.

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there are SRSs for which word problem is undecidable.

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Theorem

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This talk

if word problem is decidable, can this be shown by rewriting?

Deciding word problem by rewriting

deciding word problem $s \sim^? t$ by rewriting

reduce s,t to s',t' in normal form (no further rewrite steps). answer $s' =^? t'$.

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Example

- $121212 \sim^? 212121$. yes, since 121212 $\to s$ and 212121 $\to s$ for $s = 212212 \not\to$;

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- $121212 \sim^? 212121$. yes, since $121212 \to s$ and $212121 \to s$ for $s = 212212 \not\to$;
- $111 \sim^? 222$. no, since $111 \neq 222$ and both in normal form, $111 \not\to$ and $222 \not\to$;

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- $111 \sim^? 222$. no, since $111 \neq 222$ and both in normal form, $111 \not\to$ and $222 \not\to$;
- $21221 \sim^? 12212$. no, idem ...

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- $111 \sim^? 222$. no, since $111 \neq 222$ and both in normal form, $111 \not\to$ and $222 \not\to$;
- $21221 \sim^? 12212$. no, idem . . .
  incorrect, since $s \to 21221$ and $s \to 12212$ for $s = 12121$.

Deciding word problem by rewriting

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reduce s,t to s',t' in normal form. answer $s' =^? t'$.

Lemma

deciding by rewriting is correct if rewrite system is complete.

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Definition

- rewrite sequence is finite or infinite sequence $s \to s' \to s'' \to \ldots$ of rewrite steps;
- s reducible to t, notation $s \to^* t$, if finite rewrite sequence from s to t;
- s terminating if no infinite rewrite sequence from s;
- s confluent if $s \to^* t$, $s \to^* u$ entails t,u joinable: $t \downarrow u$ if $\exists v$, $t \to^* v$ and $u \to^* v$;
- s complete if terminating and confluent;

$\to$ is terminating/confluent/complete if every s is.

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deciding word problem $s \sim^? t$ by rewriting

reduce s,t to s',t' in normal form. answer $s' =^? t'$.

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$\downarrow$ is reflexive and symmetric, and transitive iff $\rightarrow$ confluent.

Proof.

assuming $s \downarrow t \downarrow u$, we have $s \rightarrow^* v$, $t \rightarrow^* v$, and $t \rightarrow^* w$, $u \rightarrow^* w$, for some v, w. by confluence for $t \rightarrow^* v$ and $t \rightarrow^* w$, then $v \downarrow w$, hence $s \downarrow t$.
assuming $s \rightarrow^* t$, $s \rightarrow^* u$, we have $t \downarrow s$ and $s \downarrow u$. by transitivity of joinability $t \downarrow u$. ∎

Deciding word problem by rewriting

deciding word problem $s \sim^? t$ by rewriting

reduce s,t to s',t' in normal form. answer $s' =^? t'$.

Lemma

deciding by rewriting is correct if rewrite system is complete.

Proof.

$s \sim t$ iff $s \leftrightarrow^* t$ iff $s' \leftrightarrow^* t'$ iff $s' \downarrow^* t'$ iff $s' \downarrow t'$ iff $s' = t'$. ∎

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Completion?

$\mathcal{B}_3$ is finite but not complete. can it be completed?

that is, can we find complete finite SRS $\mathcal{B}_3'$ with same convertibility as $\mathcal{B}_3$, $\sim = \sim'$?

Completion

Example

$\mathcal{S} = (\{a, b, c\}, \{(a, a), (a, b), (a, c)\})$ not complete: $\rightarrow$ neither terminating nor confluent;

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Definition

SRS irreducible if for each rule (l, r), $r \not\rightarrow$ and $l \not\rightarrow_{\mathcal{S}-\{(l,r)\}}$, normal form wrt. other rules

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$\mathcal{S} = (\{a, b, c\}, \{(a, a), (a, b), (a, c)\})$ not complete: $\rightarrow$ neither terminating nor confluent;
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SRS irreducible if for each rule (l, r), $r \not\rightarrow$ and $l \not\rightarrow_{\mathcal{S} - \{(l,r)\}}$.

Example

$\mathcal{S}'$ is complete but not irreducible. $\mathcal{S}'' = (\{a, b, c\}, \{(a, c), (b, c)\})$ is both, and $\sim' = \sim''$.

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Example

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Definition

SRS irreducible if for each rule (l, r), $r \not\rightarrow$ and $l \not\rightarrow_{\mathcal{S} - \{(l,r)\}}$.

Theorem

for every finite complete SRS $\mathcal{S}$ there is an irreducible finite SRS $\mathcal{S}'$ with $\sim = \sim'$.

