Topological Convergence in Infinitary Abstract Rewriting

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$$x \to \infty$$

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Abstract

Rewriting is a field on the border of logic, mathematics and theoretical computer science. It studies the stepwise transformation of objects and as such applies to a lot of processes in computer science, systems in mathematics and also formalized stepwise processes in other fields of science. Abstract rewriting studies properties those processes might have, independent of the concrete structure of the objects and the steps. Examples are properties like termination (modeling that processes terminate) and confluence (modeling that finite divergent processes starting from a common source can be extended to reach a common target). Infinitary abstract rewriting studies processes of infinite length and as such is concerned with convergence (modeling that the processes in some sense get arbitrarily close to some intended target).

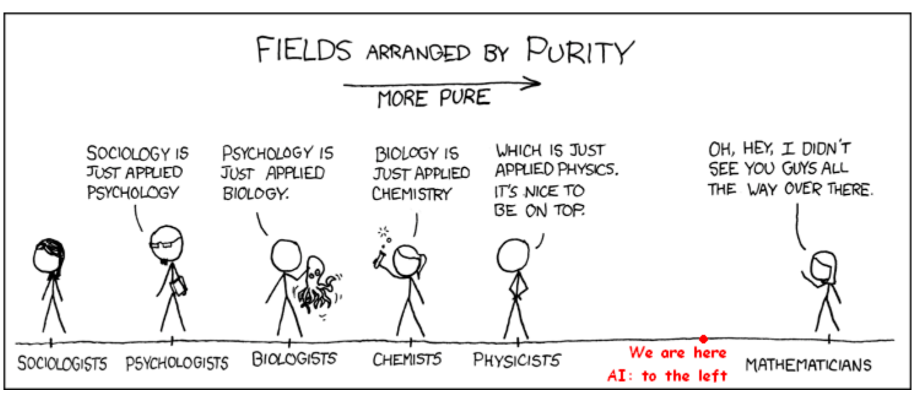
In this thesis I study the foundations of rewriting in general and infinitary abstract rewriting in particular. Infinitary abstract rewriting is still very much work-in-progress and various frameworks exist. I compare these frameworks and their formalizations and note some deficiencies. I also propose a framework of my own based on topological convergence, a notion of convergence derived from the mathematical branch of general topology. This framework is proven to encompass the two main existing frameworks and has desirable technical properties.

Contents

Chapter 1

Introduction

Rewriting is a field on the border of logic, mathematics and theoretical computer science. It studies processes of stepwise, discrete, transformation of objects and as such applies to a lot of processes in computer science, systems in mathematics and also formalized stepwise processes in other fields of science. As such it is fundamental to various topics in Cognitive Artificial Intelligence. For instance, computing at its most basic level is a process of stepwise transformation. So rewriting applies to all computing in general and as such applies to artificial intelligence in specific. It also applies to artificial intelligence systems on a higher level, since such systems can be modeled as stepwise processes. Furthermore, it might even apply to, for instance, biological systems as long as those systems are formalized and viewed and studied abstractly. Whenever a process is discrete and formalized, rewriting is foundational for it. It can be foundational for so many processes because it studies only the abstract properties of these processes. Rewriting is just concerned with formalized objects, sometimes very abstract, and the, sometimes also very abstract, transformations between them.



"Rewriting is way to the right, I saw CAI over there though. . . well, I think I did."
Alternative caption: "Computer Science is just applied rewriting."

An everyday example of a process of stepwise transformation that rewriting is concerned with would be finding the answer to a simple arithmetical equation like $4x - 64 = 2x + 20$. We can solve for x by:

$$\begin{aligned} 4x - 64 = 2x + 20 &\to 2x - 64 = 20 \\ &\to 2x = 84 \\ &\to x = 42 \end{aligned}$$

Here the objects are equations and the steps or transformations are operations like simplifying arithmetical expressions, and applying arithmetical operations to both sides of the equation. Rewriting can study the abstract properties of such a process. Does it stop? Does it matter which step we take, if multiple steps are possible? Does order in which we take steps matter?

1.1 State of the Art

The studies of rewriting itself are highly formal and can be divided in several subfields. First of all, we can choose whether to look at the internal structure of the steps and objects or to abstract over it. If we abstract over it we are in the field of **abstract rewriting**. If we don't, which field we end up in depends on how this internal structure is formalized. If our objects are formalized as graphs, we will be in **graph rewriting**. If our objects are formalized as strings, we are in **string rewriting**. If our objects are formalized as terms, we are in **term rewriting**. All these rewriting systems are non-abstract and I will refer to them as **concrete rewriting systems**. This is as far as I know not a standard term, but I think that, in the context of this thesis, this general notion of non-abstract rewriting could do with a name. Of all the concrete rewriting systems, I'll single out term rewriting systems as example of concrete rewriting in general since this is an interesting and well-developed field of study.

We can also divide the field of rewriting on whether we only want to study finite processes or if we also want to study infinite (transfinite) processes. We'll refer to the study of only finitary processes as **finitary rewriting** and we'll refer to the study of infinitary processes as **infinite rewriting**. Here, the words "finite" and "infinite" refer to the length of the processes and not to, for example, the size of the objects. If we are in the field of concrete rewriting however, then, together with this choice of length of processes, usually also comes a choice of size of objects. If we choose to study infinitary processes, it makes sense to also consider infinite objects. This is because steps in the process may increase the size of the object. So, even though objects may start out as having a finite size and even though the steps may only increase the size of the object by a finite amount; when the number of steps in the process goes to infinity then the size of the object that is being transformed also goes to infinity. Conversely, if we choose to study only finite processes, it makes sense to only consider finite objects. If we start with an object of finite size and the size of the object only increases a finite amount in a step, then the object can never get of infinite size by finitely many steps.

Justified questions to ask here are: why should we study infinite rewriting at all? What good are processes that never end? What special properties can they have aside from not ending? This last question is what much of Chapter 5 is about. There are various ways of quickly answering the first two questions though. First of all, we have the plain mathematical possibility of generalizing finite rewriting into the infinite[1]. Secondly, processes that are infinite in theory are everywhere in computing[2], infinite loops, streams, some program that calculates the list of all primes, etc. Artificial intelligence is also relevant here, many tasks in artificial intelligence can be defined without having a specific end and hence are infinite in nature. In cases such as a prime-calculating program, not only is the process formalized in a way such that it will never end, the actual task it is supposed to perform is an infinite task, there are infinitely many primes. Such a task, containing infinitely many steps, is called a supertask ([7]). That brings us to a third, astonishing and wildly speculative justification for studying infinite rewriting. In [7], papers by physicists are cited where models of general relativity are given in which supertasks can actually be carried out, in a finite amount of time! Such models are universes that obey the laws of general relativity and that apparently allow the execution of supertasks by a setup using two observers, one of whom is computing the supertask, while the other is constantly accelerating and moving around him. For the accelerating observer the task should then be completed in a finite amount of time, presumably by some general relativistic time dilation. I am totally unqualified to say what this means for our own universe, and chances are that what this means is either "not much" or "its impossible in our universe". However, I couldn't help but mentioning this outrageous possibility, where the exotic infinitely long processes that are abundant in this thesis are actually at risk of becoming executable in the face of modern physics.

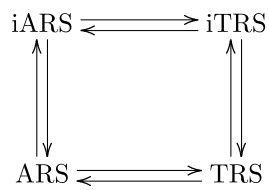
The considerations of abstractness versus concreteness and finiteness versus infiniteness already anticipate fields of rewriting where both decisions are explicitly made. We get 4 subfields of rewriting, namely:

- **Finitary abstract rewriting**, of which the formal systems, by convention, are referred to as **ARSs** (Abstract Rewrite Systems).
- **Infinitary abstract rewriting**, of which the formal systems go by various names. We will, for the moment, refer to them as **iARSs**.
- **Finitary concrete rewriting**, of we will take finitary **TRSs** (Term Rewrite Systems) as our example.
- **Infinitary concrete rewriting**, of which we will take **infinitary TRSs**, or **iTRSs**, as our example.

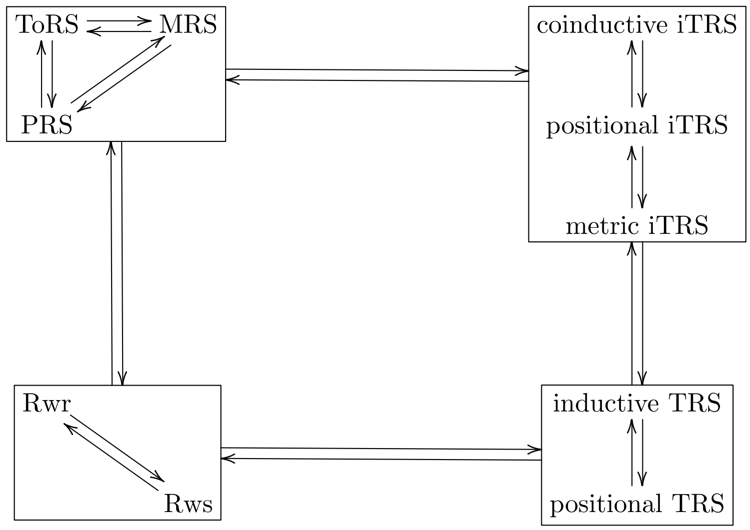
There are, or at least should be, strict and formal relations between these fields. The framework of ARSs should provide a basis for that of TRSs and abstracting

[1] This is known in the trade as generalitis.
[2] and in other fields of science and maybe even in nature

away from the internal structure of a TRS should yield an ARS. The framework of iARSs should generalize that of ARSs into the transfinite and we should be able to embed ARSs into iARSs. The framework of iARSs should provide a basis for that of iTRSs and abstracting away from the internal structure of an iTRS should yield an iARS. The framework of iTRSs should generalize that of TRSs into the transfinite and we should be able to embed TRSs into iTRSs. So these interrelations should yield the following diagram:



In practice, the relations between the various frameworks in the field are even more complicated. This is due to the fact that in the various frameworks there are multiple ways of formalizing them. In finitary abstract rewriting, we can consider formalisms that put the emphasis on the objects that are being transformed. We call the object-emphasized formalism studied in this thesis that of **rewrite relations** or **Rwr**. On the other hand, we can also consider formalisms that put the emphasis on the steps or transformations themselves. We call the step-emphasized formalism studied in this thesis that of **rewrite systems** or **Rws**. In finitary term rewriting we consider an inductive approach, **inductive TRSs**, and a positional approach, **positional TRSs**. In the field of infinitary term rewriting we consider a coinductive approach, **coinductive iTRSs**, an infinitary positional approach, **positional iTRSs**, and a metric approach, **metric iTRSs**. In infinitary abstract rewriting, known approaches are the metric approach, which we will refer to as the **MRS** approach, and the partial order approach, which we will refer to as the **PRS** approach. Also, in the field of infinitary abstract rewriting a new topological approach will be introduced which we will refer to as the **ToRS** approach. So actually our diagram can be refined as shown in the following diagram.



1.2 Objectives

In rewriting as it is, these interrelations are not all made as explicit as they could be. One aim of this thesis is to study the various formalisms and frameworks and their relations to each other. In infinitary abstract rewriting a lot of foundational work is still being done by various authors, the main aim of this thesis is to contribute there. Two existing frameworks, one based on metric spaces and one based on partial orders will be formalized. Some of their properties and relations to other frameworks and formalisms in rewriting will be studied. Most importantly, as mentioned, this thesis aims to introduce a novel approach to infinitary abstract rewriting based on topology. This approach will formalize the convergence and well-behavedness properties that infinitary abstract rewriting is concerned with using convergence in topological spaces. This brings us to the research question:

Research Question

How can infinitary abstract rewriting be formalized using topological spaces and what are the properties, advantages of such a formalization?

1.3 Methods

The methods used in this thesis are entirely analytical. I read about the mathematical background of rewriting, wrote about it and analyzed it. I read about the state of the art in rewriting, wrote about it and analyzed it. I then set up three formalisms for infinitary abstract rewriting, two slight variations on existing formalisms and one new one: the formalism of topological rewrite systems (ToRSs). I then analyzed these formalisms, proved statements about them, the relations between them and their relations with other forms of rewriting.

1.4 Structure

A lot of mathematical background is needed by the various forms of rewriting. An overview of this background is given in Chapter 2. Because of space considerations the technical details of this background have been moved to Appendix A. Chapter 3 treats finitary abstract rewriting. It will be studied from a point of view emphasizing the objects (rewrite relations) and from a point of view emphasizing the steps (rewrite systems). Their relations will be discussed. Our example of concrete rewriting and main motivation for abstract rewriting, TRSs, are treated in Chapter 4. I will give two ways of setting up the framework of terms (an inductive approach and a position-based approach) and compare them. I will discuss infinite terms, partial terms, terms of infinite width and terms of transfinite depth and will discuss the actual rewriting of those terms. Chapter 5 will be about infinitary abstract rewriting. It will be formalized using metrics (MRSs) and using partially ordered sets (PRSs) as has been done in the literature and in our novel way using topologies (ToRS). These three approaches will then be analyzed and be compared to each other, to finitary abstract rewriting (which it should generalize) and to infinitary term rewriting (which it should provide an abstract basis for). Chapter 6 will be devoted to conclusion and discussion.

As mentioned, Appendix A will be about the mathematical background of this thesis. It will include topics in the theories of ordinality and cardinality, category theory, order theory, general topology and metric topology.

Appendix B.1 will contain a start to a categorical approach of comparing rewrite relations and rewrite system. A categorical approach that is very much related to the comparison done in Chapter 3.

Appendix B.2 will contain some analysis on [13], a classic paper in transfinite abstract rewriting. This material is partly related to Chapter 5 but should also be interesting on its own.

Chapter 2

Mathematical Background

Before delving into the actual subject of this thesis, rewriting, a lot of mathematical background needs to be established. Ordinal theory, cardinal theory and order theory is needed to deal with the lengths, order and sizes of the processes and objects in rewriting and to deal with the various notions of infinity that rewriting is concerned with. General topology, metric spaces and, again, order theory are needed to deal with the formalization of the structure of the objects and the steps involved in a rewriting system and to deal with notions related to convergence. Category theory is useful to compare and relate our mathematical structures.

All of these topics are much too broad and involved to fully treat up to the level they are needed. In this chapter an introduction will be given and relevance of the topic to rewriting will be explained. While writing this thesis I worked out much of the technical detail anyway, to get a grasp of the material and because it supports the rest of this thesis. However, these technical details, as far as rewriting is concerned, are only background to the actual topic. Also, most of it is textbook material and folklore. All the material that actually supports the main body of this thesis and that is referred to throughout the thesis can be found in Appendix A. This appendix contains two types of content. For every subject, the formalisms and intuitions are set up and the important definitions and concepts are given. This is basic textbook material and is used, but not referred to, in the main body of this thesis. There are also proofs of various lemmas and theorems that the material in the main body depends on, these will be referenced where they are used. So if one is familiar with these topics, just reading the main body of the thesis and only checking the appendix when referenced is probably a good way to read this thesis. If one is not familiar with the topics, reading the entire appendix might clarify what is going on.

2.1 Ordinals and Cardinal Theory

The intuitions behind cardinal numbers and ordinal numbers (Appendix A.1) are closely related (and hence treated together) and arise from the natural notion of numbers. Natural numbers serve two purposes. They can describe size, also called the quantity aspect; the set $\{a, b, c\}$ has 3 elements. And they can describe position, also called the order aspect; the set $\{a, b, c\}$ may be ordered as is done in the sequence $\langle a, b, c\rangle$, where the position of a is first, the position of b is second and the position of c is third. These are different notions, so we should distinguish between them. We'll call the notion concerning quantity **cardinality**, we'll call the notion concerning order **ordinality**. We formalize cardinality by **cardinal numbers** and ordinality by **ordinals numbers**. For finite purposes these notions coincide and are formalized by the natural numbers. However, there is also a need for non-finite ordinals and cardinals. Here we do not simply get infinity, but whole systems of different ordinal and cardinal numbers which are better described as being **transfinite**. For transfinite purposes cardinality and ordinality do not coincide. So as we formalize both cardinality and ordinality and go into the transfinite, we need clear separation between the two (Section A.1.2).

In the appendix the ordinals are set up first (Section A.1.3). A nice characterization, which can be used as a basis for the ordinal theory needed for rewriting is obtained (Section A.1.3.1). The main application for ordinals in this thesis will be their transfinite character and their use as index sets for transfinite sequences. Using natural numbers as index sets for sequences yields only finite sequences. Also using the set of natural numbers, $\mathbb{N}$, as index sets yields the possibility of infinite sequences, but only of length ω. Such sequences might tend to a limit, but such a limit, for instance, can't be added to the end of the sequence. ω is the maximal length of such sequences. Sequences indexed by ordinals can go on beyond length ω and hence can be said to be truly transfinite, they are defined in Section A.1.5. Whenever sequences are used in this thesis, they should be understood as being ordinal indexed and hence possibly of true transfinite length. The natural numbers can be seen as an initial fragment of the ordinals (Section A.1.3.2), so transfinite sequences are actually generalizations of 'regular' sequences. Some of the arithmetic that can be done on ordinals is defined in Section A.1.3.3. The cardinal numbers can be set up within the ordinal numbers and will be developed in Section A.1.4.

2.2 General Topology

General topology (Appendix A.2) studies topological spaces and as such formalizes certain intuitions about the structure of sets in an abstract way. This structure is given by declaring which subsets of the set are **open**. A set being open can be interpreted in various ways, it can be taken as formalizing nearness or as formalizing observable properties that the members of the set share. Using open sets, general topology formalizes concepts such as **convergence** and

continuity. General topology is a very general framework for formalizing the notion of convergence. Both metric spaces and ordered sets induce a topological space, metric spaces via Definition A.3.10 and ordered sets via the Scott topology (Definition A.2.57). This way, both the notion of metric convergence and the notion of partial convergence can be expressed in terms of convergence in an induced topological space. In Section 5.2 the sets of objects and steps of rewrite systems are equipped with a topology to get a notion of converging reductions.

2.3 Metric Spaces

Metric spaces (Appendix A.3) consist of a set and a **distance measure** on the points in the set. As such, they provide an intuitive way to express the structure of a set, with it, a notion of **convergence** for sequences in that set (Proposition A.3.16). In Section 5.3 the sets of objects of rewrite systems are equipped with a metric to get a notion of converging reductions.

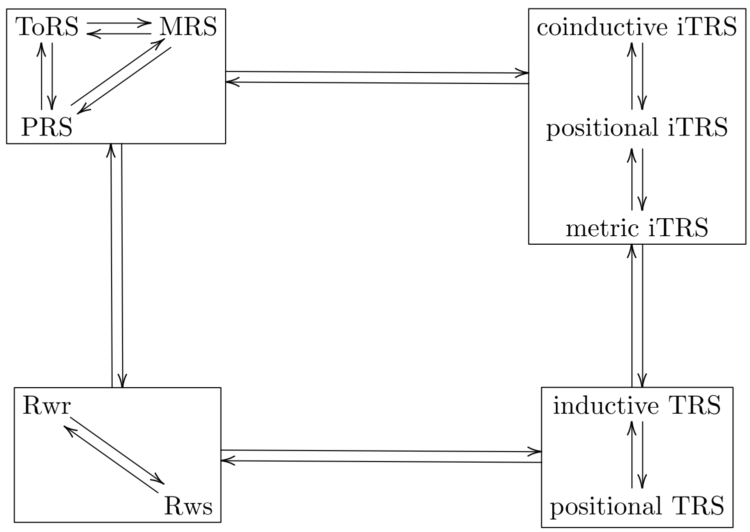
2.4 Order Theory

Order theory (Appendix A.4) is the theory of ordered sets. An **ordered set** is a set with a binary relation on it which has certain properties that one might intuitively associate with the elements of the set being ordered in a certain way. The relation is called an **order** on the set. Orders can be formalized in two ways, a non-strict way (Section A.4.1), using the reflexivity as property, and a strict way (Section A.4.2), using irreflexivity as property. In this thesis strictly ordered sets will be used as a foundation for ordinals, as is done in [16, p. 29]. Non-strict orders can be used to express the structure of sets in an order theoretical way. In this thesis they are used to express the structure of the set of objects-to-be-rewritten in the partial rewrite systems of Section 5.4. The non-strict orders used here are orders as studied by domain theory ([6]). This lets us define order theoretical notions of sequence convergence like **limit inferiors** (Definition A.4.24) and **$\mathscr{S}$-limits** (Definition A.4.27) in those rewrite systems.

2.5 Category Theory

Category theory (Appendix A.5) is the theory of mathematical structures and systems of such structures. A category consists of a collection of abstract **objects** and **morphisms** or arrows between these objects (every morphism or arrow has a source object and a target object). A category models a system of mathematical structures in the way that each structure in the system is an object in the category and each structure preserving mapping between structures in the systems is a morphism in the category. The easiest example of a category is the category of sets where sets are objects and functions between sets are the morphisms.

Category theory allows for an abstract way to talk about the internal structure of a mathematical system (like the system of set theory) and, building on that, also allows for an abstract, systematical way to talk about the relations between such systems. For example category theory can relate the category of sets to the category of topology (where the objects are topologies and the structure preserving mappings are continuous functions) in various ways. In this aspect category theory can be important to rewriting. There are various formalizations of the different fields of rewriting. In the introduction the field of rewriting was mapped as follows:

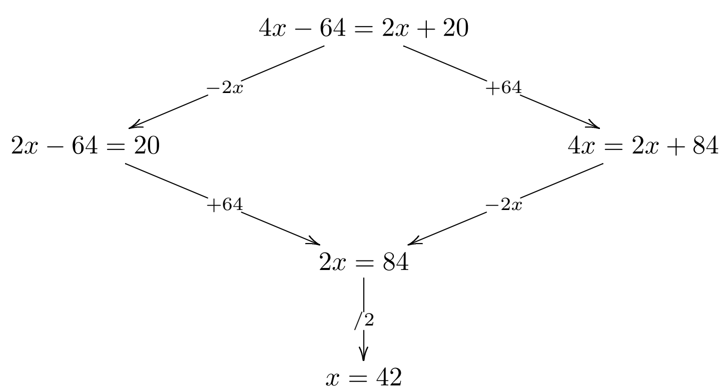


Here the arrows stand for relations between various systems of structures. We'd hope that these relations behave in certain ways. Within the boxes we'd hope that the arrows express that the systems can easily be translated into each other (since the systems are supposed to be different formalizations of the same idea). Between the boxes we'd hope that arrows express relations of generalizing into the transfinite and abstracting away from concrete structures. Category theory is the tool to study these relations.

Chapter 3

Abstract Rewriting

As mentioned in the introduction, rewriting studies processes of stepwise transformation of objects. There is a lot to these processes that only can be talked about when we look at the concrete structure of the objects being transformed and the steps doing the transforming. In the introduction we gave the example of solving the equation $4x - 64 = 2x + 20$ for x. High school algebra supplies us with some rewriting steps we can use here. For instance we can subtract $2x$ from both sides of the equation, get $4x-64-2x = 2x+20-2x$ and $2x-64 = 20$ after simplifying (in the example, these steps are compressed into one for simplicity). After that we can add 64 to both sides of the equation and then we can divide by 2 on both sides. This way, we can get to $x = 42$ and there we stop, we consider the equation to be solved. However we can only do this when we actually know what our objects (equations) and steps (subtract $2x$ from both sides of the equation, add 64, simplify ...) are. This is a very concrete rewriting system. On the other hand though, there are facets to this that we can talk about without taking concrete structure of objects and steps into account, abstracting away from them. For example, we could have taken another approach to the algebra problem and first added 64 to both sides before subtracting $2x$. This doesn't matter for the end result, we still end up with $x = 42$.



The property that is showcased here is that one could apply multiple steps to a single object, yielding different new objects, but that it is always possible to 'get back together' from these different objects (taking steps and ending up in the same object after all). It is a very abstract property, a rewriting system has it or might not have it. We can talk about rewriting systems having such a property as opposed to rewriting systems that don't without taking the internal structure of either the steps or objects of these rewriting systems into account. Another, local, example of such a property is whether or not there are steps possible from an object. In the example, we'd like to stop applying steps after we have gotten to $x = 42$. We wouldn't like to have a step that transforms $x = 42$ to $2x = 84$ again. That would bring us further away from our goal, a solved equation. We might model our system in such a way that there are no such steps. If we do so, the object $x = 42$ stands out from other objects like $4x - 64 = 2x + 20$ and $2x - 64 = 20$ and we can see that without looking at the structure of the objects or the steps. We can just note that there are steps from $4x - 64 = 2x + 20$ and $2x - 64 = 20$ and not from $x = 42$. This is an important property, it shows that $x = 42$ is an end-result and $4x - 64 = 2x + 20$ and $2x - 64 = 20$ are not. Such properties are the topic of abstract rewriting.

Abstract rewriting studies processes of stepwise transformation of objects at their most basic form, abstracting away from the concrete structure of the objects and steps. So abstract rewriting is the formal study of processes where abstract objects are transformed (or reduced or rewritten) into other objects by certain steps. This applies to all rewriting processes as a foundational framework.

There are two different approaches to formalizing abstract rewriting. One approach emphasizes the objects that might or might not be reducible to each other. The other emphasizes the steps through which the objects are reducible to each other. In [18] the first approach is called Abstract Reduction Systems ([18, chapter 1]) and the second is called Abstract Rewriting Systems ([18, chapter 8.2]). The modern terminology is to talk about rewrite relations in the first case and about rewrite systems in the second case. These formalizations are well-studied in their finite form, we will first look at these before we look at their transfinite counterparts.

3.1 Rewrite Relations

The formalization of abstract rewriting that focuses on the objects that are being transformed is that of rewrite relations:

Definition 3.1.1. A **rewrite relation** is a structure $(A, \to)$, where A is a set and $\to$ is a binary relation on A, called the reducibility relation.

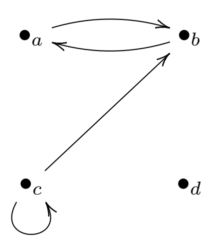
Here A models the set of objects which can be transformed, while $\to$ models the transformations.

Remark. [18, chapter 1] defines abstract reduction systems (ARSs) which are similar, but not the same. Those ARSs have a set of relations, instead of just one relation, on the set of objects and hence are defined as $(A, \{\rightarrow_\alpha \mid \alpha \in I\})$. The reason for having multiple relations is to be able to talk about commuting relations within one ARS. This has as disadvantage that we need to index our relations and need to add an index set to our definition. With the definition as given above, it is still possible to talk about commutation. It just won't be about commutating relations within one ARS, it will be about commuting rewrite relations with the same set of objects. The given definition has as clear advantage that it is less involved.

We can define:

Definition 3.1.2. In a rewrite relation $(A, \rightarrow)$ we say that $a \in A$ is **one-step reducible to** $b \in A$ if $\langle a, b\rangle \in \rightarrow$. We write $a \rightarrow b$.

The rewrite relation $(A, \rightarrow)$ can be graphically represented by a directed graph. Every object $a \in A$ is associated with a node in the graph. There is an edge from the node associated with $a \in A$ to the node associated with $b \in A$ if and only if we have $a \rightarrow b$. For instance, the rewrite relation $(\{a, b, c, d\}, \{\langle a, b\rangle, \langle b, a\rangle, \langle c, b\rangle, \langle c, c\rangle\})$ can be represented by the following graph:



In the formalism of rewrite relations, we can model the process of starting with some object, transforming it, transforming the result again and again, etc. Such a process can be thought of to as a computation. It is modeled as follows:

Definition 3.1.3. A **reduction sequence** in a rewrite relation, $(A, \rightarrow)$, is a sequence $\langle s_\beta\rangle_{\beta<\alpha}$ such that:

- $s_\beta \in A$ for all $\beta < \alpha$
- $1 \leq \alpha$ NON-EMPTY
- $\alpha \leq \omega$ BOUNDED LENGTH
- $s_\beta \rightarrow s_{\beta+1}$ for all β with $\beta + 1 < \alpha$ SUCCESSOR OBJECTS

The reduction is said to **start** in s_0. If $\alpha < \omega$ then the reduction sequence is said to **end** in $s_{\alpha-1}$ and the reduction is a reduction from s_0 to $s_{\alpha-1}$. If $\alpha = \omega$ then the reduction is said to be **non-terminating**. A **reduction step** is a pair of successive members of the reduction sequence, an instance of the reduction relation. We can write $s_\beta \rightarrow s_{\beta+1}$ for the reduction step between s_β and $s_{\beta+1}$. We can write $s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots \rightarrow s_{\alpha-1}$ for the reduction sequence as a whole.

If we interpret a reduction sequence as modeling a computation, these requirements make sense. NON-EMPTY demands that each such computation at least has a first object and hence at least starts somewhere. A step is not necessary, we might simple not perform any step on the start object. An end is also not necessary, the computation might go on indefinitely, sequence lengths of ω are allowed. We take our sequence as an ordinal indexed sequence, so in principle we might want to 'go on' after ω and take α to be some ordinal such that $\alpha > \omega$. However since we are not yet in the business of transfinite rewriting, BOUNDED LENGTH bounds the length at ω. SUCCESSOR OBJECTS is necessary to ensure that objects in the computation are actually connected by steps.

Definition 3.1.4.

- For the symmetric closure of $\rightarrow$ we write $\leftrightarrow$.
- For the reflexive-transitive closure of $\rightarrow$ we write $\twoheadrightarrow$ (sometimes also denoted by $\rightarrow^*$).
- For the reflexive-symmetric-transitive closure of $\rightarrow$ (the equivalence relation generated by $\rightarrow$) we write $\leftrightarrow^*$. If $a \leftrightarrow^* b$ then we call a and b **convertible**.

Lemma 3.1.5. *For a rewrite relation $(A, \rightarrow)$ and $a, b \in A$ we have that $a \twoheadrightarrow b$ if and only if there is a finite reduction sequence from a to b.*

Proof. Let $(A, \rightarrow)$ be a rewrite relation and let $R \subseteq A \times A$ the relation such that $\langle a, b\rangle \in R$ if and only if there is a finite rewrite sequence from a to b. We have:

- $\twoheadrightarrow \subseteq R$. We have that $\rightarrow \subseteq R$ since, if $a \rightarrow b$ then we trivially have a reduction sequence from a to b. We can prove that R is closed under transitivity and reflexivity.
  - Reflexivity. Let $a \in A$, we have a trivial reduction sequence from a to a (the sequence $\langle a\rangle$).
  - Transitivity. Let $a, b, c \in A$ and assume that we have a reduction sequence s^1 of length $n \in \mathbb{N}$ from a to b and a reduction sequence s^2 of length $m \in \mathbb{N}$ from b to c.

    Now let s^3 be the reduction sequence of length $n+m-1$ such that for all $k < n$ we have $s^3_k = s^1_n$ and for all k such that $n \leq k < n + m - 1$ we have $s^3_k = s^2_{(k-n)+1}$. The sequence s^3 is a reduction sequence because s^1 and s^2 are reduction sequences and because $s^3_{n-1} \rightarrow s^3_n$ (which holds because $s^3_{n-1} = s^1_{n-1} = b = s^2_0 \rightarrow s^2_1 = s^3_n$). Specifically, s^3 is a reduction sequence from a to b because $s^3_0 = s^1_0 = a$ and $s^3_{m+n-2} = s^2_{m-1} = b$.

  Since $\twoheadrightarrow$ is defined as the closure of $\rightarrow$ under transitivity and reflexivity, it is the smallest superset of $\rightarrow$ closed under transitivity and reflexivity. Now, because R is also a superset of $\rightarrow$ closed under transitivity and reflexivity, we have $\twoheadrightarrow \subseteq R$.

- $R \subseteq \twoheadrightarrow$. Assume $\langle a, b \rangle \in R$. There is a reduction sequence s of length $n \in \mathbb{N}$ from a to be b. We have $a = s_0$ and $b = s_{n-1}$. We can prove that for all $k < n$ we have $a \twoheadrightarrow s_k$ by induction on k.
  - Base case. We have $a = s_0$ and $\twoheadrightarrow$ is closed under reflexivity, so $a \twoheadrightarrow s_0$.
  - Step case. We get as induction hypothesis that $a \twoheadrightarrow s_k$. We have $s_k \to s_{k+1}$ because s is a reduction sequence and hence get $a \twoheadrightarrow s_{k+1}$ because $\twoheadrightarrow$ is closed under transitivity.

  So we have $a \twoheadrightarrow s_{n-1} = b$. This means $R \subseteq \twoheadrightarrow$. □

We can define:

Definition 3.1.6. A **conversion sequence** with respect to $\to$ is a reduction sequence with respect to $\leftrightarrow$.

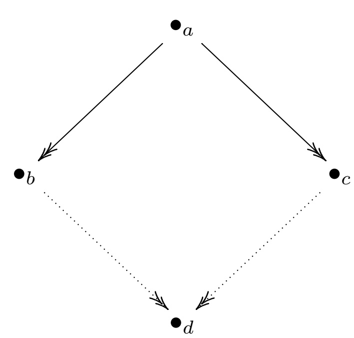
Lemma 3.1.7. *For a rewrite relation $(A, \to)$ and $a, b \in A$ we have that $a \leftrightarrow^* b$ if and only if there is a finite conversion sequence from a to b.*

Proof. This proof is similar to the proof of Lemma 3.1.5. Let $(A, \to)$ be a rewrite relation and let $C \subseteq A \times A$ be the relation such that $\langle a, b \rangle \in C$ if and only if there is a finite conversion sequence from a to b. To prove that $R = \twoheadrightarrow$:

- $\leftrightarrow^* \subseteq C$. We can prove that C is closed under reflexivity and transitivity in the same way we proved that R is closed under those operations in Lemma 3.1.5. Symmetry can be proved by showing that reversing a finite conversion sequence yields a finite conversion sequence (which is true because $\leftrightarrow$ is symmetric). Because $\leftrightarrow^*$ is the closure of $\to$ under reflexivity, symmetry and transitivity, this means $\leftrightarrow^* \subseteq C$.
- $C \subseteq \leftrightarrow^*$. We can prove this by induction in the same way we proved $R \subseteq \twoheadrightarrow$ in Lemma 3.1.5. □

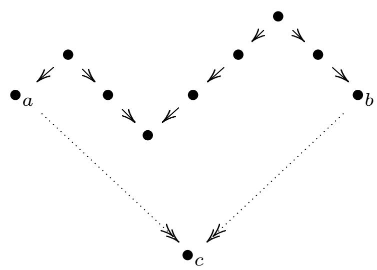
Using this, we can give two examples of the properties that abstract rewriting is concerned with:

Definition 3.1.8. A rewrite relation $(A, \to)$ is **confluent** if for all $a, b, c \in A$ such that $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$ we have a $d \in A$ such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$.



So, intuitively, that means that if we have two series of transformations (computations) that turn a into two different objects, b and c, then we also have two series of transformations (computations) on respectively b and c that turn them into the same object, d, again. This is what happened in the high school arithmetic example in the introduction of the chapter.

Definition 3.1.9. A rewrite relation $(A, \rightarrow)$ is **Church-Rosser (CR)** if for all $a, b \in A$ such that $a \leftrightarrow^* b$ we have a $c \in A$ such that $a \twoheadrightarrow c$ and $b \twoheadrightarrow c$.



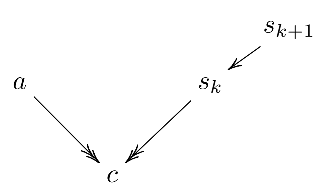
Now as example of a theorem that abstract rewriting is concerned with:

Theorem 3.1.10. *A rewrite relation is confluent if and only if it is Church-Rosser.*

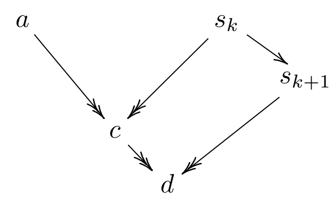
Proof. Let $(A, \rightarrow)$ be a rewrite relation.

⇒ Assume $(A, \rightarrow)$ is confluent. To prove that it is Church-Rosser, let $a, b \in A$ be arbitrary and assume that we have $a \leftrightarrow^* b$. By Lemma 3.1.7 we get that there is a conversion sequence, s of finite length n, from a to b. We can prove that for any $k < n$ there is a $c \in A$ such that $a \twoheadrightarrow c$ and $s_k \twoheadrightarrow c$ by induction on k.

- Base case. $k = 0$, $s_0 = a$ and $\twoheadrightarrow$ is closed under reflexivity. So we have $a \twoheadrightarrow a$ and $s_0 \twoheadrightarrow a$.
- Step case. We get the induction hypothesis that there is a $c \in A$ such that $a \twoheadrightarrow c$ and $s_k \twoheadrightarrow c$. Because s is a conversion sequence, we have that either $s_k \rightarrow s_{k+1}$ or $s_{k+1} \rightarrow s_k$.
  - ∗ If we have $s_{k+1} \rightarrow s_k$ we get that $s_{k+1} \twoheadrightarrow c$ by induction hypothesis and the transitivity of $\twoheadrightarrow$. We still have $a \twoheadrightarrow c$ by induction hypothesis, this proves this case.



- $*$ If we have $s_k \to s_{k+1}$ then, because $s_k \twoheadrightarrow c$, there is a $d \in A$ such that $c \twoheadrightarrow d$ and $s_{k+1} \twoheadrightarrow d$ by confluence. We get $a \twoheadrightarrow d$ by transitivity of $\twoheadrightarrow$, which proves this case.



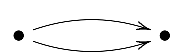
Since $b = s_{n-1}$ we get that there is a $c \in A$ such that $a \twoheadrightarrow c$ and $b \twoheadrightarrow c$. Which proves that $(A, \to)$ is Church-Rosser.