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Theorem

for every finite complete SRS $\mathcal{S}$ there is an irreducible finite SRS $\mathcal{S}'$ with $\sim = \sim'$.

Proof.

1. replace (l, r) by (l, r') with r' normal form of r; (l, r) not used since $\rightarrow$ terminating;

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Example

$\mathcal{S} = (\{a,b,c\},\{(a,a),(a,b),(a,c)\})$ not complete: $\rightarrow$ neither terminating nor confluent;
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for every finite complete SRS $\mathcal{S}$ there is an irreducible finite SRS $\mathcal{S}'$ with $\sim = \sim'$.

Proof.

1. replace (l,r) by (l,r') with r' normal form of r;
2. remove (l,r) if l rewritable by other rule; longer/more general reduction $l \rightarrow^* r$;

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1. replace (l, r) by (l, r') with r' normal form of r;
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$\mathcal{B}_3$ cannot be completed

Theorem (Kapur & Narendran)

there is no finite complete SRS $\mathcal{B}_3'$ equivalent to $\mathcal{B}_3$, i.e. such that $\sim' = \sim$

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for a proof by contradiction, suppose $\mathcal{B}_3' = (\{1, 2\}, R')$ were to exist.

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1. by the previous theorem may assume $\mathcal{B}'_3$ irreducible;
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4. set $s = 1^{L+1}2^{L+2}12$ and $\bar{s} = 212^{L+2}1^{L+1}$ its converse;

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5. $s \sim \overline{s}$; since $1^{i+1}2^{j+2}12 \sim 1^{i+1}2121^{j+1} \sim 212^{i+2}1^{j+1}$ for $i, j \geq 0$;

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4. set $s = 1^{L+1}2^{L+2}12$ and $\bar{s} = 212^{L+2}1^{L+1}$ its converse;
5. $s \sim \bar{s}$;
6. first step from s ($\bar{s}$) rewrites substring 2^k12 (2^k12); others have singleton $\sim$-class

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4. set $s = 1^{L+1}2^{L+2}12$ and $\overline{s} = 212^{L+2}1^{L+1}$ its converse;
5. $s \sim \overline{s}$;
6. first step from s ($\overline{s}$) rewrites substring 2^k12 (2^k12);
7. if $(2^k12, x) \in R'$, then $x \rightarrow$; $2^k12 \sim x$ then $x = 2^{k-i}121^i$

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5. $s \sim \overline{s}$;
6. first step from s ($\overline{s}$) rewrites substring 2^k12 (2^k12);
7. if $(2^k12, x) \in R'$, then $x \rightarrow$. contradicting $\mathcal{B}'_3$ irreducible. ∎

Extra symbol to the rescue

Example

$\mathcal{B}''_3 = (\{1, 2, {\color{red}{\rlap{1}2}}\}, \{(12, {\rlap{1}2}), ({\rlap{1}2}1, 2{\rlap{1}2}), (2{\rlap{1}2}2, {\rlap{1}2}{\rlap{1}2}), ({\rlap{1}2}{\rlap{1}2}2, 1{\rlap{1}2}{\rlap{1}2})\})$ decides $\sim$.

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Example

$\mathcal{B}_3'' = (\{1, 2, 1\!\!2\}, \{(12, 1\!\!2), (1\!\!21, 21\!\!2), (21\!\!22, 1\!\!21\!\!2), (1\!\!21\!\!22, 11\!\!21\!\!2)\})$ decides $\sim$.

E.g. $21221 \to'' 21\!\!221 \to'' 1\!\!21\!\!21 \to'' 1\!\!221\!\!2$ and $12212 \to'' 1\!\!2212 \to'' 1\!\!221\!\!2$.

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Example

$\mathcal{B}_3'' = (\{1, 2, {1\!\!\!2}\}, \{(12, {1\!\!\!2}), ({1\!\!\!2}1, 2{1\!\!\!2}), (2{1\!\!\!2}2, {1\!\!\!2}{1\!\!\!2}), ({1\!\!\!2}{1\!\!\!2}2, 1{1\!\!\!2}{1\!\!\!2})\})$ decides $\sim$.
E.g. $21221 \rightarrow'' 2{1\!\!\!2}21 \rightarrow'' {1\!\!\!2}{1\!\!\!2}1 \rightarrow'' {1\!\!\!2}2{1\!\!\!2}$ and $12212 \rightarrow'' {1\!\!\!2}212 \rightarrow'' {1\!\!\!2}2{1\!\!\!2}$.

Rescue what?

What is the relationship between $\mathcal{B}_3$ and $\mathcal{B}_3''$?

Extra symbol to the rescue

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$\mathcal{B}_3'' = (\{1, 2, 1\!\!2\}, \{(12, 1\!\!2), (1\!\!21, 21\!\!2), (21\!\!22, 1\!\!21\!\!2), (1\!\!21\!\!22, 11\!\!21\!\!2)\})$ decides $\sim$.
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Rescue what?

What is the relationship between $\mathcal{B}_3$ and $\mathcal{B}_3''$?

Answer

- $\sim \neq \sim''$ since different signatures;
  no Birkhoff's theorem (equivalent $\iff$ provably equal $\iff$ convertible)

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What is the relationship between $\mathcal{B}_3$ and $\mathcal{B}_3''$?

Answer

- $\sim \neq \sim''$ since different signatures;
  no Birkhoff's theorem (equivalent $\iff$ provably equal $\iff$ convertible)
- induce presentations of same monoid. $\langle\{1, 2\} \mid 121 = 212\rangle$ and
  $\langle\{1, 2, 1\!\!2\} \mid 12 = 1\!\!2, 1\!\!21 = 21\!\!2, 21\!\!22 = 1\!\!21\!\!2, 1\!\!21\!\!22 = 11\!\!21\!\!2\rangle$
  by generators $(1, \ldots)$ and relations $(121 = 212, \ldots)$.

Monoid

Definition

monoid is structure $\mathcal{M} = (M, e, \cdot)$ with $e \in M$ such that for all $m, n, k \in M$

$$e \cdot m = m$$

$$m \cdot e = m$$

$$(m \cdot n) \cdot k = m \cdot (n \cdot k)$$

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Examples

- natural numbers with 0 and $+$;

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  f defined by $1 \mapsto 1, 2 \mapsto 2, 12 \mapsto 12$ induces monoid isomorphism $[s]_{\sim''} \mapsto [f(s)]_\sim$

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assume $\mathcal{S} = \langle \Sigma \mid R \rangle$, $\mathcal{S}' = \langle \Sigma' \mid R' \rangle$ present same monoid; wlog $\Sigma \cap \Sigma' = \emptyset$

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3. $\langle \Sigma, \Sigma' \mid R, x' = g(x'), R' \rangle$; as $s' \sim_{\text{def}} g(s') \sim_{\mathcal{S}} g(t') \sim_{\text{def}} t'$ for $s' = t'$ in R';

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4. $\langle \Sigma, \Sigma' \mid R, x' = g(x'), R', x = f(x) \rangle$, and back again.

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No (Squier). Sketch in rest of talk.

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- contains $(\phi \cdot \psi, \psi' \cdot \phi')$ if ϕ, ψ steps orthogonal to each other, residuals ϕ', ψ'.

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Proof idea.

fdt invariant wrt each Tietze transformation. ∎

A decidable monoid that does not have fdt

Example

The monoid finitely presented by $\mathcal{S}_1 = (\Sigma_1, R_1)$ with $\Sigma_1 = \{a, b, t, x, y\}$ and $R_1 = \{(xa, atx), (xt, tx), (xb, bx), (xy, \epsilon), (ab, \epsilon)\}$

- does not have fdt;
- has complete (infinite) presentation $(\Sigma_1, R_1 \cup \{(at^k b, \epsilon) \mid k \geq 1\})$;
- has decidable word problem.

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3. suffices to show $\psi \simeq 1_v$ for $\psi : v \leftrightarrow^* v$, v normal form; conversion on normal form;

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5. follows from pasting with local confluence diagrams for local peaks:
    - if overlap then sides homotopic by critical pair homotopy;
    - otherwise by definition of homotopy.

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then suppose $\phi : s \leftrightarrow^* t$.

1. let $\underline{u} : u \to^* u'$ be path from u to its normal form u';
2. $\phi \simeq \underline{s} \cdot \underline{s}^{-1} \cdot \phi \cdot \underline{t} \cdot \underline{t}^{-1} \simeq \underline{s} \cdot \underline{t}^{-1}$; only dependent on s, t; independent of ϕ!
3. suffices to show $\psi \simeq 1_v$ for $\psi : v \leftrightarrow^* v$, v normal form;
4. suffices to show $\psi_l \cdot \psi_r \simeq 1_v$ for $\psi_i : w \to^* v$, v normal form;
5. follows from pasting with local confluence diagrams for local peaks:
    - if overlap then sides homotopic by critical pair homotopy;
    - otherwise by definition of homotopy.

hence all conversions $s \leftrightarrow^* t$ homotopic to $\underline{s} \cdot \underline{t}^{-1}$, so to each other. ∎

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Corollary

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Upshot

Contrary to expectation, rewriting is not omnipotent.

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