$\Leftarrow$ Assume $(A, \to)$ is Church-Rosser. To prove that it is confluent, let $a, b, c \in A$ be arbitrary and assume that we have $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$. We get $b \leftrightarrow^* c$ and because $(A, \to)$ is Church-Rosser we get that there is a $d \in A$ such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$, which proves confluence. □

3.2 Rewrite Systems

If we define rewriting as the study of stepwise transformations of objects, the framework of rewrite relations runs into a problem. We might imagine two transformations that both transform one fixed object into another fixed object, but that are different in how they do it. An abstract example of this would be two transformations that transform some object a into some object b, where one does it the easy way, doing little work, and the other one does it the hard way, doing a lot of work. These transformations are distinct by definition and, in this case, the transformations being distinct is definitely important since in transfinite rewriting (as treated in Chapter 5), the amount of work done by a step is important. A more concrete example stems from TRSs, which will be treated in Chapter 4. Consider the rewrite relation underlying the TRS with a single rule, $f(x) \to x$. The rule can reduce the term $f(f(a))$ to $f(a)$ in two different ways, by letting x match on $f(a)$ and by letting it match on just a ([18, p. 316]).

Such a situation, where there are two different steps between the same objects is called a **syntactic accident**. It can be represented by the following graph:



The fact that the transformations or steps involved in a syntactic incident are actually distinct can't be expressed with rewrite relations. This is because in a rewrite relation $(A, \to)$, the second member, $\to$, is a relation. If the instances of this relation are to be viewed as modeling transformations, then every transformation is uniquely defined by the pair of objects that it is between. So transformations between the same objects are exactly the same and hence there is only 'room' for one such transformation in the relation. The ARSs of [18] handle this by indexing $\to$ in such a way that transformations that cause a syntactic accident are indexed differently and hence belong to a different relation. Since there is no room for indexing in the formalism of reduction relations (there is only one relation), we can't do that. Also, as [18, p. 316] explains, it might not even be something we want to do. Doing so corresponds to the view that transformations that cause a syntactic accident are the same by default but can be made distinct. It might be better to view them as distinct transformations by default (because they *are*), while it remains observable that the objects that they are between are the same and that they hence are, in that sense, similar. This view gives rise to rewrite systems.

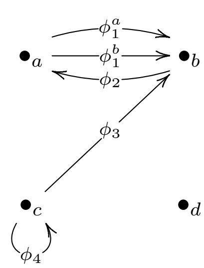
Definition 3.2.1. A **rewrite system** is a structure $(\Phi, A, \mathrm{src}, \mathrm{tgt})$, where Φ and A are sets and src and tgt are functions from Φ to A.

Here Φ is a set of steps, A is a set of objects, the steps and objects are connected by the src and tgt functions which respectively give the source and target of a given step. Φ models the transformations, A models the set of objects which can be transformed. The steps here are the main concept, they are actual, primary things, not just relations over objects. They are only connected to the objects through the src and tgt functions. In a rewrite system, the transformations involved in a syntactic accident are modelled by different steps that have the same source and target.

We also define:

Definition 3.2.2. For a rewrite system, $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ we have that $a \in A$ is **one-step reducible to** $b \in A$ if there is a $\phi \in \Phi$ such that $\mathrm{src}(\phi) = a$ and $\mathrm{tgt}(\phi) = b$. We write $a \to_\phi b$.

This allows us to also represent rewrite systems as directed graphs. We label the edges though, to emphasize that the steps are primary:



For rewrite systems we have reductions instead of reduction sequences. Like reduction sequences they might also be taken to model the process of starting with some object, transforming it, transforming the result again and again, etc: a computation.

Definition 3.2.3. A **reduction** in a rewrite system $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a tuple $\langle a, \langle \phi_\beta \rangle_{\beta < \alpha} \rangle$ such that:

- $a \in A$ and $\phi_\beta \in \Phi$ for all $\beta < \alpha$
- $\mathrm{src}(\phi_0) = a$ if $0 < \alpha$ START
- $\alpha \leq \omega$ BOUNDED LENGTH
- $\mathrm{tgt}(\phi_\beta) = \mathrm{src}(\phi_{\beta+1})$ for all β with $\beta + 1 < \alpha$ SUCCESSOR STEPS

Here α is the **length** of the reduction. The reduction is said to **start** in a. If $\alpha < \omega$ then the reduction is said to **end** in $\mathrm{tgt}(\phi_{\alpha-1})$ and the reduction is a reduction from a to $\mathrm{tgt}\,\phi_{\alpha-1}$. If $\alpha = \omega$ then the reduction is said to be **non-terminating**. The sequence $\langle\phi_\beta\rangle_{\beta<\alpha}$ is the **sequence of steps** of the reduction, it has α as length. The **sequence of objects** of the reduction is $\langle a\rangle; \langle\mathrm{tgt}(\phi_\beta)\rangle_{\beta<\alpha}$, it has $\alpha^\dagger$ (see Definition A.1.35) as length.

Remark. That a reduction of length α has a sequence of objects of length $\alpha^\dagger$ means that a reduction of length ω has a sequence of objects of length ω and that a reduction of length $n \in \mathbb{N}$ has a sequence of objects of length $n + 1$. For the reductions considered here (with a length $\leq \omega$), this can be expressed as $1 + \alpha$ ($\alpha^\dagger = 1 + \alpha$ when $\alpha \leq \omega$). However for ordinals $> \omega$ and hence for the transfinite reductions considered in Chapter 5 that no longer holds. So expressing the length of the sequence of objects as $\alpha^\dagger$ is a choice that allows for easy generalization.

This definition is a bit more involved than the one of reduction sequences in rewrite relations, but that seems necessary. We again want a start object, and don't necessarily want any steps. This way we model the computation that starts with some object and simply does nothing, don't model the computation that does nothing and doesn't even start somewhere and, most importantly, distinguish between starting in some object and doing nothing and starting in some other object and doing nothing. For this behaviour, we really need a starting object. This starting object is connected to the steps by the START. We still have BOUNDED LENGTH to bound the length of the reduction at ω. And finally, instead of requiring SUCCESSOR OBJECTS, we require SUCCESSOR STEPS to make sure that members of the reduction are connected, since, in reductions, steps are primary.

3.3 Comparison

As shown above, abstract rewriting with a focus on objects can be formalized using rewrite relations and the reduction sequences in them, while abstract rewriting with a focus steps can be formalized using rewrite systems and reductions in them. The two approaches can be compared.

3.3.1 Rewrite Systems versus Rewrite Relations

Every rewrite relation induces a rewrite system in the following way:

Definition 3.3.1. If $\mathcal{A} = (A, \to)$ is a rewrite relation then $I(\mathcal{A}) = (\{\langle x, y\rangle \mid \langle x, y\rangle \in \to\}, A, \pi_1, \pi_2)$ where $\pi_1(\langle x, y\rangle) = x$ and $\pi_2(\langle x, y\rangle) = y$. $I(\mathcal{A})$ is the rewrite system **induced** by $\mathcal{A}$.

This construction mirrors the observation that, in a rewrite relation, a reduction step is uniquely defined by the objects that it is between, but that, for a step in a rewrite system, this doesn't have to be the case. This means that not every rewrite system is induced by a rewrite relation.

Example 3.3.2. An obvious example would be the syntactic accident system $(\{\phi, \psi\}, \{a, b\}, \mathrm{src}, \mathrm{tgt})$ where $\mathrm{src}(\phi) = a$, $\mathrm{src}(\psi) = a$, $\mathrm{tgt}(\phi) = b$ and $\mathrm{tgt}(\psi) = b$. Behaviour like this simply cannot be expressed in a rewrite relation.

Example 3.3.3. A less obvious example would be simply $(\{\phi\}, \{a, b\}, \mathrm{src}, \mathrm{tgt})$ where $\mathrm{src}(\phi) = a$ and $\mathrm{tgt}(\phi) = b$. We might say that this rewrite system behaves exactly like $(\{\langle a, b\rangle\}, \{a, b\}, \pi_1, \pi_2)$ where $\pi_1(\phi) = a$ and $\pi_2(\phi) = b$, which is the rewrite system induced by the rewrite relation $(\{a, b\}, \{\langle a, b\rangle\})$. But it is technically different because in rewrite systems the steps are concrete mathematical objects, and ϕ is simply (syntacticly) different from $\langle a, b\rangle$.

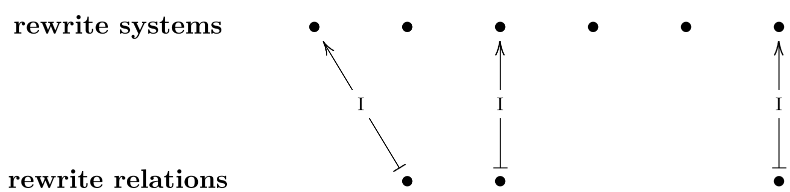
We do have:

Theorem 3.3.4. *Every two different rewrite relations induce different rewrite systems.*

Proof. Let $\mathcal{A} = (A, \to_A)$ and $\mathcal{B} = (B, \to_B)$ be different rewrite relations. Let $I(\mathcal{A}) = (\Phi, A, \mathrm{src}_\Phi, \mathrm{tgt}_\Phi)$ and $I(\mathcal{B}) = (\Psi, B, \mathrm{src}_\Psi, \mathrm{tgt}_\Psi)$.

Because $\mathcal{A} \neq \mathcal{B}$ we either have $A \neq B$, if so we get $I(\mathcal{A}) \neq I(\mathcal{B})$, or we have $\to_A \neq \to_B$. If the latter is the case we get either a $\langle x, y\rangle \in \to_A$ such that $\langle x, y\rangle \notin \to_B$ or a $\langle x, y\rangle \in \to_B$ such that $\langle x, y\rangle \notin \to_B$. In the first case we get $\langle x, y\rangle \in \Phi$ while $\langle x, y\rangle \notin \Psi$, so $\Phi \neq \Psi$ and hence $I(\mathcal{A}) \neq I(\mathcal{B})$. In the second case we get the same thing vice versa. $\square$

So the function I, formalizing our notion of rewrite relation inducing rewrite systems, is a total, injective, but non-surjective function from the class of rewrite systems to the class of rewrite relation.



We also can define the following:

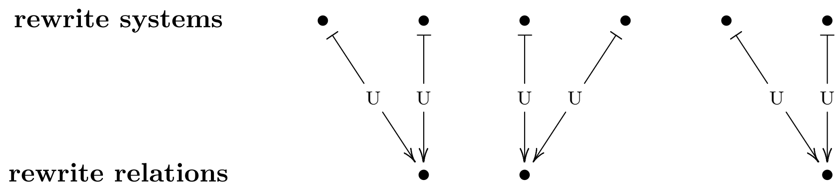
Definition 3.3.5. If $\mathbf{\Phi} = (\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a rewrite system then $U(\mathbf{\Phi}) = (A, \{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi)\rangle \mid \phi \in \Phi\})$. $U(\mathbf{\Phi})$ is the rewrite relation **underlying** by $\mathbf{\Phi}$.

We don't get that every two different rewrite systems have different underlying rewrite relations. Obvious examples would again be:

Example 3.3.6. The syntactic accident system of Example 3.3.2 and the system $(\{\phi\}, \{a, b\}, \mathrm{src}, \mathrm{tgt})$ where $\mathrm{src}(\phi) = a$ and $\mathrm{tgt}(\phi) = b$. Both have $(\{a, b\}, \{\langle a, b\rangle\})$ as underlying rewrite relation.

Example 3.3.7. A less obvious example would again be the system of Example 3.3.3 and $(\{\psi\}, \{a, b\}, \mathrm{src}, \mathrm{tgt})$ where $\mathrm{src}(\psi) = a$ and $\mathrm{tgt}(\psi) = b$. Even though the systems behave in the same way and hence might be said to be equivalent in some sense, the identity of steps matters , $\phi \neq \psi$. They are different systems with the same underlying rewrite relation.

This means that U, formalizing our notion of rewrite systems having underlying rewrite relations, is a total surjective, but non-injective function from the class of rewrite systems to the class of rewrite relation.



We do have that:

Theorem 3.3.8. *Every rewrite relation is the rewrite relation underlying the rewrite system it induces ($\mathcal{A} = U(I(\mathcal{A}))$).*

Proof. Let $\mathcal{A} = (A, \to)$ be a rewrite relation. We get:

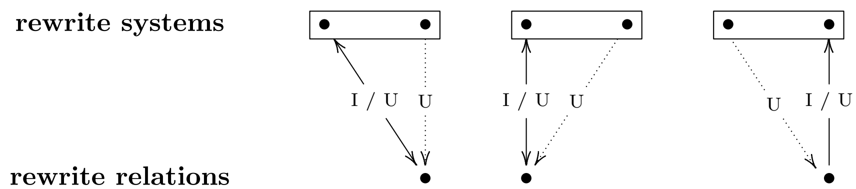
$$I(\mathcal{A}) = (\{\langle x, y\rangle \mid \langle x, y\rangle \in \to\}, A, \pi_1, \pi_2)$$

where $\pi_1(\langle x, y\rangle) = x$ and $\pi_2(\langle x, y\rangle) = y$. Now:

$$U(I(\mathcal{A})) = (A, \{\langle \pi_1(\langle x, y\rangle), \pi_2(\langle x, y\rangle)\rangle \mid \langle x, y\rangle \in \{\langle x, y\rangle \mid \langle x, y\rangle \in \to\}\})$$

which is of course just $(A, \to)$. □

So if we restrict the codomain of I and the domain of U to rewrite systems that are actually induced by some rewrite relation (that is, take I as function from rewrite relations to rewrite systems that are actually induced and take U as a function from rewrite systems that are induced), we get I and U as bijections such that $I = U^{-1}$ and $U = I^{-1}$. Also, the notion of underlying rewrite relations (U) gives rise to the method of partitioning the class of rewriting systems in equivalence classes: classes of rewrite systems having the same underlying rewrite relation (having the same underlying rewrite relation is an equivalence relation). The actual rewrite system induced by that rewrite relation then becomes natural choice as canonical representative of the equivalence class. We might even say that a rewrite relation induces an equivalence class of rewrite systems: the equivalence class that the rewrite system that it induces is a member of. Here we get a proper bijection between rewrite relations and the equivalence classes of rewrite systems.



3.3.2 Reductions versus Reduction Sequences

Continuing with reductions and reduction sequences:

Definition 3.3.9. If $\mathcal{A} = (A, \to)$ is a rewrite relation and s is a reduction sequence in $\mathcal{A}$ then $I^{\mathcal{A}}_{\to}(s) = r$, where r is a reduction sequence in $I(\mathcal{A})$ such that s is the sequence of objects of r. r is the reduction **induced** by s.

Theorem 3.3.10. *Induced reductions always exist, are unique and different reduction sequences have different induced reductions.*

Proof. Let $\mathcal{A} = (A, \to)$ be a rewrite relation. We get

$$I(\mathcal{A}) = (\{\langle x, y\rangle \mid \langle x, y\rangle \in \to\}, A, \pi_1, \pi_2)$$

where $\pi_1(\langle x, y\rangle) = x$ and $\pi_2(\langle x, y\rangle) = y$.

- Existence of induced reductions.

  Let $s = \langle a_\gamma\rangle_{\gamma<\alpha^\dagger}$ be a reduction sequence in $\mathcal{A}$. As induced reduction in $I(\mathcal{A})$ we have $r = \langle a_0, \langle a_\gamma, a_{\gamma+1}\rangle_{\gamma<\alpha}\rangle$. r is a reduction in $I(\mathcal{A})$ since:

  - By $\langle a_\gamma\rangle_{\gamma<\alpha^\dagger}$ being a reduction sequence in $\mathcal{A}$, we get that for any $\gamma < \alpha$ that $s_\gamma \to s_{\gamma+1}$. So we get $\langle s_\gamma, s_{\gamma+1}\rangle \in \{\langle x, y\rangle \mid \langle x, y\rangle \in \to\}$.
  - It respects BOUNDED LENGTH because s does.
  - It respects START and SUCCESSOR STEPS by definition.

  The sequence of objects of r is s, so r is indeed induced by s; $I^{\mathcal{A}}_{\to}(s) = r$.

- Uniqueness of an induced reductions.

  Let $s = \langle a_\gamma\rangle_{\gamma<\alpha^\dagger}$ be a reduction sequence in $\mathcal{A}$. Let $r_1 = \langle a, \langle \phi_\gamma\rangle_{\gamma<\eta}\rangle$ and $r_2 = \langle b, \langle \psi_\gamma\rangle_{\gamma<\beta}\rangle$ be two reductions in $I(\mathcal{A})$ induced by s, that is, s is the sequence of objects of both. We have:

  - $\eta = \alpha = \beta$
  - $a = a_0 = b$.
  - For any $\gamma < \alpha$ we have $\mathrm{tgt}(\phi_\gamma) = a_{\gamma+1} = \mathrm{tgt}(\psi_\gamma)$. We have $\mathrm{src}(\phi_0) = a_0 = \mathrm{src}(\psi_0)$ and for any γ such that $0 < \gamma < \alpha$ we have $\mathrm{src}(\phi_\gamma) = \mathrm{tgt}(\phi_{\gamma-1}) = a_{\gamma-1} = \mathrm{tgt}(\psi_{\gamma-1}) = \mathrm{src}(\psi_\gamma)$. So for any $\gamma < \alpha$ we have $\mathrm{src}(\phi_\gamma) = \mathrm{src}(\psi_\gamma)$. And since every $\phi \in \{\langle x, y\rangle \mid \langle x, y\rangle \in \to\}$ is uniquely identified by its source and target because $\to$ is a relation, we get that for any $\gamma < \alpha$ we have $\phi_\gamma = \psi_\gamma$.

So $r_1 = r_2$.

- Reductions induced by different reduction sequences are different.

  Let $s_1 = \langle a_\gamma \rangle_{\gamma < \alpha^\dagger}$ and $s_2 = \langle b_\gamma \rangle_{\gamma < \beta^\dagger}$ be reduction sequences in $\mathcal{A}$ such that $s_1 \neq s_2$. Depending on whether $\alpha^\dagger = \beta^\dagger$ we get:

  - $\alpha^\dagger \neq \beta^\dagger$.
    We get $\alpha \neq \beta$ and hence $I_\rightarrow^{\mathcal{A}}(s_1)$ has a different length than $I_\rightarrow^{\mathcal{A}}(s_2)$ and hence is different.
  - $\alpha^\dagger = \beta^\dagger$.
    Let $I_\rightarrow^{\mathcal{A}}(s_1)$ be of the form $\langle a, \langle \phi_\gamma \rangle_{\gamma < \alpha} \rangle$ and $I_\rightarrow^{\mathcal{A}}(s_2)$ be of the form $\langle b, \langle \psi_\gamma \rangle_{\gamma < \beta} \rangle$. Since $s_1 \neq s_2$ we get an η such that $a_\eta \neq b_\eta$. If $\eta = 0$ we get $a = a_0 \neq b_0 = b$ and hence $I_\rightarrow^{\mathcal{A}}(s_1) \neq I_\rightarrow^{\mathcal{A}}(s_2)$. Otherwise we get $\text{tgt}(\phi_{\eta-1}) = a_\eta \neq b_\eta = \text{tgt}(\psi_{\eta-1})$, so $\phi_{\eta-1} \neq \psi_{\eta-1}$ and hence $I_\rightarrow^{\mathcal{A}}(s_1) \neq I_\rightarrow^{\mathcal{A}}(s_2)$. □

Existence and uniqueness of induced reductions show that for any rewrite relation $\mathcal{A}$, $I_\rightarrow^{\mathcal{A}}$ is a function from the reduction sequences in $\mathcal{A}$ to the reductions of $I(\mathcal{A})$. That different reduction sequences have different induced reductions shows that this function is injective. Since every reduction has a sequence of objects this function is also surjective, making $I_\rightarrow^{\mathcal{A}}$ a bijection.

Reductions in $I(\mathcal{A})$

$\langle a_0, \langle \phi_0, \ldots, \phi_{n-1} \rangle \rangle$ $\langle b_0, \langle \psi_0, \ldots, \psi_{m-1} \rangle \rangle$

$\uparrow I_\rightarrow^{\mathcal{A}}$ $\uparrow I_\rightarrow^{\mathcal{A}}$

$\langle a_0, a_1, \ldots, a_n \rangle$ $\langle b_0, b_1, \ldots, b_m \rangle$

Reduction sequences in rewrite relation $\mathcal{A}$

Definition 3.3.11. If $\mathbf{\Phi} = (\Phi, A, \text{src}, \text{tgt})$ is a rewrite system and r is a reduction in $\mathbf{\Phi}$ then $U_\rightarrow^{\mathbf{\Phi}}(r) = s$, where s is the reduction sequence in $U(\mathbf{\Phi})$ such that s is the sequence of objects of r. s is the reduction sequence **underlying** by r.

We have that:

Theorem 3.3.12. *Every reduction sequence in a rewrite relation underlying a rewrite system underlies some reduction in that rewrite system.*

Proof. Let $\mathbf{\Phi} = (\Phi, A, \text{src}, \text{tgt})$ be a rewrite system, we get

$$I(\mathbf{\Phi}) = (A, \{\langle \text{src}(\phi), \text{tgt}(\phi) \rangle \mid \phi \in \Phi\})$$

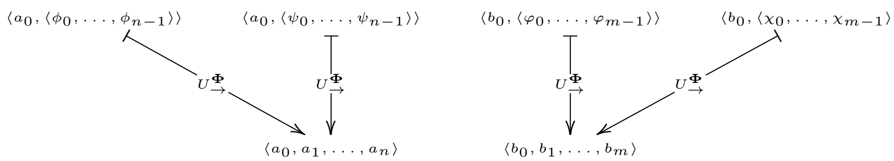
Let $s = \langle s_\gamma \rangle_{\gamma < \alpha^\dagger}$ be a reduction sequence in $I(\mathbf{\Phi})$. Because s is a reduction sequence we get that for all $\gamma < \alpha$ we have $s_\gamma \rightarrow s_{\gamma+1}$, so there is a $\phi \in \Phi$ such that $\text{src}(\phi) = s_\gamma$ and $\text{tgt}(\phi) = s_{\gamma+1}$. Let $f : \alpha \rightarrow \Phi$ be a function that choses one such ϕ for a given $\gamma < \alpha$. We get that $\langle s_0, \langle f(\gamma) \rangle_{\gamma < \alpha} \rangle$ is a proper reduction in $I(\mathbf{\Phi})$ and that its sequence of objects is s, so $I_\rightarrow^{\mathbf{\Phi}}(r) = s$. □

Two different reductions might have the same underlying reduction sequence though.

Example 3.3.13. As example, we again have the syntactic accident system of Example 3.3.2 with in it the reductions $\langle a, \langle \phi \rangle \rangle$ and $\langle a, \langle \psi \rangle \rangle$. We have $(\{a, b\}, \{\langle a, b \rangle\})$ as underlying rewrite relation and in it the reduction sequence $\langle a, b \rangle$ that underlies both $\langle a, \langle \phi \rangle \rangle$ and $\langle a, \langle \psi \rangle \rangle$. Even though the reductions have the same sequences of objects, the identity of steps matters in the rewrite system formalism and they are different reductions.

So, for any rewrite system $\mathbf{\Phi}$, $U^{\mathbf{\Phi}}_{\rightarrow}$ is a surjective, non-injective function from the reductions in $\mathbf{\Phi}$ to the reduction sequences of $U(\mathcal{A})$.

Reductions in rewrite system $\mathbf{\Phi}$



Reduction sequences in $U(\mathbf{\Phi})$

On the other hand we do have that:

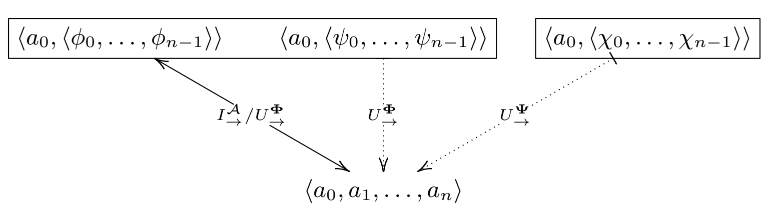
Theorem 3.3.14. *Every reduction sequence is the reduction sequence underlying the reduction it induces.*

Proof. Let $\mathcal{A} = (A, \rightarrow)$ be a rewrite relation and let $s = \langle s_\gamma \rangle_{\gamma < \alpha^\dagger}$ be a reduction sequence in it. We get $I(\mathcal{A}) = (\{\langle x, y \rangle \mid \langle x, y \rangle \in \rightarrow\}, A, \pi_1, \pi_2)$ where $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$, and $I^{\mathcal{A}}_{\rightarrow}(s) = \langle s_0, \langle s_\gamma, s_{\gamma+1} \rangle_{\gamma < \alpha} \rangle$.

Now $U(I(\mathcal{A})) = \mathcal{A}$ (Theorem 3.3.8) and in $U(I(\mathcal{A}))$ we have $U^{I(\mathcal{A})}_{\rightarrow}(I^{\mathcal{A}}_{\rightarrow}(s)) = s_0; \langle \mathrm{tgt}(\langle s_\gamma, s_{\gamma+1} \rangle) \rangle_{\gamma < \alpha}$. The fact that $s_0; \langle \mathrm{tgt}(\langle s_\gamma, s_{\gamma+1} \rangle) \rangle_{\gamma < \alpha} = \langle s_\gamma \rangle_{\gamma < \alpha^\dagger} = s$ wraps things up. □

So for any $\mathcal{A}$ we get that $I^{\mathcal{A}}_{\rightarrow}$ and $U^{I(\mathcal{A})}_{\rightarrow}$ are each others inverses, and combining these facts we get the following diagram:

Reductions in rewrite system Φ **Reductions in rewrite system Ψ**



Reduction sequence in rewrite relation $\mathcal{A}$ with $U(\mathbf{\Phi}) = \mathcal{A} = U(\mathbf{\Psi})$ and $I(\mathcal{A}) = \Psi$

3.3.3 Concluding the Comparison

The conclusion that can be drawn from this comparison is that every rewrite relation can be expressed as a rewrite system, but not vice versa. Also, every reduction sequence in a rewrite relation can be expressed as a reduction in a rewrite system (the induced rewrite system), but not vice versa. The functions I, U, $I^{\mathcal{A}}_{\rightarrow}$ and $U^{\mathbf{\Phi}}_{\rightarrow}$ express the relationship between the two approaches and are pretty well-behaved. However, an approach using rewrite systems and reductions is simply more expressive than an approach using reduction sequence and systems. This is a good reason for preferring the rewrite systems approach.

Also, the fact that in rewrite systems steps are primary, as opposed to an instance of a relation over objects (as they are in rewrite relations) is a desirable property when formalizing transfinite rewriting (Chapter 5). In transfinite rewriting we want to express whether the sequence of steps of a reduction has certain convergence properties. We can do this by expressing the structure of the steps in some way and when doing this, it is practical when the steps are actual primary things instead of instances of a relation (although it can also be done using relations ([13])).

3.3.4 A Category Theoretical View

The comparison of formalisms as done above is one for which the tools and terminology of category theory (Appendix A.5) are very well suited. Using those, the above might be expressed more concisely and compactly, and a lot of the extra analysis that category theoretic tools allow might be done. Appendix B.1 contains a start to this. However, in generality, this is difficult, time-consuming and might be considered too broad a subject and off-topic for this thesis.

Chapter 4

Term Rewriting

In abstract rewriting we abstract away from the internal structure of the objects that we are transforming and the steps that do the transforming. We view them as abstract objects with abstract steps between them. This way we can talk about the facets of stepwise transformations that don't involve the internal structure of steps and objects. However sometimes it *is* necessary or desirable to involve this internal structure. Doing so yields, what in the introduction was called concrete rewriting. Term rewriting is an instance of it, one specific rewriting paradigm where the structure of the objects and the steps is explicit. In term rewriting objects are terms and steps are instances of reduction rules, a system of such terms and rules is called a Term Rewriting System or TRS. So term rewriting studies the transformation of terms by reduction rules. Looking at this from a computational point of view, we can view terms as structured data being transformed by the computations modeled by reduction rules.

Abstract rewriting still underlies term rewriting. We can discard the internal structure of terms and reduction rules in a TRS and get a rewrite relation or rewrite system as defined in Chapter 3. The TRS is then said to be embedded in the rewrite relation or rewrite system. The same can be done in the systems for transfinite abstract rewriting defined in Chapter 5. This way, abstract rewriting gives a notion of reduction (or in the case of transfinite abstract rewriting, transfinite reduction) on TRSs.

Term rewriting is interesting and well-studied in itself, but in this thesis mainly functions as example of how a concrete rewriting system can be embedded in an abstract rewriting system and what the properties and results of such an embedding are.

To define term rewriting systems we first need to define terms. This can be done in multiple ways that all have their advantages. We define terms in an inductive and in a positional manner.

4.1 Terms

4.1.1 Signatures

Both the inductive and the positional definitions of terms depend on the notion of signature:

Definition 4.1.1. A **signature** Σ is a countable set of symbols which come equipped with a natural number called **arity**. For a symbol f with arity 1 we write $f/1$. For the function mapping a symbol, f, to its arity, 1, we write $\text{ar}(f) = 1$. For the set of all symbols with arity k we write Σ_k.

These symbols in a signature can be viewed as function symbols, their arity then is the number of arguments they take. Symbols of arity 0, $c \in \Sigma_0$, are also referred to as the constants of Σ.

We also need a set variables and we need their symbols to be disjunct from our signature.

Definition 4.1.2. If Σ is a signature then $\mathcal{X}_\Sigma$ is also a signature, consisting of a countably infinite set of symbols disjunct from Σ with arity 0.

Whenever Σ is regarded as a signature, $\mathcal{X}_\Sigma$ is regarded as the set of variables associated with that signature.

4.1.2 Inductive Terms

For any signature we can define the set of inductive terms over it:

Definition 4.1.3. For any signature Σ containing at least one constant, the set $T(\Sigma)$ of **inductive ground terms** over Σ is defined inductively by:

$$f(t_1, \ldots, t_n) \in T(\Sigma) \Leftarrow f \in \Sigma_n \wedge t_1, \ldots, t_n \in T(\Sigma)$$

The set $T(\Sigma, \mathcal{X}_\Sigma)$ of **inductive terms** over Σ is $T(\Sigma \cup \mathcal{X}_\Sigma)$. Here, symbols from $\mathcal{X}_\Sigma$ are referred to as **variables**, symbols from Σ are referred to as **function symbols**.

Remark. It is required that Σ contains at least on constant to make sure that $T(\Sigma)$ is non-empty.

It directly follows that $T(\Sigma) \subset T(\Sigma, \mathcal{X}_\Sigma)$. Also since inductive terms over a signature can be viewed as inductive ground terms over an extended signature, we can just define properties on inductive ground terms for simplicity and also get those properties on inductive (non-ground) terms. When the case where a symbol has arity 0 (is a constant) is made explicit we get:

(i) $c \in T(\Sigma) \Leftarrow c \in \Sigma_0$

(ii) $f(t_1, \ldots, t_n) \in T(\Sigma) \Leftarrow f \in \Sigma_n \wedge n > 0 \wedge t_1, \ldots, t_n \in T(\Sigma)$

The inductive definition then should be read as “the set of inductive terms over a signature is the smallest set such that (i) and (ii) hold”. Inductive terms can be constructed using (i) or from other inductive terms using (ii).

This means that inductive terms are only finitely deep. This can be made explicit with the following definition:

Definition 4.1.4. The **depth** of an inductive term $t \in T(\Sigma)$, denoted by $\text{dpt}(t)$ defined inductively as:

- If $t \in \Sigma_0$ then $\text{dpt}(t) = 1$
- If $t = f(t_1, \dots, t_n)$ with $f \in \Sigma_n$, $n > 0$ and $t_1, \dots, t_n \in T(\Sigma, \mathcal{X}_\Sigma)$ then $\text{dpt}(t) = \max\{\text{dpt}(t_k) \mid 1 \leq k \leq n\} + 1$

Every inductive term has some finite depth. When we view terms as modeling data this fact can be interpreted as stating that inductive terms model finite data, data that is in some sense of finite size. But as this thesis is concerned with transfinite rewriting we are interested in data that can grow beyond any finite size, infinite data. To get a set of terms with which we can represent infinite data we can turn the inductive definition into a coinductive one ([8]):

Definition 4.1.5. For any non-empty signature Σ, the set $T^\omega(\Sigma)$ of **coinductive ground terms** over Σ is defined coinductively by:

$$f(t_1, \dots, t_n) \in T^\omega(\Sigma) \Rightarrow f \in \Sigma_n \wedge t_1, \dots, t_n \in T^\omega(\Sigma)$$

The set $T^\omega(\Sigma, \mathcal{X}_\Sigma)$ of **coinductive terms** over Σ (or Σ-terms) is $T^\omega(\Sigma \cup \mathcal{X}_\Sigma)$.

Remark. Here, requiring that Σ is non-empty is enough to guarantee that $T(\Sigma)$ is non-empty. If $\Sigma = \{f/n\}$ with $n > 0$ we still get $t = f(\langle t \rangle_{m<n})$ as term.

We again get that $T^\omega(\Sigma) \subset T^\omega(\Sigma, \mathcal{X}_\Sigma)$. When the case where a symbol has arity 0 (is a constant) is made explicit we get:

(iii) $c/0 \in T^\omega(\Sigma) \Rightarrow c \in \Sigma_0$

(iv) $f/n(t_1, \dots, t_n) \in T^\omega(\Sigma) \wedge n > 0 \Rightarrow f \in \Sigma_n \wedge t_1, \dots, t_n \in T^\omega(\Sigma)$

The coinductive definition then should be read as “the set of coinductive terms over a signature is the *largest* set such that (iii) and (iv) hold”. Where the definition of inductive terms can be interpreted as stating how terms can be constructed (using (i) and (ii)), the definition of coinductive terms can be interpreted as stating how terms can be deconstructed (respecting (iii) and (iv)). Now, for example, the term t such that $t = f(t)$ isn't an inductive term. This is because there is a smaller set of terms without it and the set of inductive terms is defined as “the *smallest* set such that ...”. Also, it can not be constructed from other terms using (i) or (ii). However, it is a coinductive term, because having it as an coinductive term doesn't violate either (iii) or (iv), so it is in the “largest set such that...”. Also, we can deconstruct the term by removing the first f and get the same term. Also, all inductive terms are coinductive terms since, if we can build a term using one (i) and (ii), deconstructing them doesn't violate (iii) or (iv). So we get that $T(\Sigma, \mathcal{X}_\Sigma) \subset T^\omega(\Sigma, \mathcal{X}_\Sigma)$.

These terms are infinite in the sense that they have infinite depth, terms with infinite width can also be considered (see Section 4.1.6).

4.1.3 Positional Terms

We can also define positional terms over a signature:

Definition 4.1.6. For any signature Σ containing at least one constant, a **positional ground term** over Σ is a partial function, $t : \bigcup_{i<n} \mathbb{N}^i \to \Sigma)$, where:

- $\mathbb{N}^i$ is the set of strings of length i over $\mathbb{N}$
- $0 < n < \omega$ FINITE DEPTH
- For any $m < \alpha$ there is a $p \in \mathbb{N}^m$ with $t(p) \in \Sigma$ DEPTH WELL-DEFINED
- $t(p; i) \in \Sigma$ if and only if $t(p) \in \Sigma_k$ and $1 \leq i \leq k$ CONNECTEDNESS

n is called the **depth** of the term, we denote it by $\text{dpt}(t)$. DEPTH WELL-DEFINED implies that $t(\epsilon)$ is defined, we call it the **root** of the term. The domain of our partial function is the set of strings of natural numbers of length smaller than the depth of the term. The part of the domain set where the function is defined is called the set of **positions** of the term. We denote the set of positions of the term by $\text{Pos}(t)$ and get $\text{Pos}(t) = \{p \in \bigcup_{i<n} \mathbb{N}^i \mid t(p) \in \Sigma\}$. A position p is some string of length smaller than n. The length of a position denoted by $|p|$. We take the lexicographical order on strings as order on our set of positions. For any position $p \in \text{Pos}(t)$ we have that the **subterm of** t **at** p, denoted as $t|_p$, is the positional term such that $t|_p(q) = t(p; q)$ where $\text{dpt}(t|_p)$ is such that $\text{dpt}(t) = |p| + \text{dpt}(t|_p)$ (that is $\text{dpt}(t|_p) = \text{dpt}(t) - |p|$, by FINITE DEPTH).

For $s, t \in \mathcal{T}(\Sigma)$, we say that they are equivalent up to length $n \in \mathbb{N}$, denoted by $s =_{<n} t$, if for all positions, $p \in (\text{Pos}(t) \cap \text{Pos}(s))$, with a length smaller than n we have $s(p) = t(p)$. If $s =_{<\max(\text{dpt}(s), \text{dpt}(t))} t$, then s and t are syntactically equivalent and we write $s = t$.

The set of positional ground terms is denoted by $\mathcal{T}(\Sigma)$. The set $\mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ of **positional terms** over Σ is $\mathcal{T}(\Sigma \cup \mathcal{X}_\Sigma)$. Here, again, symbols from $\mathcal{X}_\Sigma$ are referred to as **variables**, symbols from Σ are referred to as **function symbols**.

As with inductive terms, since positional terms over a signature can be viewed as positional ground terms over an extended signature, properties can be defined on positional ground terms for simplicity and we also get those properties on positional (non-ground) terms.

Positional terms are of finite depth. To get terms of possibly infinite depth we can relax the axiom bounding the depth at ω.

Definition 4.1.7. For any non-empty signature Σ, an **infinite positional ground term** over Σ is a partial function, $t : \bigcup_{i<\alpha} \mathbb{N}^i \to \Sigma$, where:

- $\mathbb{N}^i$ is the set of strings of length i over $\mathbb{N}$
- $0 < \alpha \leq \omega$ INFINITE DEPTH
- For any $m < \alpha$ there is a $p \in \mathbb{N}^m$ with $t(p) \in \Sigma$ DEPTH WELL-DEFINED
- $t(p;i) \in \Sigma$ if and only if $t(p) \in \Sigma_k$ and $1 \leq i \leq k$ CONNECTEDNESS

The set of infinite positional ground terms is denoted by $\mathcal{T}^\omega(\Sigma)$. The set $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ of **infinite positional terms** over Σ is $\mathcal{T}(\Sigma \cup \mathcal{X}_\Sigma)$.

The auxiliary definitions on positional (ground) terms can be used on infinite positional (ground) terms. The only thing to note here is that, on ordinals, addition does not behave as in the finite case, it is not commutative (Section A.1.3.3). So here we still have $\mathrm{dpt}(t) = \mathrm{dpt}(t|_p) + |p|$, only this is a proper right addition now (where in the finite case there is no such thing as right addition). So $\mathrm{dpt}(t|_p) = \mathrm{dpt}(t) - |p|$ is a right subtraction. That means that we get $\mathrm{dpt}(t|_p) = \omega$ if $\mathrm{dpt}(t) = \omega$, no matter how big $|p|$ is.

We once again get that $T(\Sigma, \mathcal{X}_\Sigma) \subset T^\omega(\Sigma, \mathcal{X}_\Sigma)$ and that infinite positional terms over a signature can be viewed as infinite positional ground terms over an extended signature, so that properties on the set of infinite positional ground terms are inherited by the set of infinite positional terms.

4.1.4 Positional Terms versus Inductive Terms

For any signature Σ, we can define a mapping from positional terms to inductive terms:

Definition 4.1.8. $\mathrm{Tr} : \mathcal{T}(\Sigma) \to T(\Sigma)$ is the recursively defined mapping such that for any $t \in \mathcal{T}(\Sigma)$ we have

$$\mathrm{Tr}(t) = (t(\epsilon))(\mathrm{Tr}(t|_1), \ldots, \mathrm{Tr}(t|_{\mathrm{ar}(t(\epsilon))}))$$

We get:

Theorem 4.1.9. $\mathrm{Tr} : \mathcal{T}(\Sigma) \to T(\Sigma)$ *is a bijection.*

Proof.

- Tr is injective.

  We can order terms in $\mathcal{T}(\Sigma)$ by the proper subterm relation ($s < t \Leftrightarrow (\exists p \in \mathrm{Pos}(t).\ s = t|_p) \wedge (s \neq t)$). This is a well-founded relation.

  We prove that for any $t, s \in \mathcal{T}(\Sigma)$ if then $t \neq s$ then $\mathrm{Tr}(t) \neq \mathrm{Tr}(s)$ by induction on $\mathcal{T}(\Sigma)$ ordered by proper subterms. We get as induction hypothesis that for all proper subterms t' of t it holds that if for some $h \in \mathcal{T}(\Sigma)$ we have $t' \neq h$ then $\mathrm{Tr}(t') \neq \mathrm{Tr}(h)$.

  Let $t, s \in \mathcal{T}(\Sigma)$ and let $\mathrm{Tr}(t)$ be of the form $f(t_1, \ldots, t_{\mathrm{ar}(f)})$ and let $\mathrm{Tr}(s)$ be of the form $g(s_1, \ldots, s_{\mathrm{ar}(g)})$. Assume $t \neq s$ we get some $p \in (\mathrm{Pos}(t) \cap$

Pos(s)) such that $t(p) \neq s(p)$. We either get that $p = \epsilon$ and hence $f = t(\epsilon) \neq s(\epsilon) = g$, so $\mathrm{Tr}(t) = f(t_1, \ldots, t_n) \neq g(s_1, \ldots, s_m) = \mathrm{Tr}(s)$. Or we get $p = i; q$ for some $i \leq (\mathrm{ar}(t(\epsilon)) \cap \mathrm{ar}(s(\epsilon)))$. Now we get $t|_i \neq s|_i$ and since $t|_i$ is a subterm of t we get $\mathrm{Tr}(t|_i) \neq \mathrm{Tr}(s|_i)$. We have $t_i = \mathrm{Tr}(t|_i) \neq \mathrm{Tr}(s|_i) = s_i$ and hence $\mathrm{Tr}(t) = f(t_1, \ldots, t_n) \neq g(s_1, \ldots, s_m) = \mathrm{Tr}(s)$.

- Tr is surjective.

  Let $\mathrm{Tr}^{-1} : T(\Sigma) \to \mathcal{T}(\Sigma)$ be the inductively defined function such that for any $t \in T(\Sigma)$ of the form $f(t_1 \ldots, t_{\mathrm{ar}(f)})$ we have $\mathrm{Tr}^{-1}(t) \in \mathcal{T}(\Sigma)$ is such that $\mathrm{Tr}^{-1}(t)(\epsilon) = f$ and $\mathrm{Tr}^{-1}(t)|_i = \mathrm{Tr}^{-1}(t_i)$ for all $i \leq \mathrm{ar}(f)$.

  We prove that for any $t \in T(\Sigma)$ we have $\mathrm{Tr}(\mathrm{Tr}^{-1}(t)) = t$ by induction on the structure of t. Let t be of the form $f(t_1, \ldots, t_{\mathrm{ar}(f)})$, we get as induction hypothesis that for any $i \leq \mathrm{ar}(f)$ we get $\mathrm{Tr}(\mathrm{Tr}^{-1}(t_i)) = t_i$.

  By first unfolding the definition of Tr, then unfolding the definition of Tr^{-1} and then applying induction hypothesis we get:

$$
\begin{aligned}
&\mathrm{Tr}(\mathrm{Tr}^{-1}(t)) &=\\
&((\mathrm{Tr}^{-1}(t))(\epsilon))(\mathrm{Tr}(\mathrm{Tr}^{-1}(t)|_1), \ldots, \mathrm{Tr}(\mathrm{Tr}^{-1}(t)|_{(\mathrm{Tr}^{-1}(t))(\epsilon)})) &=\\
&f(\mathrm{Tr}(\mathrm{Tr}_\infty^{-1}(t_1)), \ldots, \mathrm{Tr}(\mathrm{Tr}_\infty^{-1}(t_{\mathrm{ar}(f)}))) &=\\
&f(t_1, \ldots, t_{\mathrm{ar}(f)}) &=\\
&t
\end{aligned}
$$

  Showing the surjectivity of Tr. □

So our mapping, $\mathrm{Tr} : \mathcal{T}(\Sigma) \to T(\Sigma)$, and its inverse, give a well-behaved back-and-forth correspondence between our two sets of terms. For every inductive term there is an associated positional term and vice versa. The inductive definition and the positional definition in some sense define the same set of terms.

We can also define a mapping from infinite positional terms to coinductive terms:

Definition 4.1.10. $\mathrm{Tr}_\infty : \mathcal{T}^\omega(\Sigma) \to T^\omega(\Sigma)$ is the corecursively defined mapping such that for any $t \in \mathcal{T}^\omega(\Sigma)$ we have

$$\mathrm{Tr}_\infty(t) = t(\epsilon)(\mathrm{Tr}_\infty(t|_1), \ldots, \mathrm{Tr}_\infty(t|_{\mathrm{ar}(t(\epsilon))}))$$

Theorem 4.1.11. $\mathrm{Tr}_\infty : \mathcal{T}^\omega(\Sigma) \to T^\omega(\Sigma)$ *is a bijection.*

Proof.

- Tr_∞ is injective.

  Let $t, s \in \mathcal{T}^\omega(\Sigma)$ and assume that $t \neq s$.

  We prove that $\mathrm{Tr}_\infty(t)$ and $\mathrm{Tr}_\infty(s)$ are not bisimilar. That is, there is no bisimulation $\mathcal{R}$ such that $\langle \mathrm{Tr}_\infty(t), \mathrm{Tr}_\infty(s) \rangle \in \mathcal{R}$. Let $\mathcal{R} : T^\omega(\Sigma) \times T^\omega(\Sigma)$

be an arbitrary bisimulation, that is, a binary relation such that for any $f(t_1, \ldots, t_n), g(s_1, \ldots, s_m) \in T^\omega(\Sigma)$ we have

$$\langle f(t_1, \ldots, t_n), g(s_1, \ldots, s_m)\rangle \in \mathcal{R} \Leftrightarrow (f = g) \wedge (\forall i \leq n.\ \langle t_i, s_i\rangle \in \mathcal{R})$$

By assumption we get a $p \in \mathrm{Pos}(t) \cap \mathrm{Pos}(s)$ such that $t(p) \neq s(p)$. We get that $\mathrm{Tr}_\infty(t|_p)$ is of the form $f(t_1, \ldots, t_n)$ and $\mathrm{Tr}_\infty(s|_p)$ is of the form $g(s_1, \ldots, s_{\mathrm{ar}(g)})$ such that $g \neq f$. So $\langle \mathrm{Tr}_\infty(t|_p), \mathrm{Tr}_\infty(s|_p)\rangle \notin \mathcal{R}$.

In general, assume $t', s' \in \mathcal{T}^\omega(\Sigma)$ and $q; i \in (\mathrm{Pos}(t') \cap \mathrm{Pos}(s'))$ and $\langle \mathrm{Tr}_\infty(t'|_{q;i}), \mathrm{Tr}_\infty(s'|_{q;i})\rangle \notin \mathcal{R}$. By the definition of Tr_∞ we get:

$$\mathrm{Tr}_\infty(t'|_q) = t'(q)(\mathrm{Tr}_\infty(t|_{q;1}), \ldots, \mathrm{Tr}_\infty(t'|_{q;\mathrm{ar}(t'(q))}))$$

and:

$$\mathrm{Tr}_\infty(s'|_q) = s'(q)(\mathrm{Tr}_\infty(s|_{q;1}), \ldots, \mathrm{Tr}_\infty(s'|_{q;\mathrm{ar}(s'(q))}))$$

Because $\mathcal{R}$ is a bisimulation we get:

$$\begin{aligned}\langle\ & t'(q)(\mathrm{Tr}_\infty(t|_{q;1}), \ldots, \mathrm{Tr}_\infty(t'|_{q;\mathrm{ar}(t'(q))})), \\ & s'(q)(\mathrm{Tr}_\infty(s|_{q;1}), \ldots, \mathrm{Tr}_\infty(s'|_{q;\mathrm{ar}(s'(q))}))\ \rangle \notin \mathcal{R}\end{aligned}$$

So:

$$\langle \mathrm{Tr}_\infty(t'|_q), \mathrm{Tr}_\infty(s'|_q)\rangle = \notin \mathcal{R}$$

Applying this argument to $\langle \mathrm{Tr}_\infty(t|_p), \mathrm{Tr}_\infty(s|_p)\rangle \notin \mathcal{R}$ and doing so $|p|$ times yields $\langle \mathrm{Tr}_\infty(t), \mathrm{Tr}_\infty(s)\rangle \notin \mathcal{R}$. This uses the fact that $|p|$ is finite, there are only finitely many positions above a given position in an infinite term, so different terms differ at finite depth. That means that we only have to 'destruct' finitely many times before get to the place where terms $\mathrm{Tr}_\infty(t)$ and $\mathrm{Tr}_\infty(s)$ differ.

- Tr_∞ is surjective.

  Let $\mathrm{Tr}_\infty^{-1} : T^\omega(\Sigma) \to \mathcal{T}^\omega(\Sigma)$ be some function such that for any $t \in T^\omega(\Sigma)$ of the form $f(t_1 \ldots, t_{\mathrm{ar}(f)})$ we have $\mathrm{Tr}_\infty^{-1}(t)$ is such that $(\mathrm{Tr}_\infty^{-1}(t))(\epsilon) = f$ and $\mathrm{Tr}_\infty^{-1}(t)|_i = \mathrm{Tr}_\infty^{-1}(t_i)$ for all $i \leq \mathrm{ar}(f)$.

  Now let $\mathcal{R} = \{\langle t, \mathrm{Tr}_\infty(\mathrm{Tr}_\infty^{-1}(t))\rangle \mid t \in T^\omega(\Sigma)\}$. To prove that $\mathcal{R}$ is a bisimulation, let $t \in T^\omega(\Sigma)$ be of the form $f(t_1, \ldots, t_{\mathrm{ar}(f)})$ and let $\mathrm{Tr}_\infty(\mathrm{Tr}_\infty^{-1}(t)) \in T^\omega(\Sigma)$ be of the form $g(t'_1, \ldots, t'_{\mathrm{ar}(g)})$. We get $g = (\mathrm{Tr}_\infty^{-1}(t))(\epsilon) = f$. Also for any $i \leq \mathrm{ar}(f)$ we get $t'_i = \mathrm{Tr}_\infty(\mathrm{Tr}_\infty^{-1}(t)|_i) = \mathrm{Tr}_\infty(\mathrm{Tr}_\infty^{-1}(t_i))$ (by the definitions of Tr_∞^{-1} and Tr_∞). We have $\langle t_i, \mathrm{Tr}_\infty(\mathrm{Tr}_\infty^{-1}(t_i))\rangle \in \mathcal{R}$ and hence $\langle t_i, t'_i\rangle \in \mathcal{R}$. So $\mathcal{R}$ is a bisimulation, hence any $t \in T^\omega(\Sigma)$ is bisimilar to $\mathrm{Tr}_\infty(\mathrm{Tr}_\infty^{-1}(t))$ which shows surjectivity of Tr_∞. □

So, our mapping $\mathrm{Tr}_\infty : \mathcal{T}^\omega(\Sigma) \to T^\omega(\Sigma)$ and its inverse again give a well-behaved back-and-forth correspondence between our two sets of infinite terms. For every coinductive term there is an associated infinite positional term and vice versa. The coinductive definition and the infinite positional definition in some sense define the same set of terms.

coinductive terms	=	infinite positional terms
$\cup$		$\cup$
inductive terms	=	positional terms

Remark. Since a lot of concepts are easier to define on positional terms (we don't have to use inductive/coinductive definitions for them). Positional terms will be used in the remainder of this thesis. Whenever terms are mentioned, positional terms meant. Still, via the association defined above, statements about those terms can also be taken as statements about inductive terms.

4.1.5 A Metric on Terms

As will become clear in chapter Chapter 5, it is useful to have some sense of structure on our set of terms. Structure that, in an abstract way, expresses which terms are alike, which are different and to which extent. The standard way to do this is by using a metric (Appendix A.3), as is done in [18, p. 670]:

Definition 4.1.12. $d_{\mathcal{T}(\Sigma)} : \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathbb{R}^+$ is the function such that any for $s, t \in \mathcal{T}(\Sigma)$:

- $s = t \Rightarrow d_{\mathcal{T}(\Sigma)}(s,t) = 0$.
- $s \neq t \Rightarrow d_{\mathcal{T}(\Sigma)}(s,t) = 2^{-k}$ where $k = \min(\{p \in (\mathrm{Pos}(s) \cup \mathrm{Pos}(t)) \mid s(p) \neq t(p)\})$ (k is the least length of the positions p where $s(p) \neq t(p)$)

We get:

Lemma 4.1.13. *$d_{\mathcal{T}(\Sigma)}$ is an ultrametric.*

Proof. Let Σ be an arbitrary signature. We need to prove three properties for $d_{\mathcal{T}(\Sigma)}$ to be an ultrametric on $\mathcal{T}(\Sigma)$:

- REFLEXIVITY: holds by the first clause of the definition of $d_{\mathcal{T}(\Sigma)}$.
- IDENTITY OF INDISCERNIBLES: holds because positions are of finite length and hence 2^{-k} where k is the least length of the positions where two terms differ is positive.
- SYMMETRY: holds by the symmetry of $=$ and $\neq$ in assumptions of the two clauses of the definition of $d_{\mathcal{T}(\Sigma)}$.
- STRONG TRIANGLE INEQUALITY: Let $t, s, u \in \mathcal{T}(\Sigma)$. Let $k_{ts} \in (\mathrm{Pos}(t) \cap \mathrm{Pos}(s))$ be the position of least length such that $t(k_{ts}) \neq s(k_{ts})$ and let $k_{su} \in (\mathrm{Pos}(s) \cap \mathrm{Pos}(u))$ be the least position of least length such that $s(k_{su}) \neq u(k_{su})$. For any position p of length $n < \min(k_{ts}, k_{su})$ we have $t(p) = s(p) = u(p)$, so if $k_{tu} \in (\mathrm{Pos}(t) \cap \mathrm{Pos}(u))$ is the position of least

length such that $t(k_{tu}) \neq u(k_{tu})$ we have $k_{tu} \geq \min(k_{ts}, k_{su})$. This means that $2^{-k_{tu}} \leq \max(2^{-k_{ts}}, 2^{-k_{su}})$ and hence

$$d_{\mathcal{T}(\Sigma)}(t,u) \leq \max(d_{\mathcal{T}(\Sigma)}(t,s), d_{\mathcal{T}(\Sigma)}(s,u)) \qquad \square$$

This metric can also directly be used on non-ground terms (containing variables). This corresponds to the view that variables in terms are just symbols. So the terms $f(x)$ and $f(y)$ differ at position 1 and $d(f(x), f(y)) = \frac{1}{2}$, where $f(x)$ and $f(x)$ do not differ, so $d(f(x), f(y)) = 0$. This seems to be the view adopted in [18, p. 670] (although terms containing variables are never explicitly mentioned). Another view would be that a term containing variables models the set of all ground instances (Definition 4.2.1) of that term. The distance between two such sets of ground instances can be given by means of the Hausdorff metric (Definition A.3.6), d_H. Since d is an ultrametric d_H is too (Definition A.3.8). We might then simply define the distance between two terms to be the Hausdorff distance between their sets of ground instances, lets call the distance function that arises from this d_h. Since $f(x)$ and $f(y)$ have the same ground instances we would get $d(f(x), f(y)) = 0$ making d_h a pseudometric instead of a metric.

Both constructions can directly be used on infinite positional terms, doing so we get $d^{\infty}_{\mathcal{T}(\Sigma)} : \mathcal{T}^{\omega}(\Sigma) \times \mathcal{T}^{\omega}(\Sigma) \to \mathbb{R}^{+}$.

This also gives us a third way to construct infinite terms ([18, p. 670]). We can use metric completion (Definition A.3.26) on the metric space $(\mathcal{T}(\Sigma), d_{\mathcal{T}(\Sigma)})$. There is an embedding of $(\mathcal{T}(\Sigma), d_{\mathcal{T}(\Sigma)})$ itself into this completion (Proposition A.3.30) and this completion is isometric to $(\mathcal{T}^{\omega}(\Sigma), d^{\infty}_{\mathcal{T}(\Sigma)})$.

Theorem 4.1.14. *The metric completion of* $(\mathcal{T}(\Sigma), d_{\mathcal{T}(\Sigma)})$ *is isometric to* $(\mathcal{T}^{\omega}(\Sigma), d^{\infty}_{\mathcal{T}(\Sigma)})$.

Proof. [18, p. 671] $\square$

If we denote our embedding of finite terms in the metric completion with $\sqsubset$ and the isometry between metrically completed terms and infinite terms with $\approx$ we get:

coinductive terms	$=$	infinite positional terms	$\approx$	metrically completed terms
$\cup$		$\cup$	$\sqsubset$	
inductive terms	$=$	positional terms		

4.1.6 Exotic Terms

After we have generalized terms to include terms with infinite depth (to be precise terms with a depth of ω), we can consider other generalizations. First of all, [17] proposes to allow for symbols with arbitrary ordinal arity. This can be viewed as another way to allow for infinite terms, modeling infinite data.

Instead of having infinite depth, terms may now have infinite width. In [17] such terms are defined using an inductive approach. It can also be done using a positional approach though. First we redefine signatures allowing for arbitrary ordinal arity.

Definition 4.1.15. A **wide signature**, Σ, is a set of symbols which come equipped with an *arbitrary* ordinal number called **arity**.

And then restating our definition for finite positional terms using wide signatures almost does the job. We define:

Definition 4.1.16. For any *wide* signature Σ containing at least one symbol, a **wide ground term** over Σ is a partial function, $t : \bigcup_{i<\alpha} On^i \to \Sigma$, where:

- On^i is the set of strings of length i over On
- $0 < \alpha < \omega$ FINITE DEPTH
- For any $m < \alpha$ there is a $p \in \mathbb{N}^m$ with $t(p) \in \Sigma$ DEPTH WELL-DEFINED
- $t(p; i) \in \Sigma$ if and only if $t(p) \in \Sigma_k$ and $1 \leq i \leq k$ CONNECTEDNESS

The set of wide ground terms is denoted by $\mathcal{T}_w(\Sigma)$. The set $\mathcal{T}_w(\Sigma, \mathcal{X}_\Sigma)$ of **wide terms** over Σ is $\mathcal{T}_w(\Sigma \cup \mathcal{X}_\Sigma)$.

This again can be generalized to terms with infinite depth:

Definition 4.1.17. For any non-empty *wide* signature, Σ, a **wide infinite ground term** over Σ is a partial function, $t : \bigcup_{i<\alpha} On^i \to \Sigma$, where:

- On^i is the set of strings of length i over On
- $0 < \alpha \leq \omega$ INFINITE DEPTH
- For any $m < \alpha$ there is a $p \in \mathbb{N}^m$ with $t(p) \in \Sigma$ DEPTH WELL-DEFINED
- $t(p; i) \in \Sigma$ if and only if $t(p) \in \Sigma_k$ and $1 \leq i \leq k$ CONNECTEDNESS

The set of wide ground terms is denoted by $\mathcal{T}_w^\omega(\Sigma)$. The set $\mathcal{T}_w^\omega(\Sigma, \mathcal{X}_\Sigma)$ of **wide infinite terms** over Σ is $\mathcal{T}_w^\omega(\Sigma \cup \mathcal{X}_\Sigma)$.

These definitions follow [17] in allowing symbols to have an arbitrary ordinal arity. Another choice could have to allow only for arities $\leq \omega$.

A useful feature of these terms having both infinite depth and transfinite width is that unlike 'regular' infinite terms (only having infinite depth), they are closed under pattern collapsing. That is, for a term containing variables, such as $f(f(f(x)))$, we might want to collapse the part not containing any variables, the pattern $f(f(f(\Box)))$, into a single symbol, ρ, and get $\rho(x)$. We can also do this with patterns containing multiple variables. For instance in $g(f(x), y)$ we can collapse the pattern $g(f(\Box), \Box)$ into ρ and get $\rho(x, y)$. Terms with finite depth are closed under this pattern collapsing (collapsing any pattern in a finite term yields a finite term). However, infinite terms (terms of infinite depth) might contain infinitely many variables. Collapsing the non-variable part of such a term does not yield a term of finite width, but a term of infinite width. For instance, collapsing the pattern $g(\Box, g(\Box, g(\Box, \ldots)))$ in the term $g(x_1, g(x_2, g(x_3, \ldots)))$ would yield $\rho(x_1, x_2, x_3, \ldots)$, a term with a width of ω. So terms with infinite depth and finite width are not closed under pattern collapsing, while terms with infinite depth and transfinite width are.

Another generalization we can make, this time to infinite positional terms, is to allow for terms with depths greater than ω. A problem here would be that positions are formalized by strings. We want transfinite positions, so we need transfinite strings. Instead of formalizing those though, it can be noted that instead of using strings to formalize positions, sequences can be used. For our purposes the two structures are the same. In the preceding text, strings were used only to conform to most the literature (for instance [9, p. 3]). A notion of positions based on sequences is identical to the one based on strings, is sometimes adopted ([17]) in the literature and is unproblematic when generalizing into the transfinite.

Definition 4.1.18. For any non-empty signature Σ, a **transfinite ground term** over Σ is a partial function, $t : \bigcup_{\beta<\alpha} \mathbb{N}^\beta \to \Sigma$, where:

- $\mathbb{N}^\beta$ is the set of possibly transfinite sequences of length β over $\mathbb{N}$
- $0 < \alpha \in On$ — TRANSFINITE DEPTH
- For any $\beta < \alpha$ there is a $p \in \mathbb{N}^\beta$ with $t(p) \in \Sigma$ — DEPTH WELL-DEFINED
- $t(p; i) \in \Sigma$ if and only if $t(p) \in \Sigma_k$ and $1 \leq i \leq k$ — CONNECTEDNESS
- If p is a position with $|p| \in \text{LimOrd}$ then $t(p)$ is defined if and only if for every strict prefix q of p we have that $t(q)$ is defined — LIMIT POSITIONS

The set of transfinite ground terms is denoted by $\mathcal{T}^\infty(\Sigma)$. The set $\mathcal{T}^\infty(\Sigma, \mathcal{X}_\Sigma)$ of **transfinite terms** over Σ is $\mathcal{T}^\infty(\Sigma \cup \mathcal{X}_\Sigma)$.

Remark. These appear to be the same transfinite terms as defined in [14]. I first devised transfinite terms as a quick example of what can be done when embedding exotic concrete rewriting systems in transfinite abstract rewriting systems. I was unaware of any existing theory related to them. While finishing this thesis, the "Liber Amicorum", a collection of writings dedicated to Roel de Vrijer, was handed to me by my supervisor, containing [14]. As can be read there and in Section 5.2.7, it turns out that transfinite terms aren't exactly suited as quick example, since their theory is riddled with problems.

A problem with transfinite terms is that the computational significance of such transfinite terms is not so obvious. However in the field of proof terms they might be useful, because proof terms for the transfinite reductions that will be defined in Chapter 5 will require terms of transfinite depth.

4.1.7 Partial Terms

When we view terms as structured data, there, in some contexts (like the Partial Rewriting Systems of Section 5.4) is a need for terms representing data that is partly undefined or unknown. This can be done using terms that are only partially defined: partial terms.

Definition 4.1.19. If Σ is a signature containing at least one constant then the set of **partial terms** over that signature is the set of terms over the signature $\Sigma \uplus \{\bot\}$. We can denote this set by $\mathcal{T}(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$.

Positions of a term with $\bot$ at it represent the positions where the term is undefined. We might define the term at these positions using replacements.

Definition 4.1.20. A **replacement**, σ, on a partial term $t \in \mathcal{T}(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ is a mapping from $\{p \in \mathrm{Pos}(t) \mid t(p) = \bot\}$ to $\mathcal{T}(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$. We define the result of the application of a replacement σ on a term $t \in \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$, denoted by t^σ, to be the term with depth $\max(\{|p| + \mathrm{dpt}(\sigma(p)) \mid p \in \mathrm{Pos}(t) \wedge t(p) = \bot\} \cup \mathrm{dpt}(t))$ such that:

- $t^\sigma(p) = t(p)$ if $t(p)$ is defined and $t(p) \neq \bot$
- $t^\sigma(p;p') = (\sigma(p))p'$ if $t(p) = \bot$ and $p' \in \mathrm{Pos}(\sigma(p))$
- $t^\sigma(p)$ is undefined otherwise

This means that the set of positions of a term is a subset of the set of positions of the term after applying a replacement. The result of applying a replacement on a term is the same as the original term on all positions not having $\bot$ at it, but possibly different on positions with $\bot$. If we have a replacement σ on a term t, then the result of applying the replacement on a subterm, $t|_p$, of t is the subterm of t^σ at position p: $t^\sigma|_p$.

Using replacements we can define the following order on partial terms:

Definition 4.1.21. If Σ is a signature then $\leq_\bot$ is the order on $\mathcal{T}(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ such that for any $t, s \in \mathcal{T}(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ we have $t \leq_\bot s$ if there is some replacement σ such that $t^\sigma = s$.

Remark. In the literature this is usually stated as something like "we have $t \leq_\bot s$ if s can be obtained from t by replacing occurrences of $\bot$ in it by new subterms" or vice versa "t can be obtained from s by replacing subterms in it by $\bot$". There, it is left implicit what the operation of replacing formally looks like. The definition of replacements above is so involved because I did try to formalize this. This might make the intuition of what is going on less clear (hence this remark for clarification), but should help in providing a high level of formalization like is used in the proves below.

This order expresses that some terms are 'more defined versions' of other terms. $s \leq_\bot t$ is to be interpreted as "t is equal to s in all places where s is defined but t might be defined in places where s isn't. Hence: t is a more defined version of s.

We can positionally characterize a term that is above another term in this order in terms of the other term:

Lemma 4.1.22. *We have $t \leq_\perp t'$ if and only if for any $p \in \mathrm{Pos}(t')$ we have $t'(p) = t(p)$ or there is some non-strict prefix q of p such that $t(q) = \perp$.*

Proof. Let $t, t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$

$\Rightarrow$ Assume that $t \leq_\perp t'$, we get a replacement σ such that $t^\sigma = t'$. Let $p \in \mathrm{Pos}(t')$. If $t(p) \neq t'(p)$ we must have some non-strict prefix q of p such that $t(q) = \perp$, because $t^\sigma = t'$ and replacements only change terms at positions of which there is a non-strict prefix with $\perp$ at it.

$\Leftarrow$ Assume that for any $p \in \mathrm{Pos}(t')$ we have $t'(p) = t(p)$ or there is some non-strict prefix q of p such that $t(q) = \perp$. Let σ be the replacement on t such that for any $p \in \mathrm{Pos}(t)$ such that $t(p) = \perp$ we have $\sigma(p) = t'|_p$. Now we have $t^\sigma = t'$ and hence $t \leq_\perp t'$. □

We can also positionally characterize a term below another term in terms of the other term:

Lemma 4.1.23. *We have $t \leq_\perp t'$ if and only if for any $p \in \mathrm{Pos}(t)$ we have $t(p) = t'(p)$ or $t(p) = \perp$.*

Proof. Let $t, t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$

$\Rightarrow$ Assume that $t \leq_\perp t'$, we get a replacement σ such that $t^\sigma = t'$. Let $p \in \mathrm{Pos}(t)$. If $t(p) \neq \perp$ then we get $t(p) = t'(p)$ because replacements do nothing on positions with a symbol other than $\perp$ at it.

$\Leftarrow$ Assume that for any $p \in \mathrm{Pos}(t)$ we have $t(p) = t'(p)$ or $t(p) = \perp$. Let σ be the replacement on t such that for any $p \in \mathrm{Pos}(t)$ such that $t(p) = \perp$ we have $\sigma(p) = t'|_p$. Now we have $t^\sigma = t'$ and hence $t \leq_\perp t'$. □

We get the following:

Theorem 4.1.24. *$(\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ is a partially ordered set.*

Proof. Let Σ be a signature.

- $\leq_\perp$ is reflexive.

  Let $t \in \mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$, we have that $t^\sigma = t$, where σ is the identity replacement; the replacement such that for any $p \in \mathrm{Pos}(t)$ with $t(p) = \perp$ we have $\sigma(p) = \perp$.

- $\leq_\perp$ is transitive.

  Let $t, s, r \in \mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$. Assume that we have that $t \leq_\perp s$ and $s \leq_\perp r$, we get replacements σ and τ such that $t^\sigma = s$ and $s^\tau = r$. Now we get the replacement υ that is such that for all $p \in \mathrm{Pos}(t)$ with $t(p) = \perp$ we have that $\upsilon(p) = \sigma(p)^\tau$, which is well-defined since $\sigma(p)$ is a subterm of r and hence we can apply τ to it. We get $t^\upsilon = r$.

- $\leq_\perp$ is antisymmetric.

  Let $t, s \in \mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$. Assume we have that $t \leq_\perp s$ and $s \leq_\perp t$. We get replacements σ and τ such that $t^\sigma = s$ and $s^\tau = t$. Suppose for contradiction that we have $t \neq s$. We get a non-empty set of positions where t and s are both defined but differ. Since the set of positions is well-founded under the prefix ordering we get a minimal position, $p \in (\text{Pos}(t) \cap \text{Pos}(s))$, such that $t(p) \neq s(p)$ but both $t(p)$ and $s(p)$ are defined. Since $t^\sigma = s$ we get $t(p) \neq s(p) = t^\sigma(p)$, so we must have $t(p) = \perp$ and since $s^\sigma = t$ we get $s(p) \neq t(p) = s^\sigma(p)$, so we must have $s(p) = \perp$. Now we get so we get $s(p) = \perp = t(p)$, so $s(p) = s(p)$ after all, contradiction. So we must have $t = s$. □

We can easily define partial infinite terms by generalizing:

Definition 4.1.25. If Σ is a non-empty signature then the set of **partial infinite terms** over that signature is the set of infinite terms over the signature $\Sigma \uplus \{\perp\}$. We can denote this set by $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$.

We get that every partial term is an partial infinite term. Replacements can be defined on partial infinite terms exactly as they are defined on partial terms (replacing $\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ by $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$) and the same goes for the order. This order on partial infinite terms is still denoted by $\leq_\perp$ and is superset (when viewed as a relation) of the order on partial terms. Theorem 4.1.24 also holds for partial infinite terms because it does not rely on the length of positions being bounded in $\mathbb{N}$. For $(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ we also get:

Theorem 4.1.26. *$(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ is a dcpo.*

Proof. $(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ is a partially ordered set by Theorem 4.1.24. Let $X \subseteq \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ be a directed set. As least upper bound we have the term, t^*, of depth $\bigsqcup(\{\text{dpt}(t) \mid t \in X\})$ such that:

- $t^*(p) = t(p)$ if there is a $t \in X$ is such that $t(p) \in \Sigma$
- $t^*(p) = \perp$ if there is no $t \in X$ such that we have $t(p) \in \Sigma$, but there is some $t \in X$ such that $t(p)$ is defined (we get $t(p) = \perp$)
- $t^*(p)$ is undefined if there is no $t \in X$ such that $t(p)$ is defined

We need to prove that this t^* is well-defined and that $t = \bigsqcup X$.

Let $t, t' \in X$ and let $p \in \text{Pos}(t)$ be such that $t(p) \in \Sigma$. By directedness of X, we have a $s \in X$ such that $t \leq_\perp s$ and $t' \leq_\perp s$, that is, there are replacements σ and τ such that $t^\sigma = s$ and $t'^\tau = s$. By Lemma 4.1.22 we get that $s(p) = t(p)$ and by Lemma 4.1.23 we get that $t'(p) = s(p) = t(p)$ or there is some non-strict prefix, q, of p such that $t'(q) = \perp$, which means that if $q = p$ we have $t'(p) = \perp$ and otherwise $t(p)$ is undefined. This all means that our first case is well-defined, that is, every choice of t gives the same symbol.

Furthermore for any position $p \in \text{Pos}(t^*)$ we have either some $t \in X$ such that $t(p) \in \Sigma$, we don't have such a t but we do have a t where $t(p)$ is defined

($t(p) = \bot$), or $t(p)$ is defined for no $t \in X$, so our definition is exhaustive. It also results is proper terms, because any $t \in X$ is a proper term.

Now let $t \in X$, for any $p \in \mathrm{Pos}(t)$ we have that $t(p) = t^*(p)$ or $t(p) = \bot$, by Lemma 4.1.23 we get that $t \leq_\bot t^*$. This means that t^* is an upper bound for X.

Let t^{**} be another upper bound for X and let $p \in \mathrm{Pos}(t^*)$. By construction of t^* we get some $t \in X$ such that $t(p) = t^*(p)$. We have $t \leq_\bot t^{**}$, so, by Lemma 4.1.23, either $t(p) = \bot$, which means that $t^* = \bot$, or $t(p) = t^{**}(p)$, which means that $t^{**}(p) = t^*(p)$. Since this holds for all $p \in \mathrm{Pos}(t^*)$, by Lemma 4.1.23 we get that $t^* \leq t^{**}$, so $t^* = \bigsqcup X$. □

Theorem 4.1.27. $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$ *is bounded complete.*

Proof. Let $X \subseteq \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ and let U be the set of upper bounds of X.As least upper bound of X we have the term t^* of depth $\bigsqcap(\{\mathrm{dpt}(u) \mid u \in U\})$ such that:

- $t^*(p) = c$ if there is some $c \in \Sigma$ such that for all $u \in U$ we have $u(p) = c$
- $t^*(p) = \bot$ if there is some $u \in U$ such that $u(p) = \bot$
- $t^*(p)$ is undefined otherwise

We need to prove that this t^* is well-defined.

t^* is well-defined by Definition 4.1.7, we only need to check if we can have 'gaps' in it: positions p such that $t^*(p)$ is undefined but for which there is a position that p is a prefix of such that t^* is defined at it. To show that we can't have such gaps in t^*: gaps might occur if $u, u' \in U$ such that there is some $p \in (\mathrm{Pos}(u) \cap \mathrm{Pos}(u'))$ with $u(p), u'(p) \in \Sigma$, but $u(p) \neq u'(p)$. Then $t^*(p)$ is undefined but maybe t^* might be defined at some position that p is a prefix of? Well, we then get the term u^* such that:

- $u^*(p) = u(p) = u'(p)$ if $u(q) = u'(q)$ for all non-strict prefixes q of p.
- $u^*(p) = \bot$ if $u(p), u'(p) \in \Sigma$, but $u(p) \neq u'(p)$
- $u^*(p)$ is undefined otherwise.

u^* is also an upper bound of X since both u and u' are. This construction shows that if $u, u' \in U$ differ at some position $p \in (\mathrm{Pos}(u) \cap \mathrm{Pos}(u'))$ we also must have an upper bound $u^* \in U$ with $u^*(p) = \bot$ and for all q such that p is a strict prefix of q, $u^*(q)$ is undefined. So the same goes for t^* and we can't have any gaps in t'.

To prove that t^* is an upper bound of X: let $t \in X$ and let $p \in \mathrm{Pos}(t)$. If $t(p) \in \Sigma$, then, by Lemma 4.1.23 we get that for all $u \in U$ we have $t(p) = u(p)$, by the construction of t^* that means that $t^*(p) = t(p)$. By Lemma 4.1.23 that means that $t \leq t^*$.

To prove that t^* is a least upper bound of X: let $u \in U$ and let $p \in \mathrm{Pos}(t^*)$, by construction we have that either $t^*(p) = u(p)$ or $t^*(p) = \bot$, so by Lemma 4.1.23 we get that $t^* \leq_\bot u$. □

Lemma 4.1.28. *A term is compact in* $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$ *if and only if it is of finite depth.*

Proof. Let $t \in \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$.

- $\Rightarrow$ Assume that t is compact in $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$.

  Suppose for contradiction that t is of non-finite depth, that is, $\mathrm{dpt}(t) = \omega$.

  Let $f : \mathbb{N} \to \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ be a function that maps any $n \in \mathbb{N}$ to the term $f(n)$ with $\mathrm{dpt}(f(n)) = n$ such that:

  - $(f(n))(p) = t(p)$ if $|p| < n - 1$ and $t(p)$ is defined
  - $(f(n))(p) = \bot$ if $|p| = n - 1$ and $t(p)$ is defined
  - $(f(n))(p)$ is undefined otherwise

  Let $n_1, n_2 \in \mathbb{N}$ be such that $n_1 \leq n_2$ and let $p \in \mathrm{Pos}(f(n_1))$. If $|p| < n_1 - 1$ then $(f(n_1))(p) = (f(n_2))(p)$ and if $|p| = n_1$ than $f(n_1) = \bot$. That means that, by Lemma 4.1.23 we get that $f(n_1) \leq_\bot f(n_2)$. This means that $\{f(n) \mid n \in \mathbb{N}\}$ is a directed set.

  We have $t \leq_\bot t$ by reflexivity of our order. Let $n \in \mathbb{N}$ and $p \in \mathrm{Pos}(f(n))$. if $|p| < n - 1$ then $(f(n))(p) = t(p)$ and if $|p| = n - 1$ then $f(n)(p) = \bot$. So by Lemma 4.1.23 we get that $f(n) \leq_\bot t$ and because $t \neq f(n)$ we get $t \not\leq_\bot f(n)$ by antisymmetry of our order. That means that we get $t \not\ll t$, so t is not compact. Contradiction. So t must be of finite depth after all.

- $\Leftarrow$ Assume that $\mathrm{dpt}(t) \in \mathbb{N}$.

  Let $X \subseteq \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ be a directed set, assume that $t \leq \bigsqcup X$. Let $f : \{p \in \mathrm{Pos}(t) \mid t(p) \neq \bot\} \to X$ be a function that maps any $p \in \mathrm{Pos}(t)$ such that $t(p) \neq \bot$ to some arbitrarily chosen $f(p) \in X$ such that $(f(p))(p) = (\bigsqcup X)(p)$. We must have some such $f(p) \in X$ by construction of $\bigsqcup X$ in Theorem 4.1.26. Since t is of finite depth, $\{p \in \mathrm{Pos}(t) \mid t(p) \neq \bot\}$ is of finite size and, by directedness of X, we get an upper bound of $\{f(p) \in X \mid p \in \mathrm{Pos}(t) \wedge t(p) \neq \bot\}$ in X, denote it by u. For any $p \in \mathrm{Pos}(t)$ such that $t(p) \neq \bot$ we get $t(p) = (f(p))(p)$ by construction of f and $(f(p))(p) = u(p)$ by Lemma 4.1.23 (since $f(p) \leq_\bot u$). This means that $t(p) = u(p)$, so by Lemma 4.1.23 we get $t \leq_\bot u$. So $t \ll t$ and t is compact. □

This lemma justifies calling compact elements of $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$ finite elements.

Theorem 4.1.29. $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$ *is algebraic.*

Proof. Let $t \in \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$, let $C \subseteq \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ be the set of all compact terms below t.

- C is directed.

  Let $c, c' \in C$. As upper bound for c and c' we get the term c^* of depth $\max(\mathrm{dpt}(c), \mathrm{dpt}(c'))$ such that:

- $c^*(p) = \bot$ if $|p| = \max(\mathrm{dpt}(c), \mathrm{dpt}(c')) - 1$ and there is some q such that $q \neq \epsilon$ and $t(p;q)$ is defined.
- $c^*(p) = t(p)$ otherwise

We have $c^* \leq_\bot t$ by Lemma 4.1.23. The depths of c and c' are finite (Lemma 4.1.28), so we get $\max(\mathrm{dpt}(c), \mathrm{dpt}(c')) = \mathrm{dpt}(c^*)$ is finite, so c^* is compact by Lemma 4.1.28. So $c^* \in C$.

We have $c, c' \leq_\bot c^*$ by Lemma 4.1.23 and the fact that $c, c' \leq_\bot t$, so c^* is indeed an upper bound for c and c'.

- $\bigsqcup C = t$.

  t is an upper bound of C by definition of C. Let t' be some other upper bound of C.

  Similarly to in Lemma 4.1.28 ($\Rightarrow$), let $f : (1 + \mathrm{dpt}(t)) \to \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ be a function that maps any $n < (1 + \mathrm{dpt}(t))$ to the term $f(n)$ with $\mathrm{dpt}(f(n)) = n$ such that:

  - $(f(n))(p) = t(p)$ if $|p| < n - 1$ and $t(p)$ is defined
  - $(f(n))(p) = \bot$ if $|p| = n - 1$ and $t(p)$ is defined
  - $(f(n))(p)$ is undefined otherwise

  For any $n \leq \mathrm{dpt}(t)$ we have that $f(n)$ is compact because its depth is n and $n < 1 + \mathrm{dpt}(t) = 1 + \omega = \omega$ hence n is finite by Lemma 4.1.28. We also have $f(n) \leq_\bot t$ by construction, so we have $f(n) \in C$. We also get $f(n) \leq t'$ because t' is an upper bound of C.

  Now let $p \in \mathrm{Pos}(t)$, we get that $t(p) = (f(|p| + 1))(p)$ by construction. We also have $p \in \mathrm{Pos}(f(|p| + 1))$ and get $(f((|p| + 1)))(p) = t'(q)$ or $(f((|p| + 1)))(q) = \bot$ because $f((|p| + 1)) \leq t'$ by Lemma 4.1.23. So we have $t(p) = (f((|p| + 1)))(p) = t'$ or $t(p) = (f((|p| + 1)))(p) = \bot$ and that means that by Lemma 4.1.23 we get $t \leq_\bot t'$. □

The maximal elements of any partially ordered set, $(A, \leq)$ are elements $m \in A$ such that for all $x \in A$ we have $m \leq x \to m = x$. In $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$, the maximal terms are the non-partial terms, terms that don't have $\bot$ at any position.

Theorem 4.1.30. *$t \in \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ is maximal in $(\mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot}), \leq_\bot)$ if and only if $t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$.*

Proof. Let $t \in \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$.

$\Rightarrow$ Assume that $t \in \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ is maximal and suppose for contradiction that $t \notin \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$, there must be some $p \in \mathrm{Pos}(t)$ such that $t(p) = \bot$. We can construct the term t' with $\mathrm{dpt}(t') = \mathrm{dpt}(t)$ such that:

- $t'(q) = x$ where $x \in \mathcal{X}_{\Sigma_\bot}$ if $q = p$

– $t'(q) = t(q)$ otherwise

We get that for any $p \in \text{Pos}(t)$ we either have $t(p) = t'(p)$ or $t(p) = \bot$, so we get $t \leq_\bot t'$ by Lemma 4.1.23. We also have $t \neq t'$ by construction, so t' is a witness to the fact that t is not maximal after all. Contradiction. We can conclude that $t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$.

$\Leftarrow$ Assume that $t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$, for all $p \in \text{Pos}(t)$ we get $t(p) \neq \bot$. Let $t' \in \mathcal{T}^\omega(\Sigma_\bot, \mathcal{X}_{\Sigma_\bot})$ be such that $t \leq_\bot t'$. By assumption and Lemma 4.1.23 we get that for any $p \in \text{Pos}(t)$ we have $t(p) = t'(p)$. By assumption and Lemma 4.1.22 we get that for any $p \in \text{Pos}(t')$ we have $t(p) = t'(p)$. So $t = t'$ and hence t is maximal. □

4.2 Rewriting Terms

Now that we have various formalizations of terms, we can formalize transformations on those terms. First we need:

Definition 4.2.1. For any signature Σ a **substitution** is a function, $\sigma : \mathcal{X}_\Sigma \to \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$. We define the result of the application of such a substitution on a term, $t \in \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$, denoted by t^σ, to be the term such that:

- $\text{dpt}(t^\sigma) = \max(\{|p| + \text{dpt}(\sigma(t(p))) \mid t(p) \in \mathcal{X}_\Sigma\} \cup \{\text{dpt}(t)\})$
- $t^\sigma(p) = t(p)$ if $t(p) \in \Sigma$
- $t^\sigma(p; p') = (\sigma(t(p)))(p')$ if $t(p) \in \mathcal{X}_\Sigma$ and $p' \in \text{Pos}(\sigma(t(p)))$
- $t^\sigma(p)$ is undefined otherwise

A term $s \in \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ is an **instance** of a term $t \in \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ if there is a substitution, σ, such that $s = t^\sigma$. When we restrict the codomain of a substitution to $\mathcal{T}(\Sigma)$, the substitution is said to be a **ground substitution**. For a term t and a ground substitution σ we have $t^\sigma \in \mathcal{T}(\Sigma)$. t^σ is said to be a ground instance of t.

So a substitution can intuitively be seen as substituting variables in terms for other terms.

Definition 4.2.2. A **reduction rule** for a set of terms, $\mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$, is a pair $\langle l, r\rangle$ such that $l, r \in \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ such that every variable occurring in r occurs in l ($\forall p \in \text{Pos}(r).\ (r(p) \in \mathcal{X}_\Sigma \to \exists q \in \text{Pos}(l)(l(q) = r(p))))$. For such a reduction rule we write $l \to r$. If we give the reduction rule a name, π, then we write $\pi : l \to r$.

Remark. In the literature, it is often also required that l is not a variable ($l(\epsilon) \notin \mathcal{X}_\Sigma$). A reason for this is that such a rule can be applied to any term and at any position and hence it would be trivial to create an infinite sequence of terms where each term is one-step reducible to the next term in the sequence (in a sense that will be defined below). From a finitary perspective this is bad, processes of reducing never end and there are no normal forms in the presence of such a rule. From an infinitary perspective however, always being able to apply some rule, having never-ending sequences of reduction and not having normal forms in the finitary sense is not really undesirable per se. From a computational perspective, not having such a demand even seems intuitive. One might easily conceive of a transformation that behaves in this way, for instance an operation that just adds a bit to some data structure.

Reduction rules define what transformations are possible between terms in the following way:

Definition 4.2.3. If $s \in \mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$, $p \in \mathrm{Pos}(s)$, $\pi : l \to r$ is a reduction rule for $\mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ and σ is a substitution such that $s|_p = l^\sigma$ then we say that s is **one-step reducible** to a term t by applying π at position p, where t is such that:

- $t(p; q) = r^\sigma(q)$ for all $q \in \mathrm{Pos}(r^\sigma)$
- $t(q) = s(q)$ for all $q \in \mathrm{Pos}(s)$ such that p is not a prefix of q

We write $s \to t$ or $s \to_{\pi,p} t$. The one-step reduction relation generated by π, $\to_\pi$, thus is the set $\{\langle s, t\rangle \mid \exists p \in \mathrm{Pos}(s).\ s \to_{\pi,p} t\}$.

Here, given a term s, a position p and reduction rule π, t is unique. Our choice of substitution σ doesn't influence how t will look, there just needs to be such a substitution. This is because every variable x in r also occurs in l. That means applying a reduction rule at a position in a term yields a unique new term.

Where terms model objects that might be transformed, one-step reducibility models which transformations are possible between these objects. From a computational perspective, terms might be viewed as modeling data structures. Reduction rules might be viewed as schemes for the one-step computations that are possible on these data structures. While one-step reducibility might be viewed as the instances of those schemes; one-step computations. Now:

Definition 4.2.4. A **term rewriting system** (TRS) is a structure $\mathcal{R} = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$. $\mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ is the set of terms of $\mathcal{R}$. The **one step reduction relation**, $\to$ or $\to_\mathcal{R}$, generated by $\mathcal{R}$ is $\bigcup\{\to_\pi \mid \pi \in R\}$.

We can directly generalize the definitions of substitution, reduction rules and one-step reducibility to infinitary, wide, wide infinitary, transfinite, partial and partial infinitary settings by replacing $\mathcal{T}(\Sigma, \mathcal{X}_\Sigma)$ by respectively $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$, $\mathcal{T}_w(\Sigma, \mathcal{X}_\Sigma)$, $\mathcal{T}_w^\omega(\Sigma, \mathcal{X}_\Sigma)$, $\mathcal{T}^\infty(\Sigma, \mathcal{X}_\Sigma)$, $\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ and $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ and get:

Definition 4.2.5. An **infinitary term rewriting system** (iTRS) is a structure $\mathcal{R}^\omega = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$. $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ is the set of terms of $\mathcal{R}^\omega$.

Definition 4.2.6. A **wide term rewriting system** (wide TRS) is a structure $\mathcal{R}_w = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}_w(\Sigma, \mathcal{X}_\Sigma)$. $\mathcal{T}_w(\Sigma, \mathcal{X}_\Sigma)$ is the set of terms of $\mathcal{R}_w$.

Definition 4.2.7. A **wide infinitary term rewriting system** (wide iTRS) is a structure $\mathcal{R}_w^\omega = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}_w^\omega(\Sigma, \mathcal{X}_\Sigma)$. $\mathcal{T}_w^\omega(\Sigma, \mathcal{X}_\Sigma)$ is the set of terms of $\mathcal{R}_w^\omega$.

Definition 4.2.8. A **transfinite term rewriting system** (transfinite TRS) is a structure $\mathcal{R}^\infty = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}^\infty(\Sigma, \mathcal{X}_\Sigma)$. $\mathcal{T}^\infty(\Sigma, \mathcal{X}_\Sigma)$ is the set of terms of $\mathcal{R}^\infty$.

Definition 4.2.9. A **partial term rewriting system** (partial TRS) is a structure $\mathcal{R}_\perp = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$. $\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ is the set of terms of $\mathcal{R}_\perp$.

Definition 4.2.10. A **partial infinitary term rewriting system** (partial iTRS) is a structure $\mathcal{R}_\perp^\omega = (\Sigma, R)$ where Σ is a signature and R is a set of reduction rules over $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$. $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ is the set of terms of $\mathcal{R}_\perp^\omega$.

4.2.1 Embedding TRSs in Rewrite Systems

As mentioned in the introduction, abstract rewriting should be an abstraction over concrete rewriting systems like TRSs, and as such provides a basis for them. To show that this is the case, TRSs can be embedded in the rewrite system formalism in the following way:

Definition 4.2.11. The rewrite system **induced** by a TRS (Σ, R) is defined as $(\Phi, \mathcal{T}(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$ where:

- $\Phi = \{\langle s, t, \pi, p\rangle \mid s \to_{\pi,p} t \text{ in } (\Sigma, R)\}$
- For all $\langle s, t, \pi, p\rangle \in \Phi$ we have $\mathrm{src}(\langle s, t, \pi, p\rangle) = s$
- For all $\langle s, t, \pi, p\rangle \in \Phi$ we have $\mathrm{tgt}(\langle s, t, \pi, p\rangle) = t$

For convenience's sake we can still write a step $\langle s, t, \pi, p\rangle \in \Phi$ as $s \to_{\pi,p} t$. For a step $\langle s, t, \pi, p\rangle$ we have that π is the **rule** that is instantiated by the step, we can write $\mathrm{rul}(\langle s, t, \pi, p\rangle)$ for π. p is the **position** where the step takes place, we can write $\mathrm{pos}(\langle s, t, \pi, p\rangle)$ for p.

This way we get a notion of reduction in a TRS, a reduction in a TRS is a reduction in the induced rewrite system. For a reduction $\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle$ in the induced rewrite system we can write

$$\mathrm{src}(\phi_0) \to_{\mathrm{rul}(\phi_0),\mathrm{pos}(\phi_0)} \mathrm{src}(\phi_1) \to_{\mathrm{rul}(\phi_1),\mathrm{pos}(\phi_1)} \mathrm{src}(\phi_2) \to$$

An example of a rewrite system being induced by a TRS would be the following:

Example 4.2.12. Let (Σ, R) with $\Sigma = \{a/0, f/1\}$ and $R = \{\pi : x \to f(x)\}$ be a TRS. The induced rewrite system is $(\Phi, \mathcal{T}(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$ where $\Phi =$

$$\{a \to_{\pi,\epsilon} f(a), f(a) \to_{\pi,1} f(f(a)), f(a) \to_{\pi,\epsilon} f(f(a)), f(f(a)) \to_{\pi,11} f(f(f(a))),\\ f(f(a)) \to_{\pi,1} f(f(f(a))), f(f(a)) \to_{\pi,\epsilon} f(f(f(a))), \ldots\}$$

A maximal reduction we get in this rewrite system would be

$$a \to_{\pi,\epsilon} f(a) \to_{\pi,1} f(f(a)) \to_{\pi,11} f(f(f(a))) \to \ldots$$

Or, looking like a Christmas tree:

$$\begin{array}{c} a \\ \big\downarrow{\scriptstyle a \to_{\pi,\epsilon} f(a)} \\ f(a) \\ \big\downarrow{\scriptstyle f(a) \to_{\pi,1} f(f(a))} \\ f(f(a)) \\ \big\downarrow{\scriptstyle f(f(a)) \to_{\pi,11} f(f(f(a)))} \\ f(f(f(a))) \\ \big\downarrow{\scriptstyle f(f(f(a))) \to_{\pi,111} f(f(f(f(a))))} \\ \vdots \end{array}$$

For iTRSs we can also define that they induce rewrite systems in this exact way, the fact that terms might now have a depth of ω is of no consequence in an abstract framework (we abstract away from the structure of objects anyway). Continuing our example:

Example 4.2.13. We can view (Σ, R) with $\Sigma = \{a/0, f/1\}$ and $R = \{\pi : x \to f(x)\}$ as an iTRS. As induced rewrite system we get $(\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$ with $\Phi =$

$$\{a \to_\epsilon f(a), f(a) \to_1 f(f(a)), f(a) \to_\epsilon f(f(a)), f(f(a)) \to_{11} f(f(f(a))),\\ f(f(a)) \to_1 f(f(f(a))), f(f(a)) \to_\epsilon f(f(f(a))), \ldots, f^\omega \to_\epsilon f^\omega, f^\omega \to_1 f^\omega, \ldots\}$$

So both the set of objects and the set of steps get extra members. On the other hand though, we still get the maximal reduction

$$a \to_{\pi,\epsilon} f(a) \to_{\pi,1} f(f(a)) \to_{\pi,11} f(f(f(a))) \to \ldots$$

There is no step that connects the finite objects to f^ω as either source or target. To involve f^ω in our reductions that start with finite objects, and hence model that infinite processes of transformation can grow finite objects into infinite objects, we need to turn to transfinite rewriting systems. In the present setting of rewrite systems and (finite) reductions, embedding iTRSs isn't as interesting.

This same approach also should work for wide TRSs, wide iTRSs, transfinite TRSs and partial TRSs, since these only differ from regular TRSs in having more terms. Notions like position and substitution can still be defined on the, in the same way. However, as has been shown for iTRSs in Example 4.2.13, embedding these formalisms that model terms that are infinite in various senses in rewrite systems and having (finite) reductions on them isn't as interesting. The interesting things happen when we turn to transfinite rewriting.

Chapter 5

Transfinite Abstract Rewriting

In the section about abstract rewriting (Chapter 3) we only allowed for reductions and reduction sequences of lengths up to ω. We might be interested though in what happens *after that*. Formalisms that consider this question might be called infinitary. The word "infinity" comes from the Latin "infinitas" also meaning unboundedness and as such signifies all sorts of phenomena that are in some sense unbounded or non-finite. Up to now we have meant just that when talking about non-finite rewriting. In the introduction we considered infinite rewriting in just about any sense of the word infinite since we didn't yet have a formalization. infinite terms are infinite in the sense that their depth is not finite, not bounded in $\mathbb{N}$, but might also be ω. Only for transfinite terms we had depths of any ordinal number and hence we could not lump it under the broad term infinite. Here we can also be more specific, transfinite rewriting is not unbounded in just any sense, it allows for reductions and reduction sequences of any ordinal length, the ordinal numbers that are not in $\mathbb{N}$ are called transfinite. Hence "transfinite" as opposed to "infinite" is the better word when we consider what happens in reductions after ω .

The fact that we even consider the question "What happens after ω?" is for two reasons, reasons that were already hinted at in the introduction of this thesis.

First of all, we have the plain mathematical possibility of generalizing reductions and reduction sequences to arbitrary ordinal lengths. We use ordinals to define sequences, ordinals go into the transfinite and hence, so do sequences, so why not allow for reductions and reduction sequences of arbitrary ordinal length?

The second reason is a more practical and pragmatic one. We are modeling processes of stepwise transformation and hence processes like computation. The computational significance of a reduction or reduction sequence of a finite length (smaller than ω) is pretty clear: in a computation we transform/reduce/rewrite an object, step by step, until we are done and then we stop. There is also computational significance to reductions or reduction sequences of length ω. We transform an object step by step by step by... and just don't stop. We get an infinite computation to which there is nothing to add since there is no last element. There are lots of computations that work this way, infinite loops, loops with no termination condition being the prime example. But what we don't model, with our 'up-to-ω' formalisms of rewrite systems and rewrite relations is what happens in these infinite computations 'as we go on'. There are a lot of computations that work towards a limit and where the in-between results look more and more like the intended result but at no point actually get there. The sequence of in-between results approximates the intended result and the intended result is a limit of the sequence of in-between results just as ω is the limit of indexes in the sequence.

We can go back to the computation of the list of all primes that was mentioned in the introduction as an example. In some abstract sense, this list exist and we can start computing. We can express this computation as a series of steps, where each step adds a number to the list, the 'next' prime. The intended result is the list of all primes. This intended result is an infinite list, we will never actually produce it, but after each step, our intermediate result looks more and more like this intended result. A rewrite relation or rewrite system has no way of modeling this behavior, though, we can't see that the objects look more and more like another (limit) object since we are abstracting away from the actual structure of the objects. Also, we have no way to express that this limit is in some sense also involved in the reduction or reduction sequence. In a reduction sequence in a rewrite relation, we might want to add the limit to the sequence, but that would yield a sequence of length $\omega+1$, which is not permitted in our up-to-ω framework. In a reduction in a rewrite system we can't even do that since sequences of steps are our main construct, we need some other way to express that the limit object is in some sense part of the reduction. When we have done so however, we can again take a step with the limit object as source and get a sequence of steps of length $\omega + 1$. When we have reductions or reduction sequences of length $\omega + 1$ we might go further and take a next step. We now have a last element in our sequence of objects (the element at index ω) and our next step is transforming that last element and repeating this process a couple of times. This might model a process like "calculating the list of all primes and then printing the first few primes out". Continuing in this fashion, our sequence of objects might eventually tend length $\omega * 2$ and towards a new limit. This might model a process like "calculating the list of all primes and then printing them all out". Continuing this, we can get sequences of arbitrary transfinite length.

The computational significance of such processes longer than ω processes might be getting lost though. When we calculate the list of all primes, every prime gets produced at some time, but when we calculate a list of all primes and then print them out, no prime ever gets printed out. The computational meaning here might be given by the compression lemma [18, 689] which states that for certain systems (left-linear iTRSs) we can transform reduction sequences [1] of arbitrary transfinite length to reduction sequences of length ω. In this case we can transform the reduction sequence that models "calculating the list of all primes and then printing them all out" into a reduction sequences that models "calculating the first prime, printing it out, calculating the second prime, printing it out, etc." these reduction sequences theoretically tend to the same limit, a state where all primes have been calculated and printed out. But in the first reduction sequences some things 'never get done' while in the second 'everything gets done eventually'. This seems like a very useful lemma. When modeling infinite processes, it states that whenever you model badly, creating reduction sequences longer than ω, where somethings never gets done, you can do better and do the same thing with a reduction sequence of length ω, such that everything eventually gets done. However to express the compression lemma it is necessary to at least be able to talk about reductions of arbitrary transfinite length.

As mentioned, if we want to allow for taking limits of reduction sequences or sequences of objects of reductions in any of our two abstract rewriting formalisms then we should at least have some notion of structure on our set of objects. This structure might be provided in multiple ways, using metrics Appendix A.3, using orders Appendix A.4 and using topologies Appendix A.2. All these disciplines might provide such a notion of structure while still keeping our framework abstract. On any set of concrete objects we can define a topology, metric or order and then abstract away from the internal structure of the objects, just leaving the structure of the set as a whole expressed by the topology/metric/order. Up to now we have been taking limits of sequences of objects, expressing that the objects in the sequence start to look more and more like some limit objects. But, especially from a rewrite system point of view, we are also interested in how the sequence of steps behave as the sequence tends towards a limit. This can be done by also giving a notion of structure on the set of steps. Again, topologies, metrics and orders can be used. These are some of the issues that transfinite abstract rewriting is concerned with.

When formalizing transfinite abstract rewriting an important choice to be made is whether to continue in the fashion of rewrite relations, emphasizing the objects that are being transformed, or in the fashion of rewrite systems, emphasizing the steps that do the transforming. Based on the argumentation and comparison in Section 3.3.3 the later seems the better choice. Another reason for this is that we want to express the structure of the set of steps. For that we'd prefer to have those steps formalized in a concrete way instead of just as a relation on objects. So the systems formalizing transfinite abstract rewriting should generalize rewrite systems. Transfinite reductions should generalize 'regular' reductions. On the other hand, as TRSs can be embed in rewrite systems Section 4.2.1, we should be able to embed the infinite counterpart of TRSs, iTRSs, in our transfinite rewrite systems.

[1] The compression lemma is expressed in terms of rewrite relations, but can also be expressed in terms of rewrite systems

5.1 Transfinite Reductions

Giving a foundation for transfinite reductions however presents a problem. Notions of convergence, limit and continuity are essential to these reductions. Without them, the computational significance of such reductions is unclear. Its hard to give meaning to a transfinite reduction where the object at index ω of the sequence of objects is unrelated to the sequence of objects up to ω. Still, due to the fact that, these notions of convergence, limit and continuity can be given in various different ways, it is useful to first define a notion of transfinite rewriting not taking them into account and then expand this notion in various different ways using different notions of convergence, limit and continuity.

We define by generalizing the reductions that we already defined for finitary rewrite systems (Definition 3.2.3). We relax the axiom BOUNDED LENGTH and get:

Definition 5.1.1. A **transfinite reduction** in a rewrite system $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a tuple $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ such that:

- $a \in A$ and $\phi_\beta \in \Phi$ for all $\beta < \alpha$
- $\alpha \in On$ — UNBOUNDED LENGTH
- $\mathrm{src}(\phi_0) = a$ if $0 < \alpha$ — START
- $\mathrm{tgt}(\phi_\beta) = \mathrm{src}(\phi_{\beta\prime})$ — SUCCESSOR STEPS

We can define the same auxiliary notions as in Definition 3.2.3, only the notion of sequence of objects poses a problem:

Definition 5.1.2. The transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ is said to **start** in a, α is the **length** of the reduction, $\langle \phi_\beta \rangle_{\beta<\alpha}$ is the **sequence of steps** of the reduction and the **sequence of objects** of the reduction is either:

- $\langle a \rangle$, if $\alpha = 0$; or
- $\langle \mathrm{src}(\phi_\beta) \rangle_{\beta<\alpha}; \langle \mathrm{tgt}(\gamma) \rangle$, if $\alpha = \gamma\prime$; or
- $\langle \mathrm{src}(\phi_\beta) \rangle_{\beta<\alpha}$, if α is a limit ordinal.

We can denote this sequence of objects by $\mathrm{src}\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$. Its length is $\alpha^\dagger$, meaning that it is α if $\alpha \in \mathrm{LimOrd}$ and $\alpha + 1$ otherwise (Definition A.1.35).

Unlike for our 'regular' reductions of Definition 3.2.3, we can not simply take the targets of all steps to get all the objects involved in the transfinite reduction. This is because there isn't necessarily a step that has as target the object that a step with limit ordinal index has as source. We also want to include that object in our sequence of objects and hence this more complicated construction. The 'regular' construction can be viewed as a shortcut to this construction for lengths $\leq \omega$. If the length of a reduction is $\leq \omega$ both constructions amount to the same and hence this is indeed a generalization of Definition 3.2.3. Defining it this way, we also get that any 'regular' reduction is a transfinite reduction.

Transfinite rewriting is mainly concerned with transfinite reductions in general, and not so much with specifically 'regular' reductions. So when, while talking about transfinite rewriting, "reductions" are mentioned, this will refer to the transfinite reductions of Definition 5.1.1. Whenever the 'regular' reductions of Definition 3.2.3 are meant, they will explicitly be referred to as "'regular' reductions".

5.2 ToRSs

As has been said, in a transfinite rewriting we want to give some notion of structure on the objects and steps of a rewrite system to be able to express issues of convergence of reductions in them. In the literature, this has been done in two ways, using metrics (for example [13], [2]) and using partial orders (for example [2]). However a topological approach has not yet been taken. Such an approach is a very intuitive option though. Topologies are the native framework for talking about issues of structure and convergence. Using topological spaces is a very general way to express structure and convergence properties. Also, as has been mentioned, both metric spaces (by Definition A.3.10) and partial ordered sets (by Definition A.2.57) induce topologies and hence we are more or less able to embed metric frameworks for transfinite rewriting (Section 5.3) and partially ordered frameworks for transfinite rewriting (Section 5.4) in a topological framework. We call our systems for topological transfinite rewriting ToRSs:

Definition 5.2.1. A **topological rewrite system (ToRS)** is a structure $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$ such that:

- $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a rewrite system.
- $(\Phi \uplus A, T)$ is a topological space.

The topology T expresses the structure of the (disjoint) union of Φ and A. We express the structure of steps and objects together because we want to be able to express that a sequence of steps converges to an object (as can be seen later). This structure can be interpreted using the nearness intuition of general topology where objects being in some open sets in the topology formalizes that the objects are near each other (to a certain degree) and where steps being in that open set means that they are also somehow near to each other and to the objects. It is perhaps better to interpret using the *finitely observable properties* intuition. There we can say that objects being in an open set formalizes that the objects in it have a certain property in common (the property that, from an extensional point of view, is that open set) and where steps being in that open set formalizes that these steps work on objects having that property but leave that property untouched.

5.2.1 Topological Convergence

Convergence of reductions comes in two flavors, strong and weak.

Definition 5.2.2. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a ToRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$, **weak topologically converges** to an object $b \in A$ if:

- $b \in \mathrm{Lim}(\mathrm{src}\,/\!(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ WTC

  (The sequence of objects of the reduction converges to b in $(\Phi \uplus A, T)$)

Here b is said to be a **weak topological limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$. If the reduction doesn't weak topologically converge to any object it is said to **weak topologically diverge**.

Remark. Note that limits in a topological space are not guaranteed to exist or be unique, so the same goes for weak topological limits. A reduction might weak topologically converge to no objects or to more than one object. More on this is said further on in this subchapter.

This definition should be straight-forward. We check for topological convergence of our sequence of objects because topologies are our chosen way to impose structure on our sets of objects (and steps). Topological convergence expresses that the limit object is approximated by the objects in the sequence of objects of the reduction, both in the sense that objects in the sequence get arbitrarily close/near to the limit object and in the sense that for every finitely observable property of the limit object, objects in the sequence eventually start to have it. We only check for convergence in the sequence of objects because it is a *weaker* notion of convergence. Convergence in the sequence of steps, that is what strong convergence is concerned with.

If a reduction models a series of stepwise transformations or a computation, then weak topological convergence of that reduction to weak topological limit should model that the in-between results of the computation approximate the limit in both senses mentioned above. For reductions with successor ordinal or 0 length this simply gives the last member of the sequence of objects (Lemma A.2.31). If the reduction is of limit ordinal length, this models what happens to the results 'as we go on'.

Definition 5.2.3. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a ToRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$, **strong topologically converges** to an object $b \in A$ if:

- $b \in \mathrm{Lim}(\mathrm{src}\,/\!(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ WTC
- $b \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ when $\alpha \in \mathrm{LimOrd}$ STC

  (The sequence of steps of the reduction converges to b in topological space $(\Phi \uplus A, T)$)

Here b is said to be a **strong topological limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$. If the reduction doesn't strong topologically converge to any object it is said to **strong topologically diverge**.

For strong topological convergence we not only require WTC as we did for weak topological convergence, we also require that whenever our sequence of steps is of limit ordinal length then it topologically converges to the same limit as the sequence of objects does. It might seem unintuitive that we require the sequence of steps to converge to an object. Doing so is possible because we have a topology on the disjoint union of objects and steps, in fact, the fact that we want to express strong convergence in this way is the technical reason for choosing to topologize the objects and steps together. It also makes sense from an intuitive point of view. For a reduction to strongly converge to a limit we not only want the in-between results of the computation to approximate the limit. We also want the steps in the computation to converge. Using the intuition of our topology on objects and steps and the finitely observable properties intuition from topology, we want, for any observable property of the limit object, that it eventually remains untouched by the steps. This also conforms with and/or extends the intuitions behind strong convergence in metric transfinite rewrite systems or partially ordered transfinite rewrite systems. In metric transfinite rewriting systems (Section 5.3), the intuition will be that for any amount of work, eventually less is done by steps (the amount of work done converges to 0). In partially ordered transfinite rewrite systems (Section 5.3), the intuition will be that for any amount of information in the limit object, it eventually remains stable during the steps. In all these cases strong convergence models that steps in some sense get simpler as the computation progresses. As such it provides a stronger sense of the computation tending to a limit than weak convergence (where we are only concerned with objects) does.

Convergence of sequences (Section A.2.2.3) in topological spaces is very liberal, it allows for sequences to converge to many limits or to none at all. Consequently, the same goes for both weak and strong topological convergence of reductions. This has as advantage that, while we still can express cases in which every reduction has a limit or where every limit is unique, this also lets us express more exotic cases where, for example, some reductions converge to multiple limits. Restricting the formalism to having unique limits can be done by requiring the topological space to be Hausdorff (Proposition A.2.42). In fact, since reductions can only converge to objects, the restriction of the topology to objects being Hausdorff (that is, having every two objects separated by open sets) is enough to guarantee uniqueness of strong and weak topological limits (Lemma A.2.43). In some topologies we do have guaranteed existence of sequence limits, examples are Scott topologies on bounded complete dcpos (as used in Section 5.4). However there does not seem to be a nice topological property that guarantees the existence of sequence limits.

Definition 5.2.4. A **weak topological reduction** is transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a ToRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$, such that for every limit ordinal $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle$ weak topologically converges to $\mathrm{src}(\phi_\lambda)$.

This means that a weak topological reduction is a reduction such that:

$$\forall \lambda \in \mathrm{LimOrd}(\alpha).\ \mathrm{src}(\phi_\lambda) \in \mathrm{Lim}(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle))$$

Weak topologicality of a reduction is the first of the properties that expresses that a reduction, in some sense, behaves properly at limit ordinal indexes.

Definition 5.2.5. A **strong topological reduction** is transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an ToRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$, such that for every limit ordinal $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle$ strong topologically converges to $\mathrm{src}(\phi_\lambda)$.

This means that a strong topological reduction is a reduction such that:

$$\forall \lambda \in \mathrm{LimOrd}(\alpha).\ \mathrm{src}(\phi_\lambda) \in \mathrm{Lim}(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle)) \wedge \mathrm{src}(\phi_\lambda) \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\lambda})$$

Strong topologicality of a reduction is the other, stronger, topological property that expresses that a reduction behaves properly at limit ordinals. So, both weak and strong topologicality of reductions are notions of well-behavedness of transfinite reductions. They formalize "what should happen in a reduction as it tends to a limit".

We can say that both weak and strong topologicality of reductions is preserved under concatenation in the following sense. If $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha_1} \rangle$ is a weak resp. strong topological reduction that weak resp. strong topologically converges to b and $\langle b, \langle \psi_\beta \rangle_{\beta<\alpha_2} \rangle$ is also a weak resp. strong topological reduction then $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha_1}; \langle \psi_\beta \rangle_{\beta<\alpha_2} \rangle$ is also a weak resp. strong topological reduction. Also, if $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak resp. strong topologically diverges then there is no $\psi \in \Phi$ such that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}; \psi \rangle$ is weak resp. strong topological.

By Lemma A.2.56 we get that the weak topological reductions are exactly the transfinite reductions for which the sequence of objects is continuous when viewed as a function from the ordinal that expresses its length (with the ordinal topology (Section A.2.4.1) on it) to the set of objects and steps (with the topology as expressed by the ToRS on it). In [13, p. 2], the weak well-behavedness concept for transfinite reductions[2] is directly defined in terms of continuity. That is, it is defined as requiring topological continuity of the reduction sequence when viewed as a function from its length to the set of objects. So Lemma A.2.56 shows that our approach is strongly related to the approach of [13]. For strong topologicality such a link with continuity isn't as clear since convergence of sequences of steps to an object is required, although Section 5.2.3 will shed some light on that.

5.2.2 Weak versus Strong Topological Convergence

We have that:

- If a transfinite reduction strong topologically converges to an object, it also weak topologically converges to that object.

And hence:

- Any strong topological limit of a transfinite reduction is also a weak topological limit of that reduction.

[2] Actually, reduction sequences in the case of [13]

- If a transfinite reduction weak topologically diverges then it also strong topologically diverges.
- Any strong topological reduction is also a weak topological reduction.

Because of this we might have defined the notion of strong topological convergence in terms of weak topological convergence and the notion of strong topological reductions in terms of weak topological reductions. We choose not to do so because, although in ToRSs, this relation between strong and weak convergence and strong and weak reductions holds, it does not hold in every formalism of transfinite rewriting. For instance, in the PRSs of Section 5.4 (inspired by the PRSs of [2]), it does not.

5.2.3 Zipped Sequences

For strong topological convergence in a ToRS we require, from an observable properties point of view, two things:

- Eventually any observable property of a strong topological limit is present in the sequence of objects.
- Eventually any observable property of a strong topological limit stays untouched during the steps.

These are two convergence processes to the same limit, but the first is only about the objects in a reduction, while the other is about steps. This is the reason for topologizing the objects and steps of our ToRS together. When conceptualizing ToRSs, this at first also seemed like a good reason to express them as one convergence process, one sequence converging to one limit. Because of this, I first had strong topological convergence formalized using zipped sequences (Section A.1.5.1) as:

Definition 5.2.6. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a ToRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$, **strong topologically converges*** to an object $b \in A$ if:

- $b \in \mathrm{Lim}(\mathrm{zip}(\mathrm{src}\,{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle), \langle \phi_\beta \rangle_{\beta<\alpha}))$ STC

  (The zip of its sequence of objects and sequence of steps converges to b in topological space $(\Phi \uplus A, T)$)

Eventually this turned out to be more complicated than necessary since Definition 5.2.3 also does the job without involving the machinery needed for zipping sequences and without needing to see how the limits of zipped sequences relate to the limits of the originals. I still mention this here because it also works and it might be considered more elegant since objects and steps are topologized together ('zipped' using a disjoint union) and hence zipping the sequences of objects and steps seems to fit in with the theme.

To prove that this definition is equivalent to Definition 5.2.3 and to "see how the limits of zipped sequences relate to the limits of the originals":

Theorem 5.2.7. *A transfinite reduction strong topologically converges to an object if and only if it strong topologically* converges.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ be a ToRS and let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$.

Depending on α we get:

- $\alpha \notin \mathrm{LimOrd}$.

  $\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ is of length $\alpha + 1$ and $\langle \phi_\beta \rangle_{\beta<\alpha}$ is of length α. So by Lemma A.2.32 we have:

  $$\mathrm{Lim}(\mathrm{zip}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle), \langle \phi_\beta \rangle_{\beta<\alpha})) = \mathrm{Lim}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$$

  So $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to an object if and only if it strong topologically* converges.

- $\alpha \in \mathrm{LimOrd}$.

  $\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ and $\langle \phi_\beta \rangle_{\beta<\alpha}$ are both of length limit ordinal length (they are of length α). So by Lemma A.2.33 we have:

  $$\mathrm{Lim}(\mathrm{zip}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle), \langle \phi_\beta \rangle_{\beta<\alpha})) = \mathrm{Lim}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$$

  and:

  $$\mathrm{Lim}(\mathrm{zip}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle), \langle \phi_\beta \rangle_{\beta<\alpha})) = \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$$

  So $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to an object if and only if it strong topologically* converges. ☐

Also, viewed this way, the link with continuity again becomes clear, where it wasn't at first. By Lemma A.2.56 we get that the weak topological reductions are exactly the transfinite reductions for which the zip of the sequence of objects and the sequence of steps is continuous when viewed as a function from the ordinal that expresses its length (with the ordinal topology (Section A.2.4.1) on it) to the set of objects and steps (with the topology as expressed by the ToRS on it).

5.2.4 Topologies on Rewrite Systems

We can endow any rewrite system with any topology on the disjoint union of the set objects and the set of steps to get a ToRS. These topologies on objects and steps are comparable (Definition A.2.46) and form a complete lattice (Lemma A.2.47). We can use the same terminology when comparing ToRSs as we do when comparing topologies:

Definition 5.2.8. If $\mathbf{\Phi} = (\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a rewrite system and T_1 and T_2 are topologies on $\Phi \uplus A$ then $(\mathbf{\Phi}, T_1)$ and $(\mathbf{\Phi}, T_2)$ are ToRSs and are ordered as: $(\mathbf{\Phi}, T_1) \leq_\Phi (\mathbf{\Phi}, T_2) \Leftrightarrow T_1 \leq_{tops_S} T_2$.

$\mathbf{\Phi}$ is said to be the rewrite system underlying $(\mathbf{\Phi}, T_1)$ and $(\mathbf{\Phi}, T_2)$ and if $(\mathbf{\Phi}, T_1) \leq_\Phi (\mathbf{\Phi}, T_2)$ then $(\mathbf{\Phi}, T_1)$ is said to be **coarser** than $(\mathbf{\Phi}, T_2)$, $(\mathbf{\Phi}, T_1)$ is said to be **finer** than $(\mathbf{\Phi}, T_2)$.

By this definition and Lemma A.2.47 we get a complete lattice on the ToRSs with the same underlying rewrite system. In such a lattice we can point out two special topologies.

The bottom element of this lattice is the rewrite system endowed with the trivial topology (Section A.2.4.4). We can call it a **trivial ToRS**. In the trivial topology every sequence trivially converges to every point because every open set containing any point also contains every other point. That means that in the trivial ToRS every transfinite reduction weak and strong topologically converges to every object and every transfinite reduction is weak and strongly topological. This means that both the weak and the strong topological reductions are actually just the transfinite reductions of Definition 5.1.1. There are no requirements at all for the behaviour of the reduction at limit ordinal indexes.

The top element of this lattice is the rewrite system endowed with the discrete topology (Section A.2.4.3). We can call it a **discrete ToRS**. In the discrete topology all points are isolated and hence the converging sequences are exactly the sequences that are eventually constant. That is, the converging sequences are sequences having a tail that has the same point at every index. Those sequences converge to that ('constant') point. This means that a transfinite reduction in the discrete ToRS weak topologically converges to a point if its sequence of objects is eventually constant. It converges to the point that it is constant in. Since every step is isolated, a sequence of steps never converges to an object and that means that a transfinite reduction in the discrete ToRS never satisfies clause STC. So, such a reduction strong topologically converges to the same object that it weak topologically converges to if it is of successor ordinal length and does not converge at all if it is of limit ordinal length. The weak topological reductions in the discrete ToRS are not of particular interest. They are the reductions whose sequence of objects are eventually constant up to any limit ordinal smaller than its length. The strong topological reductions in a discrete ToRS however are exactly the 'regular' reductions of Definition 3.2.3. In a discrete ToRS:

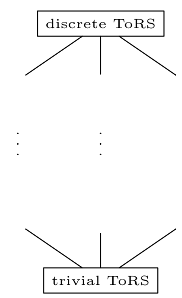
- A transfinite reduction longer than ω has a sequence of steps that up to ω that does not converge to the object at index ω and hence is not strongly topological.

- A transfinite reduction of finite length ($< \omega$) is strongly topological and it has a constant tail (the tail consisting only of its last member) and hence strong topologically converges to that last member. Just like its corresponding 'regular' reduction would be said to end there.

- A transfinite reduction of length ω is strongly topological but strong topologically diverges. Just like its corresponding 'regular' reduction would be said to be non-terminating.

This directly gives an embedding of the formalism of rewrite systems into the formalism of ToRSs. Under this embedding, any 'regular' reduction is strongly topological, if it has an end it strong topologically converges to it, and if it is non-terminating it strong topologically diverges.

So, given some rewrite system, the complete lattice of ToRSs that this rewrite system underlies looks like this:



By Lemma A.2.48 we have that every sequence that converges to a point in a topological space converges to that point in the space endowed with a coarser topology. So as an immediate consequence we get that if a transfinite reduction weak resp. strong topologically converges in a ToRS, it also weak resp. strong topologically converges in any ToRS that is coarser. Also if a transfinite reduction is weak or strong topological in a ToRS, then it is also weak resp. strong topological in any ToRS that is coarser.

5.2.5 Weak ToRSs

In our ToRS formalism we can define the following notion:

Definition 5.2.9. The **weak ToRS** induced by a ToRS $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$ is the ToRS $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T')$ where $T' = \{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$.

This yields an actually ToRS:

Theorem 5.2.10. *If T is a topology on $(\Phi \uplus A)$ then so is $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$.*

Proof. Assume that T is a topology on $(\Phi \uplus A)$.

- $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$ is closed under finite intersection.

  We prove that $\{O \cup \Phi \mid O \in T\}$ is closed under finite intersection. It will follow that $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$ is closed under finite intersection since the intersection of any set of sets having $\emptyset$ as member is $\emptyset$.

  Let $\{O_i \cup \Phi : i \in I\}$ be a finite family of open sets in $\{O \cup \Phi \mid O \in T\}$. We get $\bigcap\{O_i \cup \Phi : i \in I\} = \bigcap\{O_i : i \in I\} \cup \Phi$. Since T is closed under finite intersection we get $\bigcap\{O_i : i \in I\} \in T$. So $(\bigcap\{O_i : i \in I\} \cup \Phi) \in (\{O \cup \Phi \mid O \in T\})$.

- $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$ is closed under arbitrary union.

  We prove that $\{O \cup \Phi \mid O \in T\}$ is closed under arbitrary, 'non-empty' union. It will follow that $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$ is closed under arbitrary union. This is because the union of the empty set is $\emptyset$, which is in $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$, and because the union of any set of sets having $\{\emptyset\}$ as member is equal to the union of the same set, but with $\{\emptyset\}$ subtracted from it.

Let $\{O_i \cup \Phi : i \in I\}$ be a non-empty family of open sets in $\{O \cup \Phi \mid O \in T\}$. We get $\bigcup\{O_i \cup \Phi : i \in I\} = \bigcup\{O_i : i \in I\} \cup \Phi$. Since T is closed arbitrary union we get $\bigcup\{O_i : i \in I\} \in T$. So $(\bigcup\{O_i : i \in I\} \cup \Phi) \in (\{O \cup \Phi \mid O \in T\})$. □

So a weak ToRS is very similar to the ToRS that induces it, with the difference that every step is in every non-empty open set of the topology. This way, any sequence of steps converges to any object or step and the topology is indifferent to steps with regards to convergence of reductions. We get:

Lemma 5.2.11. *The restriction of $(\Phi \uplus A, T)$ to A is equal to the restriction of weak topology induced by $(\Phi \uplus A, T)$ to A.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$ be a ToRS. The restriction of T to A is $\{O \cap A \mid O \in T\}$. The weak ToRS induced by $(\Phi \uplus A, T)$ is $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$ and the restriction of $\{O \cup \Phi \mid O \in T\} \cup \{\emptyset\}$ to A is $\{(O \cup \Phi) \cap A \mid O \in T\} \cup (\{\emptyset\} \cap A)$. We get:

$$\begin{aligned}
\{(O \cup \Phi) \cap A \mid O \in T\} \cup (\{\emptyset\} \cap A) = \\
\{(O \cup \Phi) \cap A \mid O \in T\} \cup \{\emptyset\} = \\
\{O \cap A \mid O \in T\} \cup \{\emptyset\} = \\
\{O \cap A \mid O \in T\}
\end{aligned}$$

The last step holds because $\emptyset \in T$ since T is a topology. □

Theorem 5.2.12. *A transfinite reduction in a ToRS weak topologically converges to an object if and only if it weak topologically converges to that object in the weak ToRS induced by the ToRS.*

Proof. Weak topological convergence is only concerned with sequences of objects converging to objects and hence with convergence in the restriction of the topological space of a ToRS to objects. Now because by Lemma 5.2.11 the restrictions to objects of topological space of a ToRS and the topological space of its induced weak ToRS are the same, so Lemma A.2.69 does the job here. □

Also:

Theorem 5.2.13. *A transfinite reduction in a weak ToRS weak topologically converges to an object if and only if it strong topologically converges to that object in the weak ToRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$ be a weak ToRS, for every $O \in T$ we get $\Phi \subseteq O$. Let $\langle a, \langle \phi_\beta \rangle_{\beta < \alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$.

⇒ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak topologically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$.

By assumption we get that $\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ converges to b in $(\Phi \uplus A, T)$ (WTC). We also get that $\langle \phi_\beta \rangle_{\beta<\alpha}$ converges to b in $(\Phi \uplus A, T)$ (STC) because, let $O \in T$ be such that $b \in O$, we have $\Phi \subseteq O$ by definition of weak ToRSs. So $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to b in $(\Phi \uplus A, T)$.

⇐ Immediate from the definitions of strong and weak topological convergence. □

Simply put, in a weak ToRS the axiom STC is always true. This means that the notion of weak topological convergence in a ToRS is equivalent to the notion of strong topological convergence in the induced weak ToRS. So we can express weak topological convergence in terms of strong topological convergence in the induced weak ToRS. From an economical ('lightness' of the framework) point of view we might even consider the notion of weak topological convergence superfluous. We might only want to consider the notion of strong topological convergence, which we can consequently just call **topological convergence**. We can then study weak topological convergence through induced weak ToRS'. We can again call upon Occam's razor for justification.

The same goes for the notion of weakly topological reductions in a ToRS it is equivalent (and hence can be expressed in terms of) the notion of strong topological reductions in the induced weak ToRS. We might want to get rid of the notion of weakly topological reductions, call the strong topological reductions **topological reductions** and study weak topological reductions through induced weak ToRS'.

An idea related to this is proposed in [10]. There, notions of convergence of reductions in iTRS are given by means of a metric on terms. The paper proposes to express strong convergence in iTRSs in terms of weak convergence. It does so by associating with every iTRS an **indirected version** of that iTRS in such a way that every reduction in the iTRS strongly converges if and only if every reduction in the indirected version weakly converges ([10, p. 7]). This can be applied to individual reductions by associating with every reduction in the iTRS a reduction in the indirected version of that iTRS and modifying the metric so that a reduction strongly converges to some term in the iTRS if and only if the associated reduction in the indirected version weakly converges to that term. For the purposes of this thesis however, a problem with this approach is that it is not abstract. It depends on the structure of iTRSs and does its modifications to obtain indirected version on an iTRS level. There is no guarantee that such a trick will be available in any other framework for concrete rewriting. The weak ToRS approach has as an advantage that it is abstract in nature, all modification is done on a ToRS level. A concrete rewrite system might induce a ToRS in any way and a weak version of that ToRS, where a reduction strong topologically converges if and only if it weak topologically converges in the original ToRS, will still be available. Also, [10] modifies the terms, steps and the structure imposed on the objects (by means of the metric) and also needs an association between reductions in this original iTRS and reductions in the indirected version. The weak ToRS approach only needs to modify the structure imposed on objects and steps (by means of a topology).

The approach that formalizes both weak and strong convergence individually (Definition 5.2.2 and Definition 5.2.3) however is more in tune with the literature, hence that approach will be used in the remainder of this thesis.

5.2.6 Embedding iTRSs in ToRSs

One of the major applications of abstract rewriting systems in general is to embed concrete rewriting systems like TRSs in them (as is done in Section 4.2.1). This gives a notion of reduction on TRSs. In transfinite rewriting we have the notion of transfinite reduction and in ToRSs we have the notions of weak and strong topological reductions. Since these reductions can have transfinite length, it is interesting to also involve objects that are in some sense infinite. Combining reductions of finite length with objects of infinite size is not really interesting because objects can never grow from finite size to infinite size. Combining reductions of transfinite length with objects of finite size is not really interesting because objects can not keep growing throughout the reduction. Combining reductions of transfinite length with objects of infinite size however is interesting. Because of this, embedding iTRSs (where objects might have infinite size) in ToRSs is an obvious application; both for ToRSs in an iTRS setting and for iTRSs in an ToRS setting.

Definition 5.2.14. The ToRS **induced** by an iTRS (Σ, R) is $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T)$ where:

- $(\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$ is the rewrite system induced by (Σ, R) in the sense of Definition 4.2.11
- T is the topology generated by the subbase $\{U(p, f) \mid p \in \mathbb{N}^*, f \in \Sigma\}$ where:

$$\begin{aligned} U(p,f) = &\{\phi \in \Phi \mid (\mathrm{pos}(\phi) \not\leq p) \wedge ((\mathrm{src}(\phi))(p) = (\mathrm{tgt}(\phi))(p) = f)\} \uplus \\ &\{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\} \end{aligned}$$

To get a base for this topology we close the subbase under finite intersection and get $\{\bigcap\{U(p,f) \mid \langle p,f\rangle \in P\} \mid P \subset (\mathbb{N}^* \times \Sigma) \wedge \mathrm{card}(P) \in \mathbb{N}\}$ as a base for the topology. Here P is a finite set of position-symbol tuples and $\bigcap\{U(p,f) \mid \langle p,f\rangle \in P\}$ is the set of all terms that have those symbols at those positions and all steps that work on objects having those symbols at those positions *and* that leave those positions untouched. We can call P a **finite approximation** of any term $t \in ((\bigcap\{U(p,f) \mid \langle p,f\rangle \in P\}) \cap \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma))$ (and only of those terms) and a **static part** of any step $\phi \in ((\bigcap\{U(p,f) \mid \langle p,f\rangle \in P\}) \cap \Phi)$ (and only of those steps). This fits in well with our finitely observable properties intuition of topological spaces, a base set represents the property of having this finite amount of symbols at those positions (for terms) and leaving these symbols at these positions untouched (for steps). Open sets in the topology then are unions of these base sets, they represent disjunctive properties (having either certain symbols at certain positions or having certain other symbols at certain other positions). The restriction of this topology to terms is given by the subbase $\{U_{\mathcal{T}}(p, f) \mid p \in \mathbb{N}^*, f \in \Sigma\}$, where $U_{\mathcal{T}}(p, f) = \{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\}$, which is the subbase proposed in [17, 3].

Such a topology is not Hausdorff or even necessarily T_0. Steps are not separable from their source and target by open sets. For every subbase set that a step ϕ is in, it must hold that $(\text{src}(\phi))(p) = (\text{tgt}(\phi))(p) = f$ for given p and f putting the source and target object also in that subbase set. Also, there might be multiple steps that are topologically indistinguishable. Those are steps at the same position with the same source and target. This can happen when multiple rules give rise to steps which have the same source and target and are at the same position.

Example 5.2.15. For instance, in the TRS (Σ, R) with $\Sigma = \{g/2, a/0\}$ and $R = \{\pi_1 : g(a, x) \to a, \pi_2 g(x, a) \to a\}$ both π_1 and π_2 give rise to a step at position ϵ with $g(a, a)$ as source and a as target.

However we do have that:

Theorem 5.2.16. *Any two different terms can be separated by open sets.*

Proof. Let (Σ, R) be an iTRS and let $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \text{src}, \text{tgt}), T)$ be the ToRS that it induces. Let $t, s \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ such that $t \neq s$. We get some $p \in (\text{Pos}(t) \cup \text{Pos}(s))$ such that $t(p) \neq s(p)$. We get that $t \in U(p, t(p))$ and $s \in U(p, s(p))$. Both $U(p, t(p))$ and $U(p, s(p))$ are subbase sets and hence open. For any $r \in (U(p, t(p)) \cap \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma))$ we get $r(p) = t(p)$ and hence $r(p) \neq s(p)$, so $r \notin U(p, s(p))$. For any $\phi \in (U(p, t(p)) \cap \Phi)$ we get that $(\text{src}(\phi))(p) = (\text{tgt}(\phi))(p) = t(p)$ and hence $(\text{src}(\phi))(p) = (\text{tgt}(\phi))(p) \neq s(p)$, so $\phi \notin U(p, s(p))$. So $U(p, t(p))$ and $U(p, s(p))$ separate t and s. □

This makes the restriction of the topology to objects Hausdorff and, by Lemma A.2.43, gives unique strong and weak topological limits of reductions in the ToRS (since only objects can be limits of reductions).

Unfolding the definitions of weak and strong topological convergence for these induced ToRSs (and using Lemma A.2.26) we get:

Proposition 5.2.17. A transfinite reduction in the ToRS induced by an iTRS:

- weak topologically converges to some term t if, for every finite approximation of t, the terms in the reduction are eventually approximated by that approximation.
- strong topologically converges to some term t if, for every finite approximation P of t, the terms in the reduction are eventually approximated by P and the steps in the reduction eventually have P as remaining static.

Now, a couple of examples to show how this works out. First, continuing with our example Example 4.2.13 we get the following:

Example 5.2.18. Consider the iTRS (Σ, R) with $\Sigma = \{a/0, f/1\}$ and $R = \{x \to f(x)\}$ and denote the rewrite system that it induces by $(\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$ and hence the ToRS it induces by $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T)$. Consider the following reduction:

$$a \to_\epsilon f(a) \to_1 f(f(a)) \to_{11} f(f(f(a))) \to \dots$$

The reduction turns out to be both weakly and strongly topological (trivially, because $\mathrm{LimOrd}(\omega) = \emptyset$). It weak topologically converges to f^ω because its sequence of objects, $(\langle a, f(a), f(f(a)), f(f(f(a))), \dots\rangle)$, topologically converges to f^ω in the topological space $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$. To see this; as base for T we get:

$$B = \{\bigcap\{U(p, f) \mid \langle p, f\rangle \in P\} \mid P \subset (\mathbb{N}^* \times \Sigma) \wedge \mathrm{card}(P) \in \mathbb{N}\}$$

Where $U(p, f)$ is a subbase set as defined in Definition 5.2.14. The P that uniquely defines any set in $A \in B$ can be viewed as some finite approximation of the objects in A and as some static part of the steps in A. Let A be some base set that our limit, f^ω, is in. The P associated with A is a finite approximation of f^ω. There is some $\langle p, f\rangle \in P$ such that $|p|$ is maximal (since P is finite). From index $|p| + 1$ and on the terms in the sequence of objects in the reduction share all symbols at position p and earlier positions with each other and with f^ω. Hence P is also a finite approximation of those terms, so the terms are in A, and hence the sequence of objects converges to f^ω.

The sequence of steps of the reduction also topologically converges to f^ω. To see this; let A be some base set that f^ω is in. Again, the P associated with A is a finite approximation f^ω. The steps in A are steps such that P, is a static part of them. Let $\langle p, f\rangle \in P$ be such that $|p|$ is maximal. From index $|p|+1$ and on the steps in the sequence of steps of the reduction take place at positions of a larger length than $|p|$. That means that those steps have P as static part and hence are in A. So the sequence of steps of the reduction converges to f^ω.

For an example of a reduction of true transfinite length we can consider the following:

Example 5.2.19. Let (Σ, R) with $\Sigma = \{a/0, f/1, g/2\}$ and $R = \{a \to f(a)\}$ be an iTRS. In the induced ToRS we get the following reduction of length $\omega + 1$:

$$g(a, a) \to_1 g(f(a), a) \to_{11} g(f(f(a)), a) \to \dots \; g(f^\omega, a) \to_2 g(f^\omega, f(a))$$

This reduction is again a strong topological reduction. This time though, it is not for trivial reasons but because the part of reduction up to ω

$$g(a, a) \to_1 g(f(a), a) \to_{11} g(f(f(a)), a) \to \dots$$

strong topologically converges to the source of the step at index ω, $g(f^\omega, a)$. This in turn is because both the sequence of objects and the sequence of steps of the reduction up to ω converge to $g(f^\omega, a)$ in our induced topological space. The full reduction strong topologically converges to its last member because it is of successor ordinal length.

For an example of a weak topologically diverging (and hence strong topologically diverging) reduction:

Example 5.2.20. Consider the iTRS (Σ, R) with $\Sigma = \{a/0, b/0, f/1\}$ and $R = \{a \to b, b \to a\}$. In the induced ToRS we get the following reduction of length ω:

$$f(a) \to_1 f(b) \to_1 f(a) \to_1 f(b) \to_1 f(a) \to \dots$$

The sequence of objects of this reduction has no limit in our topological space. To see this, let any term t be a potential limit.

- If $t(\epsilon) \neq f$ then $\{\langle \epsilon, t(\epsilon)\rangle\}$ is a finite approximation of t, but not of $f(a)$ and $f(b)$. So there is no tail of the sequence of objects that is contained in the base set that this finite approximation gives rise to.
- If $t(\epsilon) = f$ then $t(1)$ is defined and $\{\langle 1, t(1)\rangle\}$ is a finite approximation of t. But it is either not a finite approximation of $f(a)$ or $f(b)$ or of both (depending on whether $t(1) = a$, $t(1) = b$ or $t(1) = f$). So there is no tail of the sequence of objects that is contained in the base set that this finite approximation gives rise to.

This means that the reduction is weak topologically diverging.

For an example of a reduction of that weak topologically converges but strong topologically diverges:

Example 5.2.21. We can consider the iTRS (Σ, R) with $\Sigma = \{a/0, f/1, g/2\}$ and $R = \{f(x) \to f(f(x))\}$. In the induced ToRS we get the following reduction of length ω:

$$g(f(a), a) \to_1 g(f(f(a), a) \to_1 g(f(f(a)), a) \to_1 g(f(f(f(a))), a) \to \dots$$

Here all the steps match the x in the reduction rule on its outermost possibility, so all the steps take place at position 1. Another possibility would have been to keep matching x on its innermost possibility, causing the steps to take place at ever longer positions (11, 111, 1111, etc.) throughout the reduction. The displayed reduction weak topologically converges to $g(f^\omega, a)$ because its sequence of objects, $\langle g(a, a), g(f(a), a), g(f(f(a)), a), g(f(f(f(a))), a), \dots\rangle$, converges to $g(f^\omega, a)$ in the induced topological space. The other possibility, with innermost steps, would have had the same sequence of objects and hence the same weak topological limit. That possible reduction would also strong topologically converge to $g(f^\omega, a)$. The reduction displayed above, with outermost steps, however does not strong topologically converge to any object. This is because all steps take place position at 1 of a term, so their only static parts are $\emptyset$ and the set $\{\langle \epsilon, g\rangle\}$. Any term with $\{\langle \emptyset, g\rangle\}$ as finite approximation also has a symbol at positions 1 and 2 and hence a finite approximation involving those positions. But in the reduction there is no step with a static part that involves those positions. This means that any term is in an open set that no step in the reduction is in. We can expand the reduction to length $\omega + 1$:

$$g(f(a), a) \to_1 g(f(f(a), a) \to_1 g(f(f(a)), a) \to \ldots g(f^\omega, a) \to g(f^\omega, f(a))$$

By the argumentation above, this is a weakly topological, but not strongly topological reduction.

5.2.7 Embedding Exotic TRSs in ToRSs

When embedding wide TRSs and wide iTRSs we can use the topology generated by the subbase $\{U(p, f) \mid p \in On^*, f \in \Sigma\}$. We still have

$$\begin{aligned} U(p, f) = &\{\phi \in \Phi \mid \mathrm{pos}(\phi) \not\leq p \wedge (\mathrm{src}(\phi))(p) = (\mathrm{tgt}(\phi))(p) = f\} \uplus \\ &\{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\} \end{aligned}$$

An interesting example of how this works out would be the following.

Example 5.2.22. Consider the wide TRS (Σ, R) with $\Sigma = \{a/0, b/0, f/\omega\}$ and $R = \{a \to b\}$. We get $f(a, a, a, \ldots)$ as term and, starting from this term, the following reduction:

$$f(a, a, a, a, \ldots) \to_1 f(b, a, a, a, \ldots) \to_2 f(b, b, a, a, \ldots) \to_3 f(b, b, b, a, \ldots) \to \ldots$$

This is a reduction of length ω. For any finite approximation P of $f(b, b, b, b, \ldots)$, we have that terms in the reduction eventually are approximated by P and that steps in the reduction eventually have P as remaining static. So we get that the reduction both weak and strong topologically converges to $f(b, b, b, b, \ldots)$. This type of convergence might be called **horizontal convergence**, since, for every part of the limit, the objects in the reduction eventually start to share it; from left to right. First $f(\ldots)$, then $f(b, \ldots)$, then $f(b, b, \ldots)$, etc. It might be opposed to the 'vertical' convergence in for instance Example 5.2.18, where, for every part of the limit f^ω, the objects in the reduction eventually start to share it; from top to bottom. First $f(\ldots)$, then $f(f(\ldots))$, then $f(f(f(\ldots)))$, etc. The fact that such a horizontal convergence process can be expressed is, at least partly due to the ToRS formalism. Horizontal convergence can not be expressed in for instance the MRSs of Section 5.3.

Partial iTRSs can also be embedded in ToRSs, but this is best done in a way that differs from the way iTRSs and wide TRSs are embedded. It is done indirectly, by first embedding an iTRS in a PRS (Section 5.4.7) and then embedding that PRS in a ToRS (Section 5.4.4). The resulting ToRS then uses the Scott topology on the partial order on terms. Section 5.4.7 expands on this.

As can be read in [14], formalizing transfinite reductions of transfinite terms and their convergence is highly problematic. The embedding of transfinite TRSs in ToRSs is how these reductions and their convergence behaviour are defined.

So these problems come into play when we want to give such an embedding, that is, now. In [14], two desirable properties of reductions of transfinite terms are mentioned. The 'push down' property and the 'pull up' property. Reductions must be able to push subterms at positions that are prefixes of limit positions (positions of limit ordinal length) down to that limit position.

Example 5.2.23. Consider the reduction rule $c \to f(c)$ and the following reduction:

$$c \to_\epsilon f(c) \to_1 f(f(c)) \to_{11} f(f(f(c))) \to \dots$$

The subterm c gets pushed down, such that, eventually, it passes any finite depth and any finite position that is a prefix of the limit position $1111\dots = 1^\omega$. Therefore it is supposed to pop up in the limit of the sequence at that limit position 1^ω, making the limit term $f^\omega(c)$.

Reductions must also be able to pull up subterms from positions at a limit ordinal depth to positions at a smaller depth.

Example 5.2.24. Consider the reduction rule $g(x) \to x$, the transfinite term $g^\omega(c)$ and the following reduction:

$$g^\omega(c) \to_\epsilon g^\omega(c) \to_\epsilon g^\omega(c) \to_\epsilon g^\omega(c) \to \dots$$

Eventually, any g in the original term is consumed by a step, leaving only c as the limit of the sequence. Thus, pulling up c from position 1^ω to position ϵ.

In [14, Definition 4.1] a topology is given on transfinite terms, but that topology turns out not to have the 'push down' and 'pull up' properties. It is mentioned that this problem might be solved or lessened by using a different topology. In ToRSs though, such a solution to the problem needs to involve the sequence of steps of the reduction and hence the problem can only be solved for strong topological convergence. Weak topological convergence remains broken. To show this, Example 5.2.23 can be adapted slightly.

Example 5.2.25. Consider the reduction rule $f(c) \to f(f(c))$ and the reduction

$$f(c) \to_\epsilon f(f(c)) \to_1 f(f(f(c))) \to_{11} f(f(f(f(c)))) \to \dots$$

Here, we are not pushing c down but $f(c)$. Eventually, $f(c)$ passes any prefix of the limit position 1^ω. So it is supposed to pop up in the limit of the sequence at 1^ω, making the limit term $f^\omega(f(c))$.

So, what is being pushed not only depends on objects, but also on steps. To maybe make the example clearer:

Example 5.2.26. Consider the reduction rule $x \to f(x)$ and the following two reductions:

$$f(c) \to_1 f(f(c)) \to_{11} f(f(f(c))) \to_{111} f(f(f(f(c)))) \to \dots$$

and

$$f(c) \rightarrow_{\epsilon} f(f(c)) \rightarrow_{1} f(f(f(c))) \rightarrow_{11} f(f(f(f(c)))) \rightarrow \dots$$

In the first reduction, steps match x on c, in the second reduction, steps match x on $f(c)$. So in the first reduction we are pushing c, while in the second reduction we are pushing $f(c)$.

Since weak topological convergence only involves sequences of objects (and the sequences of objects of the two reductions are the same), it can't 'know' which subterm is being pushed down. So any topological approach, formalizing convergence in a way that conforms to the 'push down' property, needs to involve strong topological convergence. Maybe a topology on transfinite terms can be given such that strong topological convergence in ToRSs satisfies the 'push down' property.

The problems with a topological approach and the 'pull up' property are said to be impossible to fix though. In Example 5.2.24, the sequence of objects of the reduction is the constant sequence with $g^{\omega}(c)$ as member at every position. The limit of the reduction is supposed to be c. This seemingly can't be the case using a topological approach, since any topology is going to, at least, also have $g^{\omega}(c)$ as the limit of the constant sequence with $g^{\omega}(c)$ at every position. And as showcased by Example 5.2.24, this is a limit we don't want, since every g is eventually consumed by a step and c is pulled up. To exclude this limit we should somehow be able to see that the steps in the reduction eventually remove every g in the original term, and to see this, we should be able to differentiate between the different gs in that term. If we can see that:

- The g at position ϵ in the second term of the reduction is the g at position 1 in first term of the reduction. The g at position 1 in the second term of the reduction is the g at position 11 in first term of the reduction. Etc.
- The g at position ϵ in the third term of the reduction is the g at position 1 in second term of the reduction, which is the g at position 11 in first term of the reduction. The g at position 1 in the third term of the reduction is the g at position 11 in second term of the reduction, which is the g at position 111 in first term of the reduction. Etc.
- Etc.

Then we have, both, that the sequence of objects is not constant any more, and we can see that the steps eventually remove any g in the original term. Solving both of our main problems and possibly making an topology on transfinite terms and steps that satisfies the 'pull up' property possible. In transfinite terms as they are though, it is not possible to differentiate between different occurrences of the same symbols and track those symbols across a reduction. In [14], an auxiliary notion of descendants is defined, which is used in an attempt to define convergence of reductions in a non-topological manner. However, we need the possibility to simply look at two terms in isolation and see that "the g at position

ϵ in the second term of the reduction is the g at position 1 in first term of the reduction", so that we can use this information when topologizing our terms.

As should be clear from this discussion, this thesis will not present an embedding of transfinite TRSs in ToRSs. Trying to give such an embedding is very problematic in various way. However, considering the above, it might be too early to declare topological defeat, as [14] does. Especially considering, the problems with non-topological solution that are mentioned in [14].

5.3 MRSs

The original approach to transfinite abstract rewriting has been to use metrics to express the structure of the set objects and convergence behaviour. Kennaway ([13]) defines Metric Abstract Reduction Systems (MARSs) using complete ultrametrics on a set of objects. His formalism emphasizes objects (instead of steps) and is based on the ARSs of [18, chapter 1]. He uses complete ultrametrics to make sure that every strongly continuous reduction sequence has a limit. This works, but not in the way the article says it does (Appendix B.2). Another problem with his approach is the fact that it is based on ARSs, where, as Chapter 3 explains, rewrite relations might be a preferred formalism for rewriting with an emphasis on objects. Also the argumentation against emphasizing objects instead of steps (Chapter 3 and the introduction of this chapter) comes into play here. Furthermore the formalism of MARSs does not allow for reduction sequences that do not involve a step and only one object. As such it does not generalize the 'regular' reduction sequences of Definition 3.1.3 nicely.

In [2], Metric Reduction Systems are defined based on rewrite systems. This is an improvement but has the deficiency that it models reductions as 'only' sequences of steps. It does allow for a reduction containing no steps, the empty sequence, and this reduction is said to start and end in every object of the system. But this approach is also problematic in that it can't differentiate between the process/computation where we start with object a and do nothing and the process/computation where we start with object b and do nothing. Both are modeled by the empty reduction. As such it also doesn't generalize the 'regular' reductions of Definition 3.2.3 nicely.

Based on these approach, but fixing their deficiencies, we can define metric rewrite systems as follows:

Definition 5.3.1. A **metric rewrite system (MRS)** is a structure $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$ such that:

- $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a rewrite system.
- h is a function from Φ to $\mathbb{R}^+$.
- (A, d) is a metric space.
- For all $\phi \in \Phi$ we have $d(\mathrm{src}(\phi), \mathrm{tgt}(\phi)) \leq h(\phi)$.

The metric d expresses the structure of A, it gives a measure of similarity on objects in terms of distance. The function $h : \Phi \to \mathbb{R}^+$ gives the **height** of a step $\phi \in \Phi$, intuitively, it expresses 'how much work is done' in the step. It is intuitive that the work done by a step can not be less than the distance between the source and target of the step, this distance needs to be 'crossed' and there is a minimum to the amount of work needed for that.

Remark. When interpreting h as a measure of how much work is done by a step, it might be intuitive to have the function map to $\mathbb{R}^{0+}$ instead of $\mathbb{R}^+$ since no work at all might be done (transforming an object to itself by doing nothing). However defining it this way, using $\mathbb{R}^+$, is the standard ([2], [13]). This standard is probably influenced by the main application of abstract rewriting, term rewriting, where the height of a step corresponds with the length of the position where the rule is applied (by the function 2^{-k} where k is the length of the position). A rule is always applied at some finite position, so, in the TRS case, the height of a step is always larger than 0.

Definition 5.3.2. If $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ is a transfinite reduction in an MRS then we can call $\langle h(\phi_\beta) \rangle_{\beta<\alpha}$ its **sequence of heights**.

5.3.1 Metric Convergence

We can now define convergence behaviour for reductions using metric notions.

Definition 5.3.3. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an MRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$, **weak metrically converges** to an object $b \in A$ if:

- $\lim(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)) = b$ WMC

  (The sequence of objects of the reduction converges to b in (A, d))

Here b is said to be a **weak metrical limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$. If $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ doesn't weak metrically converge to any object it is said to **weak metrically diverge**.

This definition should be straight-forward. Just like [2, p. 9], we check for metric convergence of our sequence of objects because metrics are our chosen way to impose structure on our sets of objects and steps. The metric convergence expresses that the limit object is approximated by the objects in the sequence of objects of the reduction in the sense that objects in the sequence get arbitrarily close to the limit as expressed by the metric.

Definition 5.3.4. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an MRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$, **strong metrically converges** to an object $a \in A$ if:

- $\lim(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)) = b$ WMC
- $\lim(\langle h(\phi_\beta) \rangle_{\beta<\alpha}) = 0$ when $\alpha \in \mathrm{LimOrd}$ SMC

  (If the reduction is of limit ordinal length then the sequence of heights of the reduction converges to 0 in $(\mathbb{R}^{0+}, d_{\mathbb{R}})$)

Here b is said to be a **strong metrical limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$. If $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ doesn't weak metrically converge to any object it is said to **strong metrically diverge**.

It follows that strong metrical convergence implies weak metrical convergence. But, similar to how strong topological convergence differs from weak topological convergence and following [2, p. 9], for strong metrical convergence the sequence of steps is also taken into account. This is done through the sequence of heights. We consequently get a strictly stronger notion. The height function h maps steps into a proper metric space $(\mathbb{R}^{0+}, d_{\mathbb{R}})$ and we again use metric convergence in that space. On an intuitive level, h gives the amount of work that is done in a step. So now, we not only want the objects in the sequence of objects of a reduction to be more and more like its limit, we also want the amount of work that is done by the steps to decrease further and further, such that for any amount of work eventually less than that is done. If the reduction is of successor length, this happens automatically, because, after the last step no more work is done. If the reduction is of limit ordinal length though, there is no last member, so here we really strengthen weak metric convergence and require that the amount of work done (as formalized by the sequence of heights) converges to 0.

By Lemma A.3.17 we get that any reduction weak and strong metrically converges to one object at most; metrical limits are unique.

Definition 5.3.5. A **weak metric reduction** is transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an MRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$, such that for every limit ordinal $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle$ weak metrically converges to $\mathrm{src}(\phi_\lambda)$.

This means that a weak metric reduction is a reduction such that:

$$\forall \lambda \in \mathrm{LimOrd}(\alpha).\ \lim(\mathrm{src}\,\text{'}(\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle)) = \mathrm{src}(\phi_\lambda)$$

Weak metricity of a reduction is a metric property that expresses that a reduction, in some sense, behaves properly at limit ordinal indexes.

Definition 5.3.6. A **strong metric reduction** is transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an MRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$, such that for every limit ordinal $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle$ strong metrically converges to $\mathrm{src}(\phi_\lambda)$.

This means that a strong metric reduction is a reduction such that:

$$\forall \lambda \in \mathrm{LimOrd}(\alpha).\ \lim(\mathrm{src}\,\text{'}(\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle)) = \mathrm{src}(\phi_\lambda) \land \lim(\langle h(\phi_\beta) \rangle_{\beta<\lambda}) = 0$$

Strong metricity of a reduction is the other, stronger, metric property that expresses that a reduction behaves properly at limit ordinals.

Like weak and strong topologicality of reductions, both weak and strong metricality of reductions is preserved under concatenation. Also like weakly topological reductions, a reduction is weakly metrical if and only if its sequence of objects is continuous in the topological sense when viewed as a function from the ordinal giving its length with the ordinal topology on it to the topological space induced by (A, d). For strong metrical reductions we don't have such direct a link to continuity.

5.3.2 Embedding MRSs in ToRSs

Using topological spaces is a very general way to express the structure and convergence properties of a set of objects, it is more general than using metrics. Every metric induces a topology (Definition A.3.10 and Lemma A.3.11), but not vice versa, there are non-metrizable topologies. By Lemma A.3.14, any non-Hausdorff space exemplifies this. This suggests that we can embed the MRSs into ToRSs. Getting a topology on the set of objects that we have in our MRS would be trivial (use the topology induced by the metric on objects). However we need a topology on the disjoint union of the set of objects *and* the set of steps, that displays the intended convergence behaviour, so it becomes a little more complicated.

Definition 5.3.7. The ToRS **induced by** an MRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$, is $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ where

$$T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O \mid \epsilon \in (0, \infty] \wedge O \in T_A\} \cup \{\emptyset\}$$

and T_A is the topology induced by the metric space (A, d).

To see that this is an actual ToRS we need to prove that:

Lemma 5.3.8. *$T_{\Phi \uplus A}$ is a topology on $\Phi \uplus A$*

Proof. $T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O \mid \epsilon \in (0, \infty] \wedge O \in T_A\} \cup \{\emptyset\}$, so we need to prove that:

- $T_{\Phi \uplus A}$ is closed under finite intersection.

  We show $T_{\Phi \uplus A} \setminus \{\emptyset\} = \{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O \mid \epsilon \in (0, \infty] \wedge O \in T_A\}$ is closed under finite intersection. It will follow $T_{\Phi \uplus A}$ is closed under finite intersection since the intersection of any set of sets having $\emptyset$ as member is $\emptyset$.

  Let $\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} \uplus O_i : i \in I\}$ be a finite family of open sets in $T_{\Phi \uplus A} \setminus \{\emptyset\}$. We get:

$$\bigcap\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} \uplus O_i : i \in I\} =$$
$$\bigcap\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} : i \in I\} \uplus \bigcap\{O_i : i \in I\}$$

  Since I is finite we get a $i^* \in I$ such that ϵ_{i^*} is least and hence:

$$\bigcap\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} : i \in I\} = \{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_{i^*}\}$$

Also, since T_A is a topology we get that $\bigcap\{O_i : i \in I\} \in T_A$. So we get:

$$\bigcap\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i^*\} \uplus \bigcap\{O_i : i \in I\} \in T_{\Phi \uplus A} \setminus \{\emptyset\}$$

And hence $T_{\Phi \uplus A} \setminus \{\emptyset\}$ is closed under finite intersection.

- $T_{\Phi \uplus A}$ is closed under arbitrary union.

  We show $T_{\Phi \uplus A} \setminus \{\emptyset\} = \{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O \mid \epsilon \in (0, \infty] \wedge O \in T_A\}$ is closed under arbitrary, 'non-empty' union. It will follow $T_{\Phi \uplus A}$ is closed under arbitrary union. This is because the union of $\emptyset$ is $\emptyset$, which is in $T_{\Phi \uplus A}$, and because the union of any set of sets having $\emptyset$ as member is equal to the union of the same set, but with $\emptyset$ removed from it.

  Let $\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} \uplus O_i : i \in I\}$ be a non-empty family of open sets in $T_{\Phi \uplus A} \setminus \{\emptyset\}$. We get:

$$\bigcup\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} \uplus O_i : i \in I\} =$$
$$\bigcup\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} : i \in I\} \uplus \bigcup\{O_i : i \in I\}$$

  Every non-empty subset of $(0, \infty]$ has a least upper bound, so we get that:

$$\bigcup\{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} : i \in I\} = \{\phi \in \Phi \mid 0 < h(\phi) < \bigsqcup\{\epsilon_i \mid i \in I\}\}$$

  Also, since T is a topology we get that $\bigcup\{O_i : i \in I\} \in T_A$. So we get:

$$\bigcup\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon_i\} \uplus \bigcup\{O_i : i \in I\} \in T_{\Phi \uplus A} \setminus \{\emptyset\}$$

  And hence $T_{\Phi \uplus A} \setminus \{\emptyset\}$ is closed under arbitrary, 'non-empty' union. □

We get that:

Lemma 5.3.9. *(A, T_A) is a subspace of $(\Phi \uplus A, T_{\Phi \uplus A})$*

Proof. We have:

$$\begin{aligned}
\{O_{\Phi \uplus A} \cap A \mid T_{\Phi \uplus A}\} &= \\
(\{(\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O) \cap A \mid \epsilon \in (0, \infty] \wedge O \in T_A\} \cup \{\emptyset \cap A\}) &= \\
\{O \mid O \in T_A\} \cup \{\emptyset\} &= \\
T_A \cup \{\emptyset\} &= \\
T_A &
\end{aligned}$$

And $A \subseteq (\Phi \uplus A)$. □

That means that, on objects, the induced topology on objects and steps topology inherits the convergence behaviour of the metric topology on objects:

Lemma 5.3.10. *A sequence of objects in a ToRS induced by an MRS converges to an object if and only if that sequence of objects converges to that object in the MRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$ be an MRS and let T_A be the topology induced by (A, d). We get $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ with

$$T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O \mid \epsilon \in (0, \infty] \wedge O \in T_A\} \cup \{\emptyset\}$$

as ToRS induced by $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$. Now let $\langle s_\beta \rangle_{\beta<\alpha}$ be a sequence in A and let $x \in A$.

We get that $\langle s_\beta \rangle_{\beta<\alpha}$ converges to x in $(\Phi \uplus A, T_{\Phi \uplus A})$ if and only if $\langle s_\beta \rangle_{\beta<\alpha}$ converges to x in (A, d) and hence in (A, T_A) as a consequence of Lemma 5.3.9 and Lemma A.2.69. ☐

On steps we get the following:

Lemma 5.3.11. *A sequence of steps in an ToRS induced by an MRS converges to every object if and only if the sequence of heights of steps converges to* 0 *in the MRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$ be an MRS and let T_A be the topology induced by (A, d). We get $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ with

$$T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O \mid \epsilon \in (0, \infty] \wedge O \in T_A\} \cup \emptyset$$

as ToRS induced by $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$. Now let $\langle \phi_\beta \rangle_{\beta<\alpha}$ be a sequence in Φ.

- $\Rightarrow$ Assume that $\langle \phi_\beta \rangle_{\beta<\alpha}$ converges to every $b \in A$ in $(\Phi \uplus A, T_{\Phi \uplus A})$.

  Let $b \in A$[3], let $\epsilon \in \mathbb{R}^+$ be arbitrary and let $O \in T_A$ be such that $b \in O$. Denote $\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O$ by $O_{\Phi \uplus A}$. We get $O_{\Phi \uplus A} \in T_{\Phi \uplus A}$ and $b \in O_{\Phi \uplus A}$ by the construction of $T_{\Phi \uplus A}$. By assumption $\langle \phi_\beta \rangle_{\beta<\alpha}$ converges to b and we get a β such that for all γ with $\beta \leq \gamma < \alpha$ we have $\phi_\gamma \in O_{\Phi \uplus A}$. That means $\phi_\gamma \in \{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\}$ and hence $h(\phi_\gamma) < \epsilon$. This means that $\lim(\langle h(\phi_\beta) \rangle_{\beta<\lambda}) = 0$ in $(\mathbb{R}^{0+}, d_{\mathbb{R}})$.

- $\Leftarrow$ Assume that $\langle h(\phi_\beta) \rangle_{\beta<\alpha}$ converges to 0 in $(\mathbb{R}^{0+}, d_{\mathbb{R}})$.

  Let $b \in A$ be an arbitrary object and let $\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O$, where $O \in T_A$ and $\epsilon \in (0, \infty]$, be an open set in $T_{\Phi \uplus A}$ such that b is in it. By assumption we get a β such that for all γ with $\beta \leq \gamma < \alpha$ we have $h(\phi_\gamma) < \epsilon$. This means that for all γ with $\beta \leq \gamma < \alpha$ we have $\phi_\gamma \in \{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \subseteq (\{\phi \in \Phi \mid 0 < h(\phi) < \epsilon\} \uplus O)$. This means that $b \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ in $(\Phi \uplus A, T_{\Phi \uplus A})$. ☐

5.3.3 Convergence in Induced ToRSs

For ToRSs induced by an MRSs we get:

[3]The lemma technically does not hold for the case where there are no objects in the ToRS. But in that case, there are no steps in the ToRS and no objects and steps in the MRS and all the theorems depending on this lemma trivially hold.

Theorem 5.3.12. *A transfinite reduction weak metrically converges to some object in an MRS if and only if it weak topologically converges to that object in the ToRS induced by the MRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$ be an MRS, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$. We get $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ as ToRS induced by $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$.

- $\Rightarrow$ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak metrically converges to b reduction in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$.

  By assumption, we get that $\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ converges to b in (A, d) (WMC), so by Lemma 5.3.10 we get that $\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ converges to b in $(\Phi \uplus A, T_{\Phi \uplus A})$ (WTC). So the reduction weak topologically converges to b in our ToRS.

- $\Leftarrow$ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak topologically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$.

  By assumption we get that $\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ converges to b in $(\Phi \uplus A, T_{\Phi \uplus A})$, so by Lemma 5.3.10 we get that $\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)$ converges to b. So the reduction weak metrically converges to b in our MRS. □

It directly follows that:

Proposition 5.3.13. A transfinite reduction is a weak metric reduction in a MRS if and only if it is a weak topological reduction in the ToRS induced by the MRS.

For strong metrical convergence, we get:

Theorem 5.3.14. *A transfinite reduction strong metrically converges to some object in an MRS if and only if it strong topologically converges to that object in the ToRS induced by the MRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$ be an MRS, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$. We get $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ as ToRS induced by $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$.

- $\Rightarrow$ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong metrically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$.

  We get that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak metrically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$, so weak topological convergence follows from Theorem 5.3.12.

  If $\alpha \in \mathrm{LimOrd}$ then we get $\lim(\langle h(\phi_\beta) \rangle_{\beta<\alpha}) = 0$ in $(d_{\mathbb{R}^{0+}}, d_{\mathbb{R}})$ (SMC). So by Lemma 5.3.11 we get $b \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (STC).

  So $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$.

- $\Leftarrow$ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$.

  We get that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak topologically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ so weak metrical convergence follows from Theorem 5.3.12.

  If $\alpha \in \mathrm{LimOrd}$ then we get $b \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (STC). So by Lemma 5.3.11 we get $\lim(\langle h(\phi_\beta) \rangle_{\beta<\alpha}) = 0$ in $(d_{\mathbb{R}^{0+}}, d_{\mathbb{R}})$ (SMC).

  So $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong metrically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), h, d)$. □

It directly follows that:

Proposition 5.3.15. A transfinite reduction is a strong metric reduction in MRS if and only if it is a strong topological reduction in the ToRS induced by the MRS.

This means that the embedding of MRSs in ToRSs is well-behaved in the sense that the notions of metrical convergence of a reduction in an MRS coincides with their respective notions of topological convergence in the ToRS induced by that MRS.

5.3.4 Embedding iTRSs in MRSs

We embed iTRSs in MRSs for the same reasons we embedded iTRSs in ToRSs:

Definition 5.3.16. The MRS **induced** by an iTRS, (Σ, R), is $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), h, d)$ where:

- $(\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$ is the rewrite system induced by (Σ, R) viewed as a TRS
- For all $\phi \in \Phi$ we have $h(\phi) = 2^{-|\,\mathrm{pos}(\phi)|}$.
- d is the metric on $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ from Section 4.1.5.[4]

Lemma 5.3.17. *This gives an actual MRS.*

Proof. Let $\phi \in \Phi$. The step ϕ stems from the rule $\mathrm{rul}(\phi)$ being applied to the term $\mathrm{src}(\phi)$ at position $\mathrm{pos}(\phi)$ to result in the term $\mathrm{tgt}(\phi)$. So for any $p \in \mathrm{Pos}(\mathrm{src}(\phi))$ such that $\mathrm{pos}(\phi)$ is not a non-strict prefix of p we have $(\mathrm{src}(\phi))(p) = (\mathrm{tgt}(\phi))(p)$. This means that $\min(\{|p| \mid (\mathrm{src}(\phi))(p) \neq (\mathrm{tgt}(\phi))(p)\}) \geq |\,\mathrm{pos}(\phi)|$, so $d(\mathrm{src}(\phi), \mathrm{tgt}(\phi)) \leq 2^{-|\,\mathrm{pos}(\phi)|}$. Now, because $h(\phi) = 2^{-|\,\mathrm{pos}(\phi)|}$ we get that $d(\mathrm{src}(\phi), \mathrm{tgt}(\phi)) \leq h(\phi)$. □

[4]We can also take the metric based on the Hausdorff metric alluded to in Section 4.1.5. In fact, that is arguably the better choice.

Here d is the standard choice for a measure of similarity on terms. In it, the more of a 'top part' two terms have in common the more closer to each other (according to the metric) they are considered to be. Positions of a smaller length are therefore somehow considered to be more important than positions of a bigger length. Terms with only their symbol at the root position differing are considered to be at maximum distance (1), while terms with the same root symbol but different symbols at any other position are considered to be at closer distance. For instance, using the signature $\Sigma = \{f/1, g/1, a/0, b/0\}$ we get $d(f(f(a)), g(f(a))) = 1$ while $d(f(f(a)), f(g(b))) = \frac{1}{2}$. In topologies the only notion of similarity arises from the finitely observable properties that two objects share. Here, objects are terms and observable properties arise from having a certain symbol at a certain position. Consequently, in ToRSs our notion of similarity, unlike the metric distance measure d, does not have the privileged status of positions of smaller length.

The choice for h relates 'how much work is done' in a step to the position at which a substitution takes place. This is consistent with considering positions of a smaller length more important than positions of a bigger length.

This way of embedding iTRSs in MRS follows the same principle as embeddings known from the literature ([2, Definition 5.2], [13, p. 4]) but differs slightly because the MRSs defined here differ from those in the literature.

Via this embedding of iTRSs in MRSs and the embedding of MRSs in ToRSs, iTRSs can, indirectly, be embedded in ToRSs. iTRSs also can be embedded into ToRSs directly as done in Section 5.2.6. We can compare both embeddings by comparing the ToRS induced by an iTRS with the ToRS in induced by the MRS induced by the iTRS. These are ToRSs with the same underlying rewrite system (the rewrite system induced by the iTRS), so what remains to be done is comparing the topologies of those ToRSs:

Lemma 5.3.18. *The restriction to objects of the topology of the ToRS induced by the MRS induced by an iTRS is equal to the restriction to objects of the ToRS directly induced by that iTRS.*

Proof. Let (Σ, R) be an iTRS. The set of objects of the ToRS induced by it is $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$. The topology in that ToRS restricted to objects (that is, terms) is generated by subbase

$$\{U_\mathcal{T}(p, f) \mid p \in \mathbb{N}^*, f \in \Sigma\} \text{ with } U_\mathcal{T}(p, f) = \{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\}$$

and has

$$\{\bigcap\{U_\mathcal{T}(p, f) \mid \langle p, f\rangle \in P\} \mid P \subset \mathbb{N}^* \times \Sigma \wedge \mathrm{card}(P) \in \mathbb{N}\}$$

as base. Lets denote the topology by T, the base by B and the subbase by A. The set of objects of the ToRS induced by the MRS induced by the iTRS is also $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$, the topology in that ToRS restricted to objects/term is the topology induced by the metric $(\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), d)$ where d is the standard metric on terms. Lets denote that topology by T_d. By Lemma A.3.11 we get $\{B_\epsilon(t) \mid t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \wedge \epsilon \in \mathbb{R}^+\}$ as base for this topology, denote it by B_d.

- $T_d \subseteq T$.

  We prove that $B_d \subseteq B$ and hence $T_d \subseteq T$.

  Let $B_\epsilon(t) \in B_d$ for some $t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ and $\epsilon \in \mathbb{R}^+$. We get $t' \in B_\epsilon(t)$ if and only if for all $p \in \mathrm{Pos}(t)$ with $2^{-|p|} \geq \epsilon$ we have $t'(p) = t(p)$. Since there are only finitely many of such positions p, we can set $P = \{\langle p, t(p)\rangle \mid 2^{-|p|} \geq \epsilon\}$ and we get that $B_\epsilon(t) = \bigcap\{U_\mathcal{T}(p, f) \mid \langle p, f\rangle \in P\}$ and hence $B_\epsilon(t) \in B$.

- $T \subseteq T_d$.

  We prove that for all $X \in A$ there is a set $\{Y_i \in B_d \mid i \in I\}$ such that $X = \bigcup\{Y_i \mid i \in I\}$. We'll get that the closure of A under finite intersection is a subset of B_d (since B_d is closed under finite intersection), hence the closure of A under finite intersection and arbitrary union will be a subset of the closure of B_d under arbitrary union and hence $T \subseteq T_d$.

  Let $U_\mathcal{T}(p, f) \in A$ ($p \in \mathbb{N}^*$ and $f \in \Sigma$) and use $\{B_{2^{-p}}(t') \mid t'(p) = f\}$ as the mentioned $\{Y_i \in B_d \mid i \in I\}$. Now:

  - $U_\mathcal{T}(p, f) \subseteq \bigcup\{B_{2^{-p}}(t') \mid t'(p) = f\}$
    Let $t \in U_\mathcal{T}(p, f)$ we get that $t(p) = f$, so $t \in B_t(2^{-p})$ and hence $t \in \bigcup\{B_{t'}(2^{-p}) \mid t'(p) = f\}$.
  - $\bigcup\{B_{2^{-p}}(t') \mid t'(p) = f\} \subseteq U_\mathcal{T}(p, f)$
    Let $t \in \bigcup\{B_{2^{-p}}(t') \mid t'(p) = f\}$, we get some t' with $t'(p) = f$ such that $t \in \{B_{2^{-p}}(t') \mid t'(p) = f\}$. We get that $d(t, t') < 2^{-p}$ so the least length of the positions p' where $t(p') \neq t'(p')$ is larger than p and hence $t(p) = t'(p) = f$ and hence $t \in U_\mathcal{T}(p, f)$. □

Since weak topological convergence is only concerned with convergence in the restriction of the topology of a ToRS to objects we get that:

Proposition 5.3.19. The notion of weak topological convergence in the ToRS induced by the MRS induced by an iTRS coincides with the notion of weak topological convergence in the ToRS induced directly by that iTRS.

On the other hand, with regards to steps, the topology of the ToRS induced directly by an iTRS differs greatly from the topology of the ToRS induced by the MRS induced by that iTRS. This is because, when embedding an MRS in a ToRS, steps are topologized only based on their height. This is because that is the only information about steps that is accessible in an MRS, all other structure of steps is abstracted away from. This means that, in the topology of the ToRS induced by the MRS induced by the iTRS, a step is in an open set if and only if all other steps at the same height (and smaller height) are also in it. In the ToRS induced by the iTRS directly, we have access to more information, namely the static parts of a step. So, for instance, an open set in the topology of the ToRS induced by the iTRS might contain all steps having a certain symbol as static part, but might not contain steps at the same height that do not have that symbol as static part. However, it still holds that:

Theorem 5.3.20. *The notion of strong topological convergence in the ToRS induced by an iTRS coincides with the notion of strong topological convergence in the ToRS induced by the MRS induced by that iTRS.*

Proof. Let (Σ, R) be an iTRS, let $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), h, d)$ be the MRS induced by it, let $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T_d)$ be the ToRS induced by that MRS and let $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T)$ be the ToRS induced by the iTRS directly, $\{\bigcap\{U(p, f) \mid \langle p, f\rangle \in P\} \mid P \subset \mathbb{N}^* \times \Sigma \wedge \mathrm{card}(P) \in \mathbb{N}\}$ where $U(p, f) = \{\phi \in \Phi \mid \mathrm{pos}(\phi) \not\leq p \wedge (\mathrm{src}(\phi))(p) = (\mathrm{tgt}(\phi))(p) = f\} \uplus \{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\}$ is a base for this topology, denote it by B. Let $\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle$ be a reduction in $(\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt})$

- Assume that $\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle$ strong topologically converges to b in $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T_d)$.

  We get $b \in \mathrm{Lim}(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle))$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T_d)$ and hence in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T_d)$ restricted to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$. By Lemma 5.3.18 we get $b \in \mathrm{Lim}(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle))$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$ restricted to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ and hence in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$.

  Assume that $\alpha \in \mathrm{LimOrd}$. We get that $b \in \mathrm{Lim}(\langle \phi_\beta\rangle_{\beta<\alpha})$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T_d)$. By Lemma 5.3.11 we get that $\lim(\langle h(\phi_\beta)\rangle_{\beta<\alpha}) = 0$.

  Let P be some finite approximation of b. Since $b \in \mathrm{Lim}(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle))$ we get some $\beta_1 < \alpha$ such that for all γ with $\beta_1 \leq \gamma < \alpha$ we have P is an approximation of $\mathrm{src}(\phi_\gamma)$ and $\mathrm{tgt}(\phi_\gamma)$. Which means that for all $\langle p, f\rangle \in P$ we have $(\mathrm{src}(\phi_\gamma))(p) = \mathrm{tgt}(\phi_\gamma))(p) = f$. Because P is finite we get a position, p^*, of maximal length, $|p^*|$, in P. Now because $\lim(\langle h(\phi_\beta)\rangle_{\beta<\alpha}) = 0$, we get a β_2 such that for all γ with $\beta_2 \leq \gamma < \alpha$ we have $|p^*| \leq |\mathrm{pos}(\phi_\gamma)|$. So for all $\langle p, f\rangle \in P$ we get that $\mathrm{pos}(\phi_\gamma) \not\leq p$. For $\beta = \max(\beta_1, \beta_2)$ get that for all γ with $\beta \leq \gamma < \alpha$ and $\langle p, f\rangle \in P$ both $(\mathrm{src}(\phi_\gamma))(p) = \mathrm{tgt}(\phi_\gamma))(p) = f$ and $\mathrm{pos}(\phi_\gamma) \not\leq p$. This means that P is a static part of ϕ_γ, so $b \in \mathrm{Lim}(\langle \phi_\beta\rangle_{\beta<\alpha})$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$.

- Assume that $\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle$ strong topologically converges to b in $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T)$.

  We get $b \in \mathrm{Lim}(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle))$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$ and hence in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$ restricted to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$. By Lemma 5.3.18 we get $b \in \mathrm{Lim}(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta\rangle_{\beta<\alpha}\rangle))$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T_d)$ restricted to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ and hence in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T_d)$.

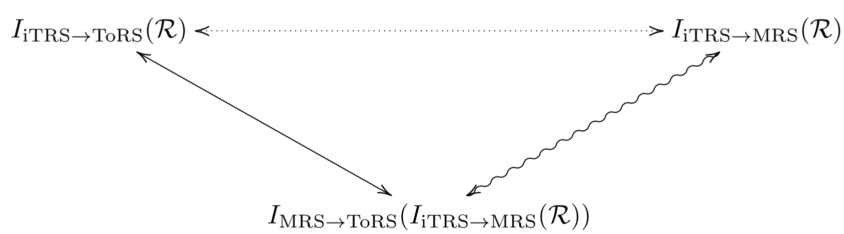
  Assume that $\alpha \in \mathrm{LimOrd}$. We get that $b \in \mathrm{Lim}(\langle \phi_\beta\rangle_{\beta<\alpha})$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T)$. Let $\epsilon \in \mathbb{R}^+$ and let $P = \{\langle p, b(p)\rangle \mid 2^{-|p|} \geq \epsilon\}$. By assumption we get a $\beta < \alpha$ such that for all γ with $\beta \leq \gamma < \alpha$ we have P as static part of ϕ_γ. So $2^{-|\mathrm{pos}(\phi)|} < \epsilon$. This means that $\lim(\langle h(\phi_\beta)\rangle_{\beta<\alpha}) = 0$, which by Lemma 5.3.11 means that $b \in \mathrm{Lim}(\langle \phi_\beta\rangle_{\beta<\alpha})$ in $(\Phi \uplus \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), T_d)$. $\square$

We now both have that:

- The notions of weak and strong (metric) convergence in an MRS coincide with the notions of weak and strong (topological) convergence in the ToRS induced by that MRS (Theorem 5.3.12, Theorem 5.3.14).
- The notions of weak and strong (topological) convergence in the ToRS induced by an iTRS coincide with the notions of weak and strong (topological) convergence in the ToRS induced by the MRS induced by that iTRS (Theorem 5.3.19, Theorem 5.3.20).

Combining, we also get that the notions of weak and strong (metric) convergence in an MRS induced by an iTRS coincide with the notions of weak and strong (topological) convergence in the ToRS induced by that iTRS.

Now, let $\mathcal{R}$ be an iTRS and write $I_{\text{iTRS}\to\text{ToRS}}$ for the function giving the ToRS induced by an iTRS, $I_{\text{iTRS}\to\text{MRS}}$ for the function giving the MRS induced by an iTRS and $I_{\text{MRS}\to\text{ToRS}}$ for the function giving the ToRS induced by an MRS. Now we get the following diagram:



Here the arrows can be interpreted as stating an equivalence of notions of convergence of reductions. The equivalence displayed as $\to$ is proven by Theorem 5.3.19 and Theorem 5.3.20. The equivalence displayed as $\rightsquigarrow$ is an instance of what is proven in Theorem 5.3.12 and Theorem 5.3.14. And the equivalence displayed by the dotted arrow is obtained combining the other two equivalences.

This shows that the embeddings of MRSs in ToRSs, iTRSs in ToRSs and iTRSs in MRSs are well-behaved with respect to each other and convergence of reductions in the sense that when combined in these various ways, convergence of reductions is preserved.

Because of this, giving examples of how metric convergence of reductions in iTRS (embedded in MRSs) works out is not as interesting. It works out it the same way as topological convergence of reductions in iTRSs (embedded in ToRSs) does. For good measure though:

Example 5.3.21. Moving example Example 5.2.18 to a metric setting, our iTRS is still (Σ, R) with $\Sigma = \{a/0, f/1\}$ and $R = \{a \to f(a)\}$. Denote the MRS induced by this iTRS by $((\Phi, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \text{src}, \text{tgt}), h, d)$. Our reduction is

$$a \to_\epsilon f(a) \to_1 f(f(a)) \to_{11} f(f(f(a))) \to \dots$$

Its sequence of objects is $\langle a, f(a), f(f(a)), f(f(f(a))), \dots\rangle$. We get $d(a, f^\omega) = 2^{-0} = 1$, $d(f(a), f^\omega) = 2^{-1} = \frac{1}{2}$, $d(f(f(a)), f^\omega) = 2^{-2} = \frac{1}{4}$, $d(f(f(f(a))), f^\omega) = 2^{-3} = \frac{1}{8}$, etc. So, for every positive distance, the distances between members of the sequence of objects and f^ω eventually get below it. That means that $\lim(\langle a, f(a), f(f(a)), f(f(f(a))), \ldots\rangle) = f^\omega$ in $(\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), d)$, so the reduction weak metrically converges to f^ω. The sequence of heights of this reduction is $\langle 1, 2^{-1}, 2^{-2}, 2^{-3}, \ldots\rangle$. We have $\lim(\langle 1, 2^{-1}, 2^{-2}, 2^{-3}, \ldots\rangle) = 0$ in $(\mathbb{R}^{0+}, d_\mathbb{R})$, so this reduction also strong metrically converges to f^ω.

Embedding wide TRSs and wide iTRSs in MRSs in the way iTRSs are embedded in MRSs is possible. The same metric on terms and the same height measure on steps can be used, but it wont have desirable results.

Example 5.3.22. Adapting the horizontal convergence example of Example 5.2.22 to a MRS setting would not yield (strong or weak) convergence of

$$f(a, a, a, a, \ldots) \to_1 f(b, a, a, a, \ldots) \to_2 f(b, b, a, a, \ldots) \to_3 f(b, b, b, a, \ldots) \to \ldots$$

to $f(b, b, b, b, \ldots)$. For all terms in the reduction we have that they differ from $f(b, b, b, b, \ldots)$ at depth 1 and hence the distance between $f(b, b, b, b, \ldots)$ and terms in the reduction is $2^{-1} = \frac{1}{2}$.

For wide iTRSs with symbols with an arity of at most ω we have:

Theorem 5.3.23. *The restriction to objects of the topology of any ToRS induced by a wide iTRSs with symbols with an arity of at most ω is T_1, regular and second-countable and hence metrizable.*

Proof. Let (Σ, R) be a wide iTRS such that for all $f \in \Sigma$ we have $\mathrm{ar}(f) \leq \omega$. Let T be the restriction to $\mathcal{T}^\omega_w(\Sigma, \mathcal{X}_\Sigma)$ of the topology of the ToRS induced by (Σ, R). $\{\bigcap\{U_\mathcal{T}(p, f) \mid \langle p, f\rangle \in P\} \mid P \subset \mathbb{N}^* \times \Sigma \wedge \mathrm{card}(P) \in \mathbb{N}\}$ where $U_\mathcal{T}(p, f) = \{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\}$ is a base for T, denote it by B.

- T is T_1.

  Let $t, s \in \mathcal{T}^\omega_w(\Sigma, \mathcal{X}_\Sigma)$ such that $t \neq s$, we get some $p \in (\mathrm{Pos}(t) \cap \mathrm{Pos}(s))$ such that $t(p) \neq s(p)$. $U_\mathcal{T}(p, t(p))$ and $U_\mathcal{T}(s, t(s))$ are (sub)base sets and hence open. We get $t \in U_\mathcal{T}(p, t(p))$ and $s \in U_\mathcal{T}(p, s(p))$, while $s \notin U_\mathcal{T}(p, t(p))$ and $t \notin U_\mathcal{T}(p, s(p))$.

- T is second-countable.

  $\mathbb{N}^*$ is countable by [5, example 1.13], Σ is countable because signatures are required to be countable, so $\mathbb{N}^* \times \Sigma$ is countable ([5, example 1.2]). This means that it also has countably many finite subsets ([5, example 1.9]), so B is countable because there is exactly one base set in B for every finite P such that $P \subset \mathbb{N}^* \times \Sigma$.

- T is regular.

  Let $C \subseteq T$ be non-empty and closed and let $t \in \mathcal{T}_w(\Sigma, \mathcal{X}_\Sigma)$ be such that $t \notin C$, we get that $T \setminus C$ is open and $t \in (T \setminus C)$. Since $\{U_\mathcal{T}(p, f) \mid p \in \mathbb{N}^* \wedge f \in \Sigma\}$ is a subbase for T, we get some $p \in \mathbb{N}^*$ and $f \in \Sigma$ such that $U_\mathcal{T}(p, f) \subseteq (T \setminus C)$ and $x \in U_\mathcal{T}(p, f)$. Now the complement of

$U_\mathcal{T}(p, f)$ is the set of all terms t' such that $t'(p) \neq f$. To prove that this set is the union of base sets and hence open:

We get that either $t'(p) \in (\Sigma \setminus \{f\})$ or $t(p)$ is undefined. The set of all t' such that $t'(p) \in (\Sigma \setminus \{f\})$ is the union of all $U_\mathcal{T}(p, f')$ for $f' \in (\Sigma \setminus \{f\})$ which are base sets. The set of all t' such that $t'(p)$ is undefined is the set of terms for which there is some index n of the string p with q as the prefix of p up to n such that $\mathrm{ar}(t'(q)) < p(n)$. This is the union of all sets $U_\mathcal{T}(q, f')$ where q is again the prefix of p up to some n and $f' < p(n)$.

So $T \setminus U_\mathcal{T}(p, f)$ is the union of base sets and hence open. Since $U_\mathcal{T}(p, f) \subseteq (T \setminus C)$ we get $C \subseteq (T \setminus U_\mathcal{T}(p, f))$ and hence $T \setminus U_\mathcal{T}(p, f)$ is a neighborhood of C. So $U_\mathcal{T}(p, f)$ and $T \setminus U_\mathcal{T}(p, f)$ separate t and C.

Now by Theorem A.3.13 we get that T is metrizable. □

We also get that the entire topology of a ToRS induced by a wide iTRS with symbols with an arity of at most ω is regular, second-countable and hence metrizable. That means that we could find a way in which these restricted wide iTRSs induce MRSs that does give desirable (equal to ToRS case) convergence behaviour. Although the metric involved is not necessarily intuitive or easy to find. For wide iTRSs in general though, the situation is worse.

Theorem 5.3.24. *The topology of a ToRS induced by a wide iTRS restricted to objects (terms) might not be first-countable and hence not metrizable.*

Proof. Let (Σ, R) be an wide iTRS where $\Sigma = \{f/\omega_1, a/0, b/0\}$ (ω_1 is the first uncountable ordinal). Let T be the restriction to $\mathcal{T}^\omega_w(\Sigma, \mathcal{X}_\Sigma)$ of the topology of the ToRS induced by (Σ, R). As base for T we get

$$\{\bigcap\{U_\mathcal{T}(p, f) \mid \langle p, f\rangle \in P\} \mid P \subset \mathbb{N}^* \times \Sigma \wedge \mathrm{card}(P) \in \mathbb{N}\}$$

where $U_\mathcal{T}(p, f) = \{t \in \mathcal{T}^\omega_w(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\}$.

Let $t \in \mathcal{T}^\omega_w(\Sigma, \mathcal{X}_\Sigma)$ be such that $t(\epsilon) = f$ and for any $\alpha < \omega_1$ we have $t(\alpha) = a$. For any $\alpha < \omega_1$ we have that $t \in U_\mathcal{T}(\alpha, a)$ and $U_\mathcal{T}(\alpha, a) \neq U_\mathcal{T}(\beta, a) \Leftrightarrow \alpha \neq \beta$, giving uncountably many distinct subbase sets having t as member, these sets are also base sets. Any base set is defined by some $P \subset \mathbb{N}^* \times \Sigma$ such that P is finite. For any base set $\bigcap\{U_\mathcal{T}(p, f) \mid \langle p, f\rangle \in P\}$ we get $\bigcap\{U_\mathcal{T}(p, f) \mid \langle p, f\rangle \in P\} \subseteq U_\mathcal{T}(\alpha, a) \Leftrightarrow \langle \alpha, a\rangle \in P$. Since P is finite, any base set can only be contained in finitely many sets in $\{U_\mathcal{T}(\alpha, a) \mid \alpha < \omega_1\}$. Now suppose for contradiction that T is first-countable, we get a countable local basis for x. So, by Lemma A.2.50 we get a set Y of countably many base sets such that for any $Y_i \in Y$ we have $x \in Y_i$ and for any base set A with $x \in A$ there is a $Y_i \in Y$ such that $Y_i \subseteq A$. Now $\{U_\mathcal{T}(\alpha, a) \mid \alpha < \omega_1\}$ gives uncountably many of such base sets A, while each of the countably many $Y_i \in Y$ can only be in finitely many such base sets. Contradiction, T is not first-countable. So we get that T is not metrizable. □

This means that the topology of a ToRS induced by a wide iTRS might not be first-countable. Which in turn means that, in general, wide iTRSs can not be embedded in MRSs in the same way they are embedded in ToRSs.

5.4 PRSs

In [2, 12] an approach to transfinite abstract rewriting using partial orders is defined. As mentioned though the general approach in [2] is slightly lacking in that reductions are sequences of steps where the empty reduction starts and ends in every object. Very similar to this approach (and fixing the deficiency with our already defined transfinite reductions which have an explicit start object), we can define:

Definition 5.4.1. A **partial rewrite system (PRS))** is a structure $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ such that:

- $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ is a rewrite system.
- ctx is a function from Φ to A.
- $(A, \leq)$ is a continuous bounded complete dcpo.
- For all $\phi \in \Phi$ we have $\mathrm{ctx}(\phi) \leq \mathrm{src}(\phi), \mathrm{tgt}(\phi)$.

The order $\leq$ expresses the structure of A, it can be interpreted as an informational ordering on objects. Objects above other objects contain at least the same informational content as the other object (and possibly more). $a \leq b$ is to be interpreted as "b contains at least all the information that a contains". $\mathrm{ctx}(\phi)$ gives the **context** of a step $\phi \in \Phi$, it expresses "what part of the object transformed in the step is not changed by the step", "what part of the object remains untouched by the step" or "what part of the object remains stable during the step". Requiring the context of a step to be smaller than both source and target of the step is consistent with this interpretation as what remains unchanged in a step is necessarily of lesser informational value than both the object 'before' and 'after' the step.

Requiring $(A, \leq)$ to be a continuous bounded complete dcpo is a stronger demand than has been made in the literature. In [2, p. 12], for instance, $(A, \leq)$ is only required to be a partial order, though it is required to be a bounded complete dcpo for several properties to hold. Still, I think these requirements are very justified. Much of this justification comes from domain theory ([6]) where, indeed, orders are interpreted as informational orderings.

Requiring partial orderedness of our set of objects is an obvious first step. Having reflexivity (and hence using non-strict orders instead of strict orders) is the standard in the literature, but given that for any partially ordered set we have an associated strictly ordered set (Lemma A.4.29) we could probably also work with strictly ordered sets and hence with irreflexivity. An advantage of the reflexive approach is that, in the presence of reflexivity, the basic statement $a \leq b$ is to be interpreted as "b contains all the information that a contains", where, using the irreflexive approach, $a < b$ would have to be interpreted as "b contains all the information that a contains and more", which is obviously more complex.

Since we are in the business of ordering we are committed to transitivity. Also, the relation "containing at least all the information that X contains" is a transitive one, no matter how we look at it. If a contains at least the information that b contains and b contains at least the information that c contains then a contains at least the information that c contains by any natural interpretation of what it is to contain information. What this comes down to is that containment, in any sense, is a transitive notion.

Also, interpreting our order as an informational ordering, anti-symmetry also comes naturally. If we have $x \leq y$ and $y \leq x$ then x and y contain the exact same information. Anti-symmetry states that they then *are* the same, and hence actually is an extensionality-like principle, stating that objects *are* the information they contain.

To justify our other properties, let us first explain what a couple of notions that those properties rely on come down to when using our intuition of an informational ordering:

- Upper bound. An upper bound of a set is an object that contains at least all the information that is contained in the members of the set (and possibly more). Therefore the existence of an upper bound of a set guarantees that the information contained in the objects in the set is non-contradictory and can be conglomerated. This means that a set with an upper bound is consistent in the information it provides in a strong sense (the upper bound being a witness to its consistency).

- Least upper bound. A least upper bound of a set is a minimal (as formalized by the order) object that contains all the information that is contained in the members.

- Directed set. In a directed set any two objects, a and b, have an upper bound, c, in the set ($a, b \leq c$). So the information contained in c, contains the information contained in both a and b. First of all, like having an upper bound, this also guarantees the consistency of the information in the set in a sense, albeit a weaker sense. No two pieces of information in the set directly contradict each other. The sort of consistency guaranteed by a set having an upper bound implies this type of consistency but not vice versa. Secondly, since c is also in the directed set, the information in the set builds up in some way (and is directed in that way). The information contained in any two objects (a and b) in the set is extended by another object (c). In this sense (directed sets being consistent in some way and directed or building up information) a directed set can be thought of as being coherent in the information it provides, and as converging in the sense that more and more information in it can be combined by taking upper bounds of pairs of objects. A least upper bound of such a set is an object that contains the information provided by this set and nothing less than necessary. That way, it can be viewed as limit of the set.

- Way-belowness. If directed sets are coherent sets that are converging to their least upper bound then directed sets which do not contain their least upper bound (and hence are infinite by definition) can be said to properly converge to it. In such a set there are infinitely many objects strictly above any object and hence between that object and the least upper bound. Now, some object a is way below some other object b if, for any least upper bound of a directed set that b is below, a is already below some object actually in the set. Directed sets that are not properly converging to their least upper bound contain their least upper bound (their greatest member) so, in that case, a being way below b is simply the case if $a \leq b$. But for properly converging directed sets it is also required that a is *way* below b, meaning that the gap between a and b is 'big enough' to bridge any infinity between least upper bounds of directed sets above b and the objects in those sets. With our informational ordering intuition this means that: if we have some coherent, converging set, containing information such that the total information (in its least upper bound) contains the information contained in b, then the information in a is already contained in some object in that set. Even if the convergence is proper and the set infinite. So, if the information in b is contained in the information that is the result of some converging process, then the information in a is already there at some finite stage of the process. In this sense a is *way* simpler than b.

- Compactness. Where way-belowness is a relative notion of simpleness of objects, compactness is an absolute notion of simpleness. An object is compact if it is way below itself, meaning that if it is below the least upper bound of a directed set then it is also below some object in the directed set. So, in this sense compact items are so low in the order that there is no process of (possibly proper) convergence needed to get to them. They are the objects containing the basic pieces of information.

With these intuitions, requiring our set of objects, ordered by informational containment, to be a dcpo seems intuitive. It requires that all directed sets have a least upper bound and hence requires that every set which looks as if it is coherent and converging in some structural way (is directed) actually has a limit (least upper bound). Since we are in the business of modeling convergence behaviour, this is a very desirable property. It states simply that we can conglomerate information that is known to be coherent and converging, which also seems intuitive. In fact, in domain theory, having structures that are at least dcpos as objects of interest is standard.

Requiring bounded completeness is requiring that any set with an upper bound also has a least upper bound. With the intuitions above, this means that we can conglomerate the information in a set that we know is consistent in a minimal way. This seems to conform with intuitions we have about information and is very useful when we are interested in convergence. It gives us a specific piece of information with a nice property (being minimal) when we conglomerate information.

Requiring continuity of our ordered set is requiring that the set of objects way below any object is directed and that this object is its least upper bound.

With the intuitions above, this means that we can express any object as the limit of the (directed) set of objects simpler than it. So, due to our ordered set being a dcpo, we can always conglomerate information that is known to be coherent and converging and hence get more complex information. Due to continuity we can also disassemble this more complex information into a coherent and converging set of simpler information by continuity. This also shows that no hidden information was introduced when conglomerating in the first place. All the information in a complex piece of information actually comes from the more simple pieces of information that the complex piece is built of. In this sense, requiring continuity is requiring a very handy well-behavedness property on our convergence structures. If we have the result of some proper convergence process, we want to be able to get the objects that converged to it back from that result.

From a more technical point of view, these requirements shouldn't be too strong since the intended applications (most importantly, the embedding of TRSs) seem to meet these demands. In the PRSs of [2], continuity of the ordered set of objects is not required (or mentioned). However, it seems necessary to be able to use Proposition A.2.64, which provides a link between partial orders and general topology. Furthermore, our partially ordered sets being dcpos seems elegant because we will be using Scott topologies (which are intended as topologies on dcpos) on them. And finally, bounded completeness seems necessary because it implies existence of all limit inferiors of sequences (Lemma A.4.25) which we need for Lemma 5.4.7.

We could also have chosen to require algebraicity. Algebraicity means that any object can be expressed as the least upper bound of all compact objects below it. So, with our informational ordering intuitions, we can disassemble information (no matter how complex and infinite) into coherent and converging sets of only basic information pieces. This is another pretty well-behavedness property, even stronger than that of continuity (as formally proven by Lemma A.4.23). In fact it relates to continuity as compactness relates to way-belowness. Continuity lets us disassemble complex information into relatively simpler information. Algebraicity lets us disassemble complex information into information that is simple in an absolute sense. If we had required algebraicity, our ordered sets would have been **Scott Domains** (algebraic bounded complete dcpo's). I choose not to require it though, since I do not seem to need it and want to keep our requirements down to a minimum. Continuity seems to be sufficient to keep convergence well-behaved enough for our purposes, so by Occam's razor only continuity is demanded for PRSs.

Finally, one more definition on PRSs:

Definition 5.4.2. If $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ is a transfinite reduction in a PRS then we can call $\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha}$ its **sequence of contexts**.

5.4.1 Partial Convergence

Now we can define weak and strong convergence using order theoretic tools.

Definition 5.4.3. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, **weak partially converges** to an object $b \in A$ if:

- $\liminf(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)) = b$ WPC

  (The limit inferior of the sequence of objects of the reduction is b)

Here b is said to be a **weak partial limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$.

This convergence definition is again pretty straight-forward. Because PRSs use partial orders to express the structure of the set of objects, the convergence definition is in terms of this partial order. Taking limit inferiors is a standard way to get limits of sequences in an ordered set. Because the definition is about weak convergence, like we did for weak topological convergence and weak metric convergence, we only involve the sequence of objects of the reduction. However, when we define in terms of limit inferiors the intuition behind the definition is not that sequence members look to be more and more like the limit (as it is in our metric case). Using our informational ordering intuitions, it is that the *part* of the information that the objects in some tail of the sequence have in common gets more and more like the limit as we take tails further and further on in the sequence. In other words we define the limit as the collection of the parts of information that eventually stay present in the objects in the sequence, hence the partiality. The ('total') objects in the sequence might not get very similar to the limit at all, only the collection of the parts of the information that eventually stay present in those objects does. In this sense the sequence approximates the information in the limit inferior.

Definition 5.4.4. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, **strong partially converges** to an object $b \in A$ if:

- $\liminf(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)) = b$ when $\alpha \notin \mathrm{LimOrd}$ SPC-S

  (If the reduction is of 0 or successor ordinal length then b is the limit inferior of the sequence of objects of the reduction)

- $\liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) = b$ when $\alpha \in \mathrm{LimOrd}$ SPC-L

  (If the reduction is of limit ordinal length then b is the limit inferior of the sequence of contexts of the reduction)

Here b is said to be a **strong partial limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$.

This definition is less obvious than our previous notions of convergence of a reduction. One might have expected the first clause to have been exactly WPC, that is, have the restriction to non-limit ordinals omitted. Then we would get that any reduction that strong partially converges to an object also weak partially converges to that object. Analogously to how strong topological and strong metric convergence relate to weak topological and weak metric convergence. However, this would pose a problem. Definition 5.4.4 and a possible definition analogous to the topological and metric cases (with the restriction to limit ordinals omitted in the first clause) would agree for reductions where the limit inferior of the sequence of contexts is equal to the limit inferior of the sequence of objects and for reductions of 0 or successor ordinal length. However, as proven later in this section (Lemma 5.4.7), we have that the limit inferior of the sequence of contexts is always smaller to than or equal to the limit inferior of the sequence of objects. Indeed, the limit inferior of the sequence of contexts might be strictly smaller that the limit inferior of the sequence of objects. If that is the case and if those sequences are of limit ordinal length, then the effect of the proposed definitions differ (as will be exemplified in Example 5.4.40). In that case, by Definition 5.4.4, the reduction would strong partially converge to the limit inferior of the sequence of contexts. However, since the limit inferior of the sequence of contexts is different from the limit inferior of the sequence of objects, the reduction wouldn't strongly converge at all according to the proposed definition analogous to our topological and metric cases.

This isn't very intuitive. Intuitively, the limit inferior of the sequence of objects is the collection of the parts of information in the objects in the sequence that eventually stay present in the reduction. The limit inferior of the sequence of contexts is the collection of the parts of information in the objects in the sequence that eventually stay untouched by the steps. In the case where the limit inferior of the sequence of contexts is strictly smaller is than the limit inferior of the sequence of objects, the limit inferior of the sequence of objects contains all the information that the limit inferior of the sequence of contexts does and more. That means that the information that eventually stays untouched by steps in the reduction eventually stays present in the reduction (as per Lemma 5.4.7), while there is 'extra' information that eventually stays present, but not untouched by steps. So by our intuition of an informational ordering, the limit inferior of the sequence of contexts is exactly the collection of the parts of information that eventually stay present in the object *and* stay untouched by the steps in the reduction. It seems reasonable to consider that to be enough to demand from a strong partial limit. Definition 5.4.4 does so. We shouldn't want the reduction not to strong partially converge simply because this limit does not contain *all* the information that eventually stays present in the reduction (as the definition analogous to the topological/metric case would). Especially not because we are trying to express *partial* convergence.

By Lemma A.4.25 we get that every reduction in a PRS has a weak and a strong partial limit. This means that there is no notion of partial divergence that is directly analogous to the notions of metric and topological divergence. Also, limits are unique by definition of limit inferiors, so any reduction in a PRS weak partially converges to exactly one object and strong partially converges to exactly one object. As can be gathered from the preceding discussion, these limit-objects are not necessarily the same.

Since the limit inferior of any sequence with successor length is its last member (Lemma A.4.26), the definitions of partial convergence can be simplified to:

- If a reduction is of length 0, its weak and strong partial limit is the only and last member of its sequence of objects, its start.

- If a reduction is of successor ordinal length, its weak and strong partial limit is the last member of its sequence of objects, the target of its last step.
- If a reduction is of limit ordinal length, its weak partial limit is the limit inferior of its sequence of objects and its strong partial limit is the limit inferior of its sequence of steps.

Definition 5.4.5. A **weak partial reduction** is transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, such that for every limit ordinal $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle$ weak partially converges to $\mathrm{src}(\phi_\lambda)$.

This means that a weak partial reduction is a reduction such that:

$$\forall \lambda \in \mathrm{LimOrd}(\alpha).\ \liminf(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle)) = \mathrm{src}(\phi_\lambda)$$

Weak partiality of a reduction is a partial property that expresses that a reduction, in some sense, behaves properly at limit ordinal indexes.

Definition 5.4.6. A **strong partial reduction** is transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in an PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, such that for every limit ordinal $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\langle a, \langle \phi_\beta \rangle_{\beta<\lambda} \rangle$ strong partially converges to $\mathrm{src}(\phi_\lambda)$.

This means that a strong partial reduction is a reduction such that:

$$\forall \lambda \in \mathrm{LimOrd}(\alpha).\ \liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\lambda}) = \mathrm{src}(\phi_\lambda)$$

Strong partiality of a reduction is the other, stronger, partial property that expresses that a reduction behaves properly at limit ordinals.

Like weak and strong topologicality and weak and strong metricality of reductions, both weak and strong partiality of reductions is preserved under concatenation.

5.4.2 Weak versus Strong Partial Convergence

As mentioned, unlike with weak and strong metrical convergence, we don't get a result as strong as "any strong partial reduction is a weak partial reduction". This is because elements of the sequence of contexts of a reduction might be strictly smaller than elements of the sequence of objects of the reduction. And because of this, the sequence of contexts might have a strictly smaller limit inferior than the sequence of objects. As mentioned, this will be exemplified by Example 5.4.40. We do have that:

Lemma 5.4.7. $\liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) \leq \liminf(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$.

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ be a PRS and let $\langle \phi_\beta \rangle_{\beta<\alpha}$ be a sequence in Φ.

For every $\beta < \alpha$ we have $\mathrm{ctx}(\phi_\beta) \leq \mathrm{src}(\phi_\beta), \mathrm{tgt}(\phi_\beta)$. So for all $\beta < \alpha$ we get that $\prod\{\mathrm{ctx}(\phi_\gamma) \mid \beta \leq \gamma < \alpha\} \leq \prod\{\mathrm{src}(\phi_\gamma), \mathrm{tgt}(\phi_\gamma) \mid \beta \leq \gamma < \alpha\}$. And hence $\liminf(\langle \mathrm{ctx}(\phi_\beta)\rangle_{\beta<\alpha}) \leq \liminf(\mathrm{src}\,{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle))$. □

Remark. Had we chosen not to require bounded completeness for our ordered set of objects, limit inferiors of sequences would not need to exist in general. Specifically, the existence of the limit inferior of the sequence of contexts of a reduction would not imply the existence of a limit inferior of the sequence of objects (and vice versa). As such, the above lemma would have to be heavily quantified. This also motivates the choice of requiring bounded completeness.

So, lacking the result that any strong partial reduction is a weak partial reduction, we do get that:

Theorem 5.4.8. *Any transfinite reduction in a PRS that strong partially converges to some maximal object also weak partially converges to that object.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ be a PRS and let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle$ be a transfinite reduction in it that strong partially converges to a maximal object $b \in A$. Depending on α we get one of the two following cases:

- $\alpha \in \mathrm{LimOrd}$. We get that $b = \liminf(\langle \mathrm{ctx}(\phi_\beta)\rangle_{\beta<\alpha})$ by SPC-L. We get $\liminf(\langle \mathrm{ctx}(\phi_\beta)\rangle_{\beta<\alpha}) \leq \liminf(\mathrm{src}\,{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle))$ by Lemma 5.4.7 so $b \leq \liminf(\mathrm{src}\,{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle))$. Now because b is maximal we get $b = \liminf(\mathrm{src}\,{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle))$ and we have WPC.
- $\alpha \notin \mathrm{LimOrd}$. We directly get WPC from SPC-S. □

Also, if we define the following:

Definition 5.4.9. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle$ in a PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, is said to be **maximal at limits** if for all $\lambda \in \mathrm{LimOrd}(\alpha)$ we have that $\mathrm{src}(\phi_\lambda$ is maximal with respect to $\leq$.

It directly follows from Theorem 5.4.8 that:

Proposition 5.4.10. Any strong partial reduction in a PRS is a weakly partial if it is maximal at limits.

The property of being maximal at limits is a weak property. In [2, 13] the following is defined:

Definition 5.4.11. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha}\rangle$ in a PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, is said to be **total** if all elements of its sequence of objects are maximal with respect to $\leq$.

This is a stronger property than a reduction being maximal at limits. So we also have that any strong partial reduction in a PRS is a weakly partial if it is total, as [2, 13] shows. The property of totality of a reduction is interesting in itself since every reduction of non-partial terms in the PRS induced by an iTRS is total (Section 5.4.7). Still, only assuming the weaker requirement of maximality at limits, as done in Proposition 5.4.10, seems the better choice as requirement for a "any strong partial reduction is weakly partial"-like theorem. This is motivated by the fact that in abstract rewriting we aren't necessarily concerned with TRSs, so the fact that a weaker requirement yields a stronger theorem might be useful in other applications of transfinite abstract rewriting.

5.4.3 $\mathscr{S}$-convergence

In the discussion of strong partial convergence the following intuition was the reason for having Definition 5.4.4 instead of a definition analogous to the topological/metric case. "We shouldn't want to exclude an object as possible limit of a reduction (and hence have the reduction not converge at all) simply because the possible limit does not contain *all* the information that eventually stays present in the reduction, but rather all the information that eventually stays present *and* stays untouched in the reduction." This gives rise to an interesting alternative to the notion of partial convergence of reductions. Instead of using limit inferiors we might use $\mathscr{S}$-limits (Definition A.4.27).

Definition 5.4.12. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, **weak $\mathscr{S}$-converges** to an object $b \in A$ if:

- $b \leq \liminf(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ WSC

  (b is an $\mathscr{S}$-limit of the sequence of objects)

b is said to be a **weak $\mathscr{S}$-limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$.

Definition 5.4.13. A transfinite reduction $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ in a PRS, $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, **strong $\mathscr{S}$-converges** to an object $b \in A$ if:

- $b \leq \liminf(\mathrm{src}{\prime}(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ WSC
- $b \leq \liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) = b$ when $\alpha \in \mathrm{LimOrd}$ SSC

  (If the reduction is of limit ordinal length then b is an $\mathscr{S}$-limit of the sequence of contexts)

b is said to be a **strong $\mathscr{S}$-limit** of $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$.

Using our informational ordering intuition, the limit inferior of a sequence of objects is the collection of the parts of information contained in the objects that eventually stay present through the sequence. $\mathscr{S}$-limits are the objects smaller than it. Hence, a $\mathscr{S}$-limit is an object such that the information it contains eventually stays present in the objects of the sequence (while there might be information that eventually stays present in the objects of the sequence that is not contained in the $\mathscr{S}$-limit). Sequences approximate information in $\mathscr{S}$-limits in a similar sense they approximate information in limit inferiors. The only difference is that a sequence might strictly overapproximate the information in $\mathscr{S}$-limits, where it approximates the information in limit inferiors exactly. This means that a weak $\mathscr{S}$-limit of a reduction is an object such that the information it contains eventually stays present in the reduction, while the strong $\mathscr{S}$-limit of a reduction is an object such that the information it contains eventually stays both present and untouched in the reduction. Comparing this to the definitions of weak partial and strong partial convergence: the weak partial limit of a reduction is the object such that the information it contains is *all* the information eventually stays present in the reduction and the strong partial limit of the reduction is the object such that the information it contains is *all* the information that eventually stays both present and untouched in the reduction.

The advantage of these alternative notions of convergence based on $\mathscr{S}$-limits is that they are analogous to the topological and metric notions in that every strong partial limit of a reduction is also a weak partial limit of the reduction by definition, while they retain the partiality of the notion and the fact limits always exist. Also, we are looking for notions of partial convergence and partial limits. In that light, having a notion of convergence with as intuition that the information that a limit contains is information that eventually stays present in the sequence, but might only be *a part* of the total of information that eventually stays present in the sequence, is not so strange. Furthermore, using $\mathscr{S}$-limits instead of limit inferiors fits in a bit better with our new topological approach (of ToRSs). This is because the $\mathscr{S}$-limits of a sequence are exactly the topological limits of the sequence in the induced Scott-topological space (Lemma A.2.62), while the limit inferior of a sequence is just one of its topological limits. However, since a sequence might have multiple $\mathscr{S}$-limits, this approach does lose uniqueness of limits. Also, I'm not aware of literature using $\mathscr{S}$-limits in partial rewriting contexts. In the literature ([2]), limit inferiors are used.

5.4.4 Embedding PRSs in ToRSs

Like metrics, orders induce topologies. Every dcpo has a Scott topology associated with it (Definition A.2.57) and two different dcpos have different associated Scott topologies (Lemma A.2.61). Every topological space also has an order associated with it, its specialization quasi-order (Definition A.2.36), but different topologies might have the same specialization quasi-order (Example A.2.37). We do however have that the specialization quasi-order induced by a Scott topology induced by a dcpo is that dcpo (Lemma A.2.60). And also, the $\mathscr{S}$-limits of a sequence in a continuous dcpo are exactly the topological limits of this sequence in the induced Scott topology. This means that we can go back-and-forth between continuous dcpos and Scott topologies with a well-behaved one-to-one mapping that preserves sequence limits. Mapping orders to topologies comes down to taking the Scott topology of the order, mapping topologies to orders comes down to taking the specialization quasi-order of the topology. In general though, the mapping is not one-on-one because of the different topologies that have the same specialization quasi-order associated with it. However since we are only concerned with continuous dcpos in the PRS formalism, this suggests that we can have a well-behaved embedding of PRSs in the ToRS formalism. This is one of the reasons mentioned above for restricting PRSs to continuous dcpos.

A PRS induces a ToRS in the following way:

Definition 5.4.14. The ToRS **induced by** a PRS $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$ is $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$ where $T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ and $T_{\mathscr{S}}$ is the Scott topology induced by $(A, \leq)$.

Involving the Scott topology on the ordered set of objects should be an obvious choice in light of the discussion above. Topologizing steps via their context objects should also be straightforward since steps are ordered through these context objects in PRSs.

To see that this is an actual ToRS we need to prove that:

Lemma 5.4.15. *$T_{\Phi \uplus A}$ is a topology on $\Phi \uplus A$*

Proof. We have $T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$, so we need to prove that:

- $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ is closed under finite intersection.

  Let $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\}$ be a finite family of open sets in $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$. We get:

  $$\begin{aligned}\bigcap\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\} =& \bigcap\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} : i \in I\} \\ &\uplus \bigcap\{O_i : i \in I\}\end{aligned}$$

  Since $T_{\mathscr{S}}$ is a topology it is closed under finite intersection, so there is an $O^* \in T_{\mathscr{S}}$ such that $\bigcap\{O_i : i \in I\} = O^*$. Also:

  $$\begin{aligned}\bigcap\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} : i \in I\} =& \{\phi \in \Phi \mid \text{ctx}(\phi) \in \bigcap\{O_i : i \in I\}\} \\ =& \{\phi \in \Phi \mid \text{ctx}(\phi) \in O^*\}\end{aligned}$$

  So we get:

  $$\bigcap\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\} = \{\phi \in \Phi \mid \text{ctx}(\phi) \in O^*\} \uplus O^*$$

  Which means that:

  $$\begin{aligned}&\bigcap\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\} \in \\ &\qquad \{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}\end{aligned}$$

  And hence $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ is closed under finite intersection.

- $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ is closed under arbitrary union.

  Let $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\}$ be an arbitrary family of open sets in $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$. We get:

  $$\begin{aligned}\bigcup\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\} =& \bigcup\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} : i \in I\} \\ & \uplus \bigcup\{O_i : i \in I\}\end{aligned}$$

  Since $T_{\mathscr{S}}$ is a topology it is closed arbitrary union, so there is an $O^* \in T_{\mathscr{S}}$ such that $\bigcup\{O_i : i \in I\} = O^*$. Also:

  $$\begin{aligned}\bigcup\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} : i \in I\} =& \{\phi \in \Phi \mid \text{ctx}(\phi) \in \bigcup\{O_i : i \in I\}\} \\ =& \{\phi \in \Phi \mid \text{ctx}(\phi) \in O^*\}\end{aligned}$$

  So we get:

  $$\bigcup\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\} = \{\phi \in \Phi \mid \text{ctx}(\phi) \in O^*\} \uplus O^*$$

  Which means that:

  $$\begin{aligned}\bigcup\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O_i\} \uplus O_i : i \in I\} \in \\ \{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}\end{aligned}$$

  And hence $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ is closed under arbitrary union. □

We get that:

Lemma 5.4.16. *$(A, T_{\mathscr{S}})$ is a subspace of $(\Phi \uplus A, T_{\Phi \uplus A})$*

Proof. We have $A \subseteq (\Phi \uplus A)$ and:

$$\begin{aligned}T_{\Phi \uplus A} \cap A =& (\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}) \cap A \\ =& \{O \mid O \in T_{\mathscr{S}}\} \\ =& T_{\mathscr{S}}\end{aligned}$$

□

That means that, on objects, the induced topology on objects and steps topology inherits the convergence behaviour of the Scott topology induced by the order on objects:

Lemma 5.4.17. *An object in a PRS is the limit of a sequence of objects in the Scott topology on the set of objects if and only if it is the limit of this sequence in the topology of objects and steps in the ToRS induced by the PRS.*

Proof. Let $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$ be a PRS, let $T_{\mathscr{S}}$ be the Scott topology on $(A, \leq)$, let $T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$, let $\langle s_\beta \rangle_{\beta < \alpha}$ be a sequence in A and let $b \in A$.

We get that $\langle s_\beta \rangle_{\beta < \alpha}$ converges to b in $(A, T_{\mathscr{S}})$ if and only if $\langle s_\beta \rangle_{\beta < \alpha}$ converges to b in $(\Phi \uplus A, T_{\Phi \uplus A})$ as a consequence of Lemma 5.4.16 and Lemma A.2.69. □

Here we directly get:

Lemma 5.4.18. *The $\mathscr{S}$-limits of a sequence of objects in a PRS are precisely the topological limits of the sequence in the induced ToRS.*

Proof. Let $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$ be an PRS and let $T_{\mathscr{S}}$ be the Scott topology induced by $(A, \leq)$. We get $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$ with

$$T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$$

as ToRS induced by $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$. Now let $\langle s_\beta \rangle_{\beta<\alpha}$ be a sequence in A and let $b \in A$.

$\Rightarrow$ Assume that b is an $\mathscr{S}$-limit of $\langle s_\beta \rangle_{\beta<\alpha}$ in $(A, \leq)$.

By Proposition A.2.64 we get that b is a topological limit of $\langle s_\beta \rangle_{\beta<\lambda}$ in $(A, T_{\mathscr{S}})$ and by Lemma 5.4.17 we get that b is a topological limit of $\langle s_\beta \rangle_{\beta<\lambda}$ in $(\Phi \uplus A, T_{\Phi \uplus A})$.

$\Leftarrow$ Assume that b is a topological limit of $\langle s_\beta \rangle_{\beta<\alpha}$ in $(\Phi \uplus A, T_{\Phi \uplus A})$.

By Lemma 5.4.17 we get that b is a topological limit of $\langle s_\beta \rangle_{\beta<\alpha}$ in $(A, T_{\mathscr{S}})$ and by Proposition A.2.64 we get that b is an $\mathscr{S}$-limit of $\langle s_\beta \rangle_{\beta<\alpha}$ in $(A, \leq)$. □

So:

Lemma 5.4.19. *The limit inferior of a sequence of objects in a PRS is the greatest object that the sequence converges to in the ToRS induced by the PRS.*

Proof. This follows from Lemma 5.4.18 because the greatest $\mathscr{S}$-limit of a sequence is the limit inferior of a sequence by definition. □

Furthermore:

Lemma 5.4.20. *The limit inferior of a sequence of contexts of steps in a PRS is a limit of the sequence of context objects of that sequence of steps in the ToRS induced by that PRS.*

Proof. Let $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$ be a PRS, let $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$ be the ToRS that it induces and let $\langle \phi_\beta \rangle_{\beta<\alpha}$ be a sequence of steps in $(\Phi, A, \text{src}, \text{tgt})$.

We get $\liminf(\langle \text{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) \leq \liminf(\text{src}\,\prime(\langle \phi_\beta \rangle_{\beta<\alpha}))$ by Lemma 5.4.7. So $\liminf(\langle \text{ctx}(\phi_\beta) \rangle_{\beta<\alpha})$ is a $\mathscr{S}$-limit of $\langle \text{src}\,\prime(\phi_\beta) \rangle_{\beta<\alpha}$ and $\liminf(\langle \text{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) \in \text{Lim}(\langle \text{src}\,\prime(\phi_\beta) \rangle_{\beta<\alpha})$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ by Lemma 5.4.18. □

Lemma 5.4.21. *The limit inferior of a sequence of context of steps in a PRS is the greatest object that the sequence of those steps converges to in the ToRS induced by that PRS.*

Proof. This follows from Lemma 5.4.19 because steps and their context objects are topologically indistinguishable in $(\Phi \uplus A, T_{\Phi \uplus A})$ ($T_{\Phi \uplus A}$ is defined as $\{\{\phi \in \Phi \mid \text{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$) □

5.4.5 Convergence in Induced ToRSs

Unlike metrical convergence, partial convergence does not coincide with topological convergence. We do get:

Theorem 5.4.22. *If a transfinite reduction weak partially converges to some object in a PRS then it also weak topologically converges to that object in the ToRS induced by the PRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ be a PRS, let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ be the ToRS induced by it, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$.

Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak partially converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$, we get that $\liminf(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle)) = b$ in $(A, \leq)$. By Lemma 5.4.19 we get that $b \in \mathrm{Lim}(\mathrm{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, T_{\Phi \uplus A})$. So the reduction weak topologically converges to b in our ToRS. □

It directly follows that:

Proposition 5.4.23. If a transfinite reduction in PRS is a weak partial reduction then it is a weak topological reduction in the ToRS induced by the PRS.

However, the converses of these statements do not hold. Reductions in a ToRS induced by a PRS might have weak topological limits that they do not weak partially converge to. This is because a non-greatest limit of the sequence of objects of the reduction is (by Lemma 5.4.18) not the limit inferior of that sequence in the PRS but a $\mathscr{S}$-limit strictly smaller than it.

An example of this is Example 5.4.40 (upcoming, when the embedding of iTRSs in PRSs is defined). There, the sequence of objects is:

$$\langle g(a, a), g(f(a), a), g(f(f(a)), a), g(f(f(f(a))), a), \ldots \rangle$$

The term $g(\bot, a)$ is a topological limit of this sequence. But this is 'only' an $\mathscr{S}$-limit and not the limit inferior of the sequence, which is $g(f^\omega, a))$. So the reduction

$$g(f(a), a) \to_1 g(f(f(a), a) \to_1 g(f(f(a)), a) \to_1 g(f(f(f(a))), a) \to \ldots$$

has a unique weak partial limit, $g(f^\omega, a))$. It has many weak topological limits though: $\{\bot, g(\bot, \bot), g(\bot, a), g(f(\bot), \bot), g(f(\bot), a), \ldots, g(f^\omega, \bot), g(f^\omega, a)\}$.

So, we 'only' get the following theorem:

Theorem 5.4.24. *A transfinite reduction weak partially converges to some object in a PRS if and only if this object is the largest object that the reduction weak topologically converges to in the ToRS induced by the PRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ be a PRS, let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ be the ToRS induced by it, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$.

⇒ Theorem 5.4.22

⇐ Assume that is the largest object that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ weak topologically converges in $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$.

We get that $b \in \text{Lim}(\text{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (WTC). Since b is the largest object that the reduction weak topologically converges to, we can apply Lemma 5.4.19 and we get that $b = \liminf(\text{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, \leq_{\Phi \uplus A})$ (WPC). So the reduction weak partially converges to b in our PRS. □

It directly follows that:

Proposition 5.4.25. A transfinite reduction that is maximal at limits is a weak partial reduction in a PRS if and only if it is a weak topological reduction in the ToRS induced by the PRS.

For strong partial convergence we get:

Theorem 5.4.26. *If a transfinite reduction strong partially converges to some object in a PRS then it also strong topologically converges to that object in the ToRS induced by the PRS.*

Proof. Let $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$ be a PRS, let $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$ be the ToRS induced by it, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \text{src}, \text{tgt})$ and let $b \in A$.

Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong partially converges to b in $((\Phi, A, \text{src}, \text{tgt}), \text{ctx}, \leq)$. Based on whether $\alpha \in \text{LimOrd}$ we get 2 cases:

- $\alpha \notin \text{LimOrd}$.

  We get that $b = \liminf(\text{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(A, \leq)$ (SPC-S) by assumption, so by Lemma 5.4.19 we get that $b \in \text{Lim}(\text{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (WTC). Now because $\alpha \notin \text{LimOrd}$ we get that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to b in $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$.

- $\alpha \in \text{LimOrd}$.

  We get $b = \liminf(\langle \text{ctx}(\phi_\beta) \rangle_{\beta<\alpha})$ in $(A, \leq)$ (STC-L) by assumption, so by Lemma 5.4.20 we get that $b \in \text{Lim}(\text{src}\,\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (WTC). Also, by Lemma 5.4.21, we get that $b \in \text{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (STC). So we get that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to b in $((\Phi, A, \text{src}, \text{tgt}), T_{\Phi \uplus A})$. □

It directly follows that:

Proposition 5.4.27. If a transfinite reduction in PRS is a strong partial reduction then it is a strong topological reduction in the ToRS induced by the PRS.

Again, the converses of these statements do not hold. A reduction in a ToRS induced by a PRS might have a strong topological limit that it does not strong partially converge to. Again, this is because a non-greatest limit of the sequence of objects or contexts of the reduction is not the limit inferior of that sequence in the PRS, but a $\mathscr{S}$-limit strictly smaller than it. We only get get:

Theorem 5.4.28. *A transfinite reduction strong partially converges to some object in a PRS if and only if this object is the largest object that the reduction strong topologically converges to in the ToRS induced by the PRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ be a PRS, let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ be the ToRS induced by it, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$.

$\Rightarrow$ Theorem 5.4.26

$\Leftarrow$ Assume that b is the largest object that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$. Based on whether $\alpha \in \mathrm{LimOrd}$ we get 2 cases:

- $\alpha \notin \mathrm{LimOrd}$.
  We get that $b \in \mathrm{Lim}(\mathrm{src}\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (WTC). Since b is the largest object that the reduction strong topologically converges to, we can apply Lemma 5.4.19 and we get that $b = \liminf(\mathrm{src}\prime(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ in $(\Phi \uplus A, \leq_{\Phi \uplus A})$ (SPC-S).
- $\alpha \in \mathrm{LimOrd}$.
  We get $b \in \mathrm{Lim}(\langle \phi_\beta \rangle \beta < \alpha)$ in $(\Phi \uplus A, T_{\Phi \uplus A})$ (STC). Since b is the largest object that the reduction strong topologically converges to we can apply Lemma 5.4.21 and we get that $b = \liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha})$ in $(A, \leq)$ (SPC-L).

So the reduction strong partially converges to b in our PRS. ☐

It directly follows that:

Proposition 5.4.29. A transfinite reduction that is maximal at limits is a strong partial reduction in a PRS if and only if it is a strong topological reduction in the ToRS induced by the PRS.

On the other hand, $\mathscr{S}$-limits of reductions do coincide with topological limits. By Lemma 5.4.18 we immediately get:

Proposition 5.4.30. A transfinite reduction weak $\mathscr{S}$-converges to some object in a PRS if and only if it weak topologically converges to that object in the ToRS induced by the PRS.

Also:

Theorem 5.4.31. *A transfinite reduction strong $\mathscr{S}$-converges to some object in a PRS if and only if it strong topologically converges to that object in the ToRS induced by the PRS.*

Proof. Let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ be a PRS, let $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$ be the ToRS induced by it, let $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ be a transfinite reduction in $(\Phi, A, \mathrm{src}, \mathrm{tgt})$ and let $b \in A$.

- $\Rightarrow$ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong S-converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq$). We get $b \leq \liminf(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ (WSC), so by Lemma 5.4.18 we get $b \in \mathrm{Lim}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ (WTC). If $\alpha \in \mathrm{LimOrd}$ we also get $b \leq \liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) = b$ (SSC). By Lemma 5.4.18 this means $b \in \mathrm{Lim}(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha})$. And because $T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid \mathrm{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ we get $b \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ (STC).
- $\Leftarrow$ Assume that $\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle$ strong topologically converges to b in $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T_{\Phi \uplus A})$. We get $b \in \mathrm{Lim}(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ (WTC), so by Lemma 5.4.18 we get $b \leq \liminf(\mathrm{src}'(\langle a, \langle \phi_\beta \rangle_{\beta<\alpha} \rangle))$ (WSC). If $\alpha \in \mathrm{LimOrd}$ we also get $b \in \mathrm{Lim}(\langle \phi_\beta \rangle_{\beta<\alpha})$ (STC). Because $T_{\Phi \uplus A} = \{\{\phi \in \Phi \mid \mathrm{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ we get $b \in \mathrm{Lim}(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha})$. And hence by Lemma 5.4.18 we get $b \leq \liminf(\langle \mathrm{ctx}(\phi_\beta) \rangle_{\beta<\alpha}) = b$ (SSC). ☐

This is an extra argument for considering $\mathscr{S}$-convergence instead of partial convergence. $\mathscr{S}$-convergence coincides with topological convergence in a topological space that is derived in a straight-forward way when one considers the intuitions behind ToRSs and PRSs. That is, the topological space with as topology $\{\{\phi \in \Phi \mid \mathrm{ctx}(\phi) \in O\} \uplus O \mid O \in T_{\mathscr{S}}\}$ where $T_{\mathscr{S}}$ is the Scott topology on objects of a PRS. In this sense $\mathscr{S}$-convergence can be thought of as being topological, where partial convergence is not.

5.4.6 Partial Divergence

Because every reduction has a weak and strong partial limit there is no notion of divergence of reductions in PRSs analogous to the one in ToRSs and MRSs. However, we can do better. Considering maximal objects, we get an alternative, more fine-grained notion of divergence in PRSs.

If we interpret $\leq$ as an informational ordering on A, then maximal objects contain, in some sense, a maximal amount of information and non-maximal objects contain a non-maximal amount of information. Now a reduction might be considered to be weak or strong **partially divergent** if it respectively weak or strong partially converges to a non-maximal object. This might even be considered to yield a measure of the level of convergence or divergence of a reduction:

- A reduction that converges (weak partially or strong partially) to a maximal object can be said to fully converge.

- A reduction that converges to an object below an object that another reduction converges to can be said to converge less or diverge more than the other reduction.
- A reduction that converges to a minimal object (bounded completeness of our partial order yields a minimal object in the ordered set) can be said to be fully diverging.

So, as [2, p. 13] states, in a metric framework we can only distinguish between converging and diverging reductions, in a partial framework we get a more fine-grained distinction. We can quantify how much a reduction converges or diverges by how its limit is ordered compared to other objects in the ordered set of objects. This partial divergence is the main reason for considering PRSs in the first place.

In ToRSs topological divergence has been defined as non-convergence, however it is also possible to reconstruct this partial approach to divergence in ToRSs. Using the specialization quasi-order (Definition A.2.36) of the restriction of the topology of a ToRS to objects, we get an order on our objects. If a reduction converges to an object strictly below the object another reduction converges to, we might say the first reduction converges less or diverges more than the other reduction. This makes sense because an object being strictly below another object in the specialization quasi-order means that the second object is in all the open sets the first object is in, and in more. Hence, by our finitely observable properties intuition, the second object has all the properties the first object has and more, and hence can be seen as a 'better specified version' of the first object. Likewise, if a reduction converges to a limit object that is non-maximal under this order, we might say it only partially converges and hence partially diverges. Since that object is not maximal, it is 'not specified as much as it could have been'.

When considering weak topological convergence in embedded PRSs we can see that this topological reconstruction of partial divergence using specialization quasi-orders works correctly. If $((\Phi, A, \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq)$ is a PRS and $((\Phi, A, \mathrm{src}, \mathrm{tgt}), T)$ the ToRS it induces, then by Lemma 5.4.16 the restriction of T to A is $T_{\mathscr{S}}$, the Scott topology on A and by Lemma A.2.60 the specialization quasi-order on the Scott topology is again our original order, $\leq$.

A useful feature of ToRSs is that they allow to express partial convergence without having the need for the set of objects to always form a bounded complete dcpo. The specialization quasi-order of the restriction of its topology to objects might even be the identity order ($a \leq b \Leftrightarrow a = b$), as it is in the ToRS induced by any iTRS. In this case, a reduction never converges to some object below another object, so the notions of partial convergence in this ToRS are just as expressive as the regular notions of convergence and hence rather useless. But in other cases they will not be. An example is the ToRS induced by the PRS induced by a partial iTRS. In that ToRS, the notion of partial convergence is just as expressive as the notion of partial convergence in the PRS induced by the iTRS is. So ToRSs allow for more freedom in the structure that is given on the set of objects than PRSs do (arbitrary topologies versus bounded complete dcpos), while they retain the advantage of having a nice notion of partial convergence when sufficient structure *is* present.

5.4.7 Embedding iTRSs in PRSs

We want to embed iTRSs in PRSs for the same reasons we embedded iTRSs in ToRSs and in MRSs. In the PRS formalism however, we need our set of objects to be ordered and to form a continuous bounded complete dcpo. We don't have such an order on the set of infinite terms over some signature, but we do have such an order on the set of partial infinite terms over the same signature (Definition 4.1.21). Using sets of partial terms and the orders on them, we can embed iTRSs in the PRS framework in the following way:

Definition 5.4.32. The PRS **induced** by an iTRS (Σ, R) is $((\Phi, \mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq_\perp)$ where:

- $(\Phi, \mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \mathrm{src}, \mathrm{tgt})$ is the rewrite system induced by the *partial* iTRS $(\Sigma_\perp, R)$
- $\leq_\perp$ is the standard order on the set of *partial* terms $\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$
- For all $\phi \in \Phi$ we have:
  - $(\mathrm{ctx}(\phi))(p) = \perp$ if $p = \mathrm{pos}(\phi)$
  - $(\mathrm{ctx}(\phi))(p)$ is undefined if $\mathrm{pos}(\phi)$ is a strict prefix of p
  - $(\mathrm{ctx}(\phi))(p) = s(p) = t(p)$ otherwise

Lemma 5.4.33. *This gives an actual PRS.*

Proof.

- $(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ is a partial ordered set by Theorem 4.1.24, a dcpo by Theorem 4.1.26, bounded complete by Theorem 4.1.27 and algebraic by Theorem 4.1.29 and hence continuous by Lemma A.4.23.
- Let $\phi \in \Phi$. For any $p \in \mathrm{Pos}(\mathrm{ctx}(\phi))$ we have that $(\mathrm{ctx}(\phi))(p) = (\mathrm{src}(\phi))(p) = (\mathrm{tgt}(\phi))(p)$ if $p \neq \mathrm{pos}(\phi)$ or $(\mathrm{ctx}(\phi))(p) = \perp$ if $p = \mathrm{pos}(\phi)$, so by Lemma 4.1.23 we get $\mathrm{ctx}\,\phi \leq_\perp \mathrm{src}(\phi)$. □

This way of embedding iTRSs in PRSs is, in principle, the same as the embedding in [2], but differs slightly due to the PRSs defined here differing from the PRSs in [2, Definition 6.2].

The order $\leq_\perp$ expresses the structure of $\mathcal{T}(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$. We have $t \leq_\perp s$ if s can be obtained from t by replacing occurrences of $\perp$ in it by new subterms. So s is a 'more specified' version of t. Hence the order can be interpreted as an informational ordering on terms. Terms above other terms contain at least the same informational content (and possibly more), and $a \leq_\perp b$ is to be interpreted as b *contains at least all the information that* a *contains*. When interpreting the order on partial terms this way, it fits in perfectly with the informational ordering intuitions that PRSs are based on.

Taking partial terms over some signature as objects instead of taking 'regular' terms also makes sense from an intuitional point of view. In a PRS, the order on objects is interpreted as an informational ordering, some object might contain less information than some other object. This fits in well with partial terms that might be seen as representing data that might be partly undefined (and hence contains less information than it 'could have'). By Theorem 4.1.30 the original, 'non-partial', terms are maximal elements of the set of partial terms. So for those terms we get that our PRS formalism is more well-behaved than it is in general. A reduction of terms that strong partially converges to some term also weak partially converges to that term (Theorem 5.4.8) and any strong partial reduction of terms is also weak partial (Proposition 5.4.10).

Via this embedding of iTRSs in PRSs and the embedding of PRSs in ToRSs, iTRSs can be embedded in ToRSs. iTRSs can also be embedded directly into ToRSs, as is done in Section 5.2.6. We can compare these embeddings by comparing the ToRS induced by the PRS induced by an iTRS with the ToRS induced by the iTRS directly. These ToRSs differ, they do not even have the same underlying rewrite system. The ToRS induced by the PRS induced by the iTRS has partial terms as objects, where the ToRS induced by the iTRS directly has 'regular', non-partial terms as objects. Also, the ToRS induced by the PRS induced by the iTRS has steps between partial terms in its set of steps, where the ToRS induced by the iTRS directly does not. However, the topologies of the two ToRSs can still be compared by restricting the topology of the ToRS induced by the PRS induced by the iTRS to non-partial terms and steps between non-partial terms.

First, we have:

Lemma 5.4.34. *The Scott topology on $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ has the following base*

$$B_{\mathscr{S}} = \{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq t'\} \mid t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$$

Proof. Let Σ be some signature. $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ is the set of partial infinite terms over it, it is ordered by $\leq_\perp$ and $(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ is an algebraic bounded complete dcpo. Let $T_{\mathscr{S}}$ be the Scott topology on $(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$. To prove that $B_{\mathscr{S}} = \{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$ is a base of $T_{\mathscr{S}}$:

- The closure of $B_{\mathscr{S}}$ under arbitrary union is a subset of $T_{\mathscr{S}}$.

  Let $\{t_i \mid i \in I\}$ be an arbitrary set of partial terms of finite depth. For any $i \in I$ we have that $\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$ is upper by definition and hence $\bigcup_{i \in I}\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$ is upper.

  Also, let $i \in I$, t_i is of finite depth and hence compact in the order theoretic sense (Lemma 4.1.28). Let D be a directed set of terms with $\bigsqcup D \in \{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$, we get $t_i \leq_\perp \bigsqcup D$ and by the definition of compactness (Definition A.4.18), there is a $d \in D$ such that $t_i \leq_\perp d$

and hence $d \in \{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$. So every directed set that has a least upper bound in $\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$ is eventually in $\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$ and hence every directed set that has a least upper bound in $\bigcup_{i \in I}\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$ is eventually in $\bigcup_{i \in I}\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\}$.

So $\bigcup_{i \in I}\{t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t_i \leq_\perp t\} \in T_{\mathscr{S}}$ and the closure of $B_{\mathscr{S}}$ under arbitrary union is a subset of $T_{\mathscr{S}}$.

- $T_{\mathscr{S}}$ is a subset of the closure of $B_{\mathscr{S}}$ under arbitrary union.

  Let $O \in T_{\mathscr{S}}$. We prove that

  $$O = \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$$

  - We get $\bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\} \subseteq O$ because O is upper.
  - $O \subseteq \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$.

    Let $t \in O$. If t is of finite depth, we directly get

    $$t \in \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$$

    If t is of infinite depth, let $f : \mathbb{N} \to \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ be a function that maps any $n \in \mathbb{N}$ to the term $f(n)$ with $\mathrm{dpt}(f(n)) = n$ such that:

    * $(f(n))(p) = t(p)$ if $|p| < n - 1$ and $t(p)$ is defined
    * $(f(n))(p) = \perp$ if $|p| = n - 1$ and $t(p)$ is defined
    * $(f(n))(p)$ is undefined otherwise

    We get $f(n_1) \leq_\perp f(n_2) \Leftrightarrow n_1 \leq n_2$, so $\{f(n) \mid n \in \mathbb{N}\}$ is a directed set. We get $\bigsqcup\{f(n) \mid n \in \mathbb{N}\} = t$. Now because O is Scott open we get that $\{f(n) \mid n \in \mathbb{N}\}$ is eventually in O, so there is an $n \in \mathbb{N}$ such that $f(n) \in O$. $f(n)$ is of finite depth (of depth n) and $f(n) \leq_\perp t$, and hence $t \in \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$. So $O \subseteq \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$.

  So $O = \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq_\perp t'\} \mid t \in O \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$ and hence $T_{\mathscr{S}}$ is a subset of the closure of $B_{\mathscr{S}}$ under arbitrary union. □

Theorem 5.4.35. *The topology of the ToRS induced by the PRS induced by an iTRS restricted to non-partial terms is equal to the topology of the ToRS induced by the iTRS restricted to non-partial terms.*

Proof. Let (Σ, R) be an iTRS. Let $((\Phi, \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \mathrm{src}, \mathrm{tgt}), \mathrm{ctx}, \leq_\perp)$ be the PRS induced by that iTRS and let $((\Phi, \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \mathrm{src}, \mathrm{tgt}), T_{\leq_\perp})$ be the ToRS induced by that PRS. The restriction of $T_{\leq_\perp}$ to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ is the Scott topology on $(\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \leq_\perp)$ restricted to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$. Denote this restriction of the Scott topology on $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ by $T^*_{\mathscr{S}}$. By Lemma 5.4.34, we get $\{\{t' \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \mid t \leq t'\} \mid t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$ as base of the Scott topology on $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$. So, as base of the Scott topology on $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ restricted to $\mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ we get

$$B^*_{\mathscr{S}} = \{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq t'\} \mid t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}) \wedge \mathrm{dpt}(t) \in \mathbb{N}\}$$

Let $((\Phi^*, \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma), \mathrm{src}, \mathrm{tgt}), T)$ be the ToRS directly induced by (Σ, R). Denote the restriction of the topology T to $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$ by T^*. As base of T^*, we get $B^* = \{\bigcap\{U_{\mathcal{T}}(p, f) \mid \langle p, f\rangle \in P\} \mid P \subset \mathbb{N}^* \times \Sigma\}$ where P is a finite and $U_{\mathcal{T}}(p, f) = \{t \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t(p) = f\}$.

T^* and $T^*_{\mathscr{S}}$ are both topologies on $\mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma)$, we want to prove that they are equal.

- $T^*_{\mathscr{S}} \subseteq T^*$.

  We prove that $B^*_{\mathscr{S}} \subseteq B^*$. Let $A \in B^*_{\mathscr{S}}$. We get that there is some $t \in \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp})$ with $\mathrm{dpt}(t) \in \mathbb{N}$ such that $A = \{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\}$. Let $P = \{\langle p, f\rangle \mid t(p) \in \Sigma\}$, that is the set of all tuples of positions and symbols such that t is defined at the position as that symbol and is not $\perp$. We get that:

  - For any $s \in A$ we have $t \leq_\perp s$, so by Lemma 4.1.23 we get $s \in \bigcap\{U_{\mathcal{T}}(p, f) \mid \langle p, f\rangle \in P\}$.
  - For any $s \in \bigcap\{U_{\mathcal{T}}(p, f) \mid \langle p, f\rangle \in P\}$ we get $t \leq_\perp s$ by Lemma 4.1.23, so $s \in A$.

  So get $A = \bigcap\{U_{\mathcal{T}}(p, f) \mid \langle p, f\rangle \in P\}$ and hence $A \in B$.

- $T^* \subseteq T^*_{\mathscr{S}}$.

  Let $A \in B^*$, we'll prove that $A \in T^*_{\mathscr{S}}$. There is some P such that $A = \bigcap\{U_{\mathcal{T}}(p, f) \mid \langle p, f\rangle \in P\}$. Let X be a set of partial terms of finite depth where $t \in X$ if and only if:

  - For any $\langle p, f\rangle \in P$ we have $t(p) = f$.
  - For any $\langle p, f\rangle \in P$ such that there is no $\langle q, g\rangle \in P$ with $q < p$ and any $1 \leq i \leq \mathrm{ar}(f)$ we have $t(p; i) = \perp$.

  We can prove that $A = \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\}$:

  - $A \subseteq \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\}$.

    Let $s \in A$. There is some $t \in X$ such that $t \leq_\perp s$, namely the finite, partial term t such that:

    * $t(p) = s(p)$ if there is no $\langle q, f\rangle \in P$ such that $q \leq p$.
    * $t(p) = \perp$ if $\langle p, f\rangle \in P$ for some f and there is no $\langle q, f\rangle \in P$ such that $q \leq p$.
    * $t(p)$ is undefined otherwise.

    We get $t \leq_\perp s$ by Lemma 4.1.23. So $s \in \{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\}$ and hence $s \in \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\}$.

– $\bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\} \subseteq A$.
  Let $s \in \bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\}$, we get some $t \in X$ such that $t \leq_\perp s$. For any $\langle p, f\rangle \in P$ we have $t(p) = f$ and hence $s(p) = f$ by Lemma 4.1.23. So $s \in \bigcap\{U_{\mathcal{T}}(p, f) \mid \langle p, f\rangle \in P\}$ and hence $s \in A$.

Now, because $\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\} \subseteq B^*_{\mathscr{S}}$ we get that $\bigcup\{\{t' \in \mathcal{T}^\omega(\Sigma, \mathcal{X}_\Sigma) \mid t \leq_\perp t'\} \mid t \in X\} \in T^*_{\mathscr{S}}$, and hence $A \in T^*_{\mathscr{S}}$.

This means we get $B^* \subseteq T^*_{\mathscr{S}}$, and because $T^*_{\mathscr{S}}$ is closed under arbitrary union, $T^* \subseteq T^*_{\mathscr{S}}$. □

Steps in a PRS induced by an iTRS are topologized through their context objects. Steps in an iTRS are directly topologized through their static parts. Context objects in PRSs express exactly what static parts express. We have $\text{ctx}(\phi)(p) \in \Sigma \Leftrightarrow (\text{pos}(\phi) \not\leq p) \wedge ((\text{src}(\phi))(p) = (\text{tgt}(\phi))(p) = f)$ and $\text{ctx}(\phi)(p) = \perp \Leftrightarrow (\text{pos}(\phi) = p)$. So, by Theorem 5.4.35, we get:

Proposition 5.4.36. The topology of the ToRS induced by the PRS induced by an iTRS restricted to non-partial terms and steps between non-partial terms is equal to the topology of the ToRS induced by the iTRS.

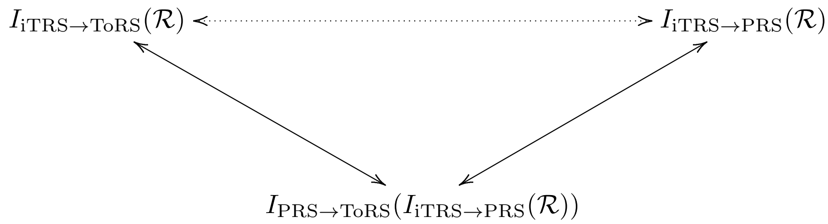
Remark. The proof of Theorem 5.4.35 could have been expanded to prove this, but that would have made it even longer and messier. Thats why I chose to just prove the statement of Theorem 5.4.35, where all the actual work is done, and give the statement of Proposition 5.4.36 as a proposition.

From Proposition 5.4.36 it follows that:

Proposition 5.4.37. The notions of weak and strong topological convergence in the ToRS induced by an iTRS coincide with, respectively, the notions of weak and strong topological convergence of reductions of non-partial terms to non-partial limits in the ToRS induced by the PRS induced by that iTRS.

So for PRSs the relations between convergence in the PRS induced by an iTRS, convergence in the ToRS induced by the iTRS and the ToRS induced by the PRS induced by the iTRS are much messier then they are for MRSs. Partial convergence in a PRS only coincides with topological convergence in a ToRS induced by the PRS in the case of convergence to a maximal limit in the ToRS induced by the PRS (Theorem 5.4.24, Theorem 5.4.28). Convergence in the ToRS induced by an iTRS only coincides with convergence in the ToRS induced by the PRS induced by the iTRS in the case of convergence of a reduction of non-partial terms to a non-partial limit. However, since maximal objects in the PRS induced by an iTRS are non-partial terms Lemma 4.1.30. When restricted to reductions of non-partial terms and non-partial limits, convergence coincides for the ToRS induced by an iTRS, the ToRS induced by the PRS induced by the iTRS and the PRS induced by the iTRS.

So, if $\mathcal{R}$ is an iTRS and we write $I_{\text{iTRS}\to\text{ToRS}}$ for the function giving the ToRS induced by an iTRS, $I_{\text{iTRS}\to\text{PRS}}$ for the function giving the PRS induced by an iTRS and $I_{\text{PRS}\to\text{ToRS}}$ for the function giving the ToRS induced by an PRS. Now we get the following diagram:



Here the arrows can be interpreted as stating an equivalence of notions of convergence of reductions of non-partial terms to non-partial limits.

To give some examples of how convergence in the PRS induced by an iTRS works out:

Example 5.4.38. Moving example Example 5.2.18 to a PRS setting, our iTRS is still (Σ, R) with $\Sigma = \{a/0, f/1\}$ and $R = \{a \to f(a)\}$. The PRS induced by this iTRS is $((\Phi, \mathcal{T}^\omega(\Sigma_\perp, \mathcal{X}_{\Sigma_\perp}), \text{src}, \text{tgt}), \text{ctx}, \leq_\perp)$ where Φ is the set of steps between partial terms that R gives rise to. Our reduction is:

$$a \to_\epsilon f(a) \to_1 f(f(a)) \to_{11} f(f(f(a))) \to \ldots$$

Its sequence of objects is $\langle a, f(a), f(f(a)), f(f(f(a))), \ldots\rangle$, the limit inferior of this sequence of objects is f^ω. So the reduction weak partially converges to f^ω. The sequence of contexts of the reduction is $\langle \perp, f(\perp), f(f(\perp)), f(f(f(\perp))), \ldots\rangle$, the limit inferior of this sequence of contexts is f^ω. So the reduction also strong partially converges to f^ω.

The reduction weak $\mathscr{S}$-converges to all partial terms below f^ω, that is, $\{\perp, f(\perp), f(f(\perp)), f(f(f(\perp))), \ldots, f^\omega\}$. The reduction strong $\mathscr{S}$-converges to all partial terms below f^ω, that is, again, $\{\perp, f(\perp), f(f(\perp)), f(f(f(\perp))), \ldots, f^\omega\}$.

Example 5.4.39. Moving Example 5.2.20 of a diverging reduction in the ToRS induced by an iTRS to a PRS setting; we have (Σ, R) with $\Sigma = \{a/0, b/0, f/1\}$ and $R = \{a \to b, a \to b\}$ as iTRS and in the induced PRS we get the following reduction of length ω;

$$f(a) \to_1 f(b) \to_1 f(a) \to_1 f(b) \to_1 f(a) \to \ldots$$

Its sequence of objects is $\langle f(a), f(b), f(a), f(b), f(a), \ldots\rangle$, the limit inferior of this sequence of objects is $f^\perp$ which is a partial term, but a perfectly fine weak partial limit nonetheless. So our reduction weak partially converges to $f(\perp)$. The sequence of contexts of the reduction is $\langle f(\perp), f(\perp), f(\perp), f(\perp), f(\perp), \ldots\rangle$, the limit inferior of this sequence of contexts is $f^\perp$. So the reduction also strong partially converges to $f^\perp$. This shows that the reduction, that weak topologically diverges, doesn't weak partially diverge in the sense that it has no limit. It has a limit, it is just a partial limit. So the reduction partially diverges in the sense that its limit contains a non-maximal amount of information, or, is not as "specified as it could have been". The weak and strong partial limits give, respectively, the information that eventually stays present in the reduction and the information that eventually stays present and untouched in the reduction.

The reduction weak $\mathscr{S}$-converges to all partial terms below $f^{\perp}$, that is, $\{\perp, f(\perp)\}$. The reduction strong $\mathscr{S}$-converges to all partial terms below $f^{\perp}$, that is, again, $\{\perp, f(\perp)\}$. The weak and strong $\mathscr{S}$-limits give, respectively, *some part* of the information that eventually stays present in the reduction and *some part* of the information that eventually stays present and untouched in the reduction.

Example 5.4.40. Moving Example 5.2.21, of a weakly converging but strongly diverging reduction in the ToRS induced by an iTRS, to PRS setting; We can consider the iTRS (Σ, R) with $\Sigma = \{a/0, f/1, g/2\}$ and $R = \{f(x) \to f(f(x))\}$. In the induced PRS we get the following reduction of length ω:

$$g(f(a), a) \to_1 g(f(f(a), a) \to_1 g(f(f(a)), a) \to_1 g(f(f(f(a))), a) \to \ldots$$

Its sequence of objects is $\langle g(a, a), g(f(a), a), g(f(f(a)), a), g(f(f(f(a))), a), \ldots \rangle$, the limit inferior of this sequence of objects is $g(f^{\omega}, a)$. So the reduction weak partially converges to $g(f^{\omega}, a)$. The sequence of contexts of the reduction is $\langle g(\perp, a), g(\perp, a), g(\perp, a), g(\perp, a), \ldots \rangle$, the limit inferior of this sequence of contexts is $g(\perp, a)$. So the reduction strong partially converges to $g(\perp, a)$. So this is an example of a reduction that strong partially converges to a limit that it does not weak partially converge to. The strong partial limit $g(\perp, a)$ is indeed not maximal as the contraposition of Lemma 5.4.8 implies. As implied by Lemma 5.4.7 the weak partial limit $g(f^{\omega}, a)$ is strictly larger than it.

The following shows that $\mathscr{S}$-converges indeed causes less trouble of this kind. The reduction weak $\mathscr{S}$-converges to all partial terms below $g(f^{\omega}, a)$, that is, $\{\perp, g(\perp, \perp), g(\perp, a), g(f(\perp), \perp), g(f(\perp), a), \ldots, g(f^{\omega}, \perp), g(f^{\omega}, a)\}$. The reduction strong $\mathscr{S}$-converges to all partial terms below $g(\perp, a)$, that is, $\{\perp, g(\perp, \perp), g(\perp, a)\}$. So every strong $\mathscr{S}$-limit of the reduction is indeed a weak $\mathscr{S}$-limit of the reduction.

Trying to embed wide TRSs or wide iTRSs, using respectively what could be called partial wide terms or partial wide infinite terms (obtained by adding $\perp$ to the signature) and the natural orders on them (obtained by generalizing $\leq_{\perp}$), yields problems. First of all, it needs to be checked whether these partially ordered sets of terms are continuous bounded complete dcpos. If not, it needs to be checked whether the PRS formalism can be relaxed to allow them as sets of objects anyway. Allowing less well-behaved ordered sets in the PRS formalism could cause problems elsewhere.

To start, sets of partial non-infinite wide terms won't form a dcpo when endowed with an order generalizing $\leq_{\perp}$. Under such an order, $\{f(\perp), f(f(\perp)), f(f(f(\perp))), \ldots\}$ is directed but doesn't have a partial non-infinite wide term as least upper bound. Its upper bound f^{ω} is an infinite term. Also, the base of the Scott topology on partial terms as given in Lemma 5.4.34

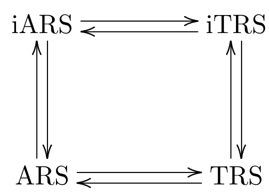
will need to be adapted for partial wide infinite terms to talk about compact terms instead of terms with finite depth.

Chapter 6

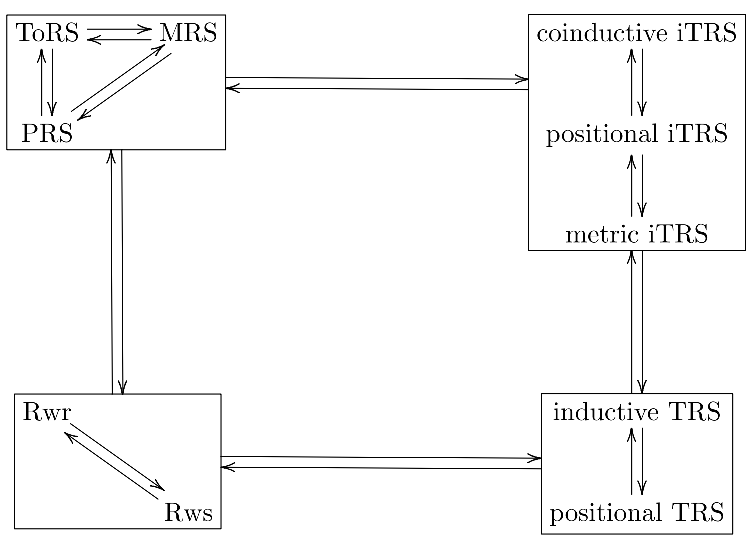
Conclusion and Discussion

6.1 Conclusion

We mapped the subfields of rewriting as:



Where ARS is the field of abstract rewriting, TRS is the field of term rewriting, our chosen concrete rewrite system, iTRS is the field of infinitary term rewriting and iARS is the field of infinitary abstract rewriting. Due to the various formalizations of these fields the extended diagram looks like:

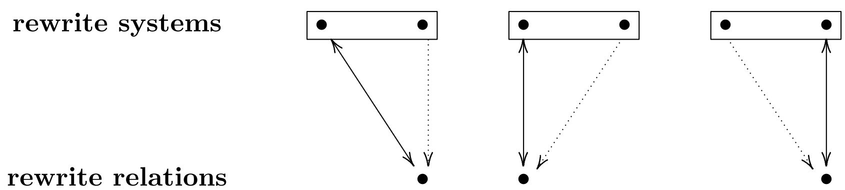


First abstract rewriting and term rewriting, both finite and infinite, were studied and after that the main topic of interest, transfinite rewriting.

6.1.1 Abstract Rewriting

We have two ways of setting up abstract rewriting. We can focus on objects and get rewrite relations (Section 3.1) or we can focus on steps and get rewrite systems (Section 3.2). Both have their own notion of reduction (reductions in rewrite systems, reduction sequences in rewrite relations).

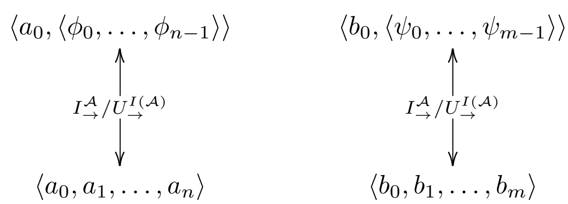
Every rewrite system has an underlying rewrite relation, but multiple rewrite systems might have the same underlying rewrite systems. Every rewrite relation induces a rewrite system and underlies that rewrite system. Not all rewrite systems are induced by some rewrite relation. (Section 3.3.1)



This means that the formalism of rewrite systems allows for expressing more detail than that of rewrite relations. The syntactic incident system (Example 3.3.2) is a prime example here, rewrite relations can not express such behaviour.

If a rewrite relation induces a rewrite system then every reduction in the rewrite system has exactly one underlying reduction sequence in the rewrite relation. This reduction sequence induces the reduction. Every reduction sequence underlies a reduction. (Section 3.3.2)

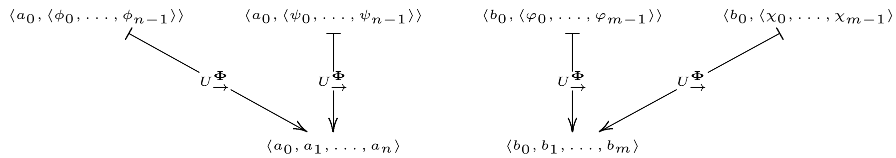
Reductions in $I(\mathcal{A})$



Reduction sequences in rewrite relation $\mathcal{A}$

If a rewrite relation underlies a rewrite system then every reduction in the rewrite system has an underlying reduction sequence. Every reduction sequence underlies at least one reduction, but might underlie more. (Section 3.3.2)

Reductions in rewrite system $\mathbf{\Phi}$



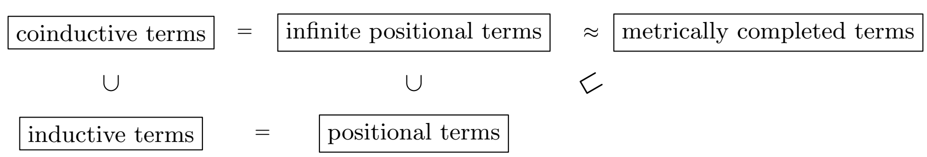
Reduction sequences in $U(\mathbf{\Phi})$

This means that the formalism of rewrite systems also is more expressive in its reductions than the formalism of rewrite relation is in its reduction sequences.

These are a good reasons to use rewrite systems and reductions, instead of rewrite relations and reduction sequences, as a basis for transfinite abstract rewriting. (Section 3.3.3)

6.1.2 Term Rewriting

We considered two approaches to set up terms, a positional and an inductive one. They yield the same set of terms (Theorem 4.1.9). These approaches were generalized to infinite terms using, respectively, infinite sets of positions and coinduction, yielding again the same set of terms (Theorem 4.1.11). A metric was given on positional terms and completing the resulting metric space yields a set of infinite terms that is isometric to the set of infinite positional terms (Theorem 4.1.14).



Furthermore, we gave various related or generalized formalisms of terms. We defined partial terms to generalize 'regular' terms. They model terms that are not completely defined. They are extremely well behaved (Theorem 4.1.24, Theorem 4.1.26, Theorem 4.1.27, Lemma 4.1.28, Theorem 4.1.29, Theorem 4.1.30). They are used when embedding iTRSs in PRSs (Section 5.4.7). We also defined more exotic sets of terms, namely:

- Wide terms: terms not only having a potentially infinite depth, but also an infinite (even transfinite) width (Definition 4.1.16, Definition 4.1.17).
- Transfinite terms: terms potentially having a depth of any ordinal (transfinite) length (Definition 4.1.18).

Using any of those formalisms of terms, rewrite systems can be created (Section 4.2). Notions of transfinite reduction on these terms then might be obtained by trying to embed these rewrite systems in the various formalisms for transfinite abstract rewriting.

6.1.3 Transfinite Abstract Rewriting

The research question posed in the introduction of this thesis was:

Research Question

> How can infinitary abstract rewriting be formalized using topological spaces and what are the properties and advantages of such a formalization?

It was noted that the type of infinitary abstract rewriting that we are interested in is better characterized as transfinite abstract rewriting. To answer the research question, a new topological formalism for transfinite abstract rewriting was proposed, that of ToRSs (Section 5.2). It was noted that topologies seem to be the natural choice for modeling transfinite abstract rewriting since notions like convergence, limit and continuity, which transfinite abstract rewriting is concerned with, are native to topology.

Two existing frameworks were also considered. One based on metric spaces ([13],[2]) and the other based partial orders ([2]). We have argued that the existing formalisms that instantiate these frameworks have some deficiencies. We gave variants of these frameworks that should fix these deficiencies. MRSs (Section 5.3, based on metrics and PRSs (Section 5.4, based on partial orders. For PRSs we considered the notion of $\mathscr{S}$-convergence (Definition 5.4.12, Definition 5.4.12) as an alternative of the 'regular' notion of partial convergence and argued why this might be considered a better notion. Both MRSs (Definition 5.3.7, Lemma 5.3.8) and PRSs (Definition 5.4.14, Lemma 5.4.15) can be embedded in ToRSs yielding:

- Equality of topological and metric convergence of reductions in embedded MRSs (Theorem 5.3.12, Theorem 5.3.14),
- Similarity of topological and partial convergence of reductions in embedded PRSs (Theorem 5.4.24, Theorem 5.4.28), and
- Equality of topological and $\mathscr{S}$-convergence of reductions in embedded PRSs (Proposition 5.4.30, Theorem 5.4.31)

However, ToRSs also allow for expressing more exotic convergence behaviour than MRSs and PRSs do, like convergence to multiple limits. Also, the notion of partial divergence, convergence to a term that is not maximal, that exists for PRSs can be reconstructed in ToRSs (Section 5.4.6). This can be done without requiring the set of objects to form a bounded complete dcpo, like PRSs do.

ToRSs (but also MRSs and PRSs) can be seen as generalizing rewrite systems, not rewrite relations, as is preferable by the argumentation given. iTRSs can be embedded in all three formalisms (Definition 5.2.14, Definition 5.3.16, Definition 5.4.32), where for MRSs and PRSs the embeddings are standard in the literature. For ToRSs the given embedding seems intuitive and yields the same notions of convergence as for MRSs and PRSs (Proposition 5.3.19, Theorem 5.3.20, Proposition 5.4.36). Wide TRSs and wide iTRSs can be embedded in ToRSs in the same intuitive way in which iTRSs are embedded in ToRSs. This yields a topology that is not metrizable (Theorem 5.3.24) and hence a notion of convergence that is not metric, giving evidence for the claim that the property of topological spaces that they allow to express more exotic behaviour than metric spaces do is useful in a rewriting context.

We also argued that the distinction between strong and weak convergence is actually unnecessary in ToRSs since weak convergence can be expressed in terms of strong convergence in an induced weak ToRS (Section 5.2.5).

Concluding: the ToRS formalism appears to be a well-behaved generalization of the MRS and PRS formalisms and as such should be a very useful framework for transfinite abstract rewriting.

6.2 Relevance and Novelty

The results in this thesis might be relevant in multiple ways. Foremost, the idea of using topologies (instead of metrics or partial orders) to formalize transfinite abstract rewriting is, as far as I know, new and as such should be interesting. It should be especially interesting due to its relation with existing frameworks. It encompasses both metric and partially ordered frameworks, but has some notable advantages over both. As mentioned in the introduction, the field of rewriting applies to many discrete processes and, building on that, the field of transfinite rewriting applies to infinitary discrete processes where convergence plays a role. These processes are aplenty in computation, so the theory that describes them and proves their properties can use a solid foundation. The introduction of a topological approach and the comparison of this approach to (patched up) existing approaches should help solidify this foundation. The same goes for the patching up of these existing formalisms and observations made about them.

Furthermore, the overview given over the various subfields of rewriting and how they relate, while not novel and at least known in folklore, might be structural and clear enough to be useful. For instance, I have not seen a category theoretical approach to studying rewrite relations, rewrite systems and their relation as given in Appendix B.1. Such a way of studying these topics should be very useful in choosing between frameworks and seeing how they relate. On a smaller scale, there are proofs and observations in this thesis that can be useful in the various fields of rewriting that I at least have not yet seen. Examples are the observation in Appendix B.2 about the existence of limits of strongly continuous transfinite reductions in [13], and the proof of algebraicity of infinite partial terms (Theorem 4.1.29).

6.3 Discussion and Further Research

There are various topics that might have been added to this thesis or could be considered missing.

For one, using ToRSs to model topological convergence of reductions grants much freedom, reductions might have one, no or many limits. In cases where we do not want such a wide array of possibilities, we'd hope to be able to put some restrictions on the topologies used and consequently get more well-behaved convergence behaviour of our reductions. For instance, the topology of a ToRS being Hausdorff is a sufficient condition for the reductions in the ToRS having unique limits, but we might also be interested in what condition is necessary for having unique limits of reductions. Also we might be interested in topological properties necessary and/or sufficient for guaranteeing the existence of limits of reductions.

Related to this is the fact that, in rewriting, we model our computations as reductions, using sequences. Sequences can be considered weak in a topological framework. Convergence of sequences is sufficient to describe some but not all topologies. For filters and nets this is the case for all topologies, a topology is defined by how its filters, or nets, converge. If we model our computations, not as reductions using sequences, but using filters or nets, we get a more general framework that should be more fitting in a topological setting. We then get that a topology being Hausdorff is not only sufficient, but also necessary for having unique filter and net limits. Also a topological space is compact if and only if every filter base on it is a subset of a convergent filter base and, equivalently, if every net on it has a convergent subnet. So for filters and nets we would have our topological properties for more well-behaved convergence behaviour. The question that arises though is: is an intuitive/useful/intelligible notion of rewriting based on filters or nets possible?

Taking this idea a bit less far: sequences are ordinal indexed, and hence they inherit well-foundedness and a notion of successor (both on indexes) from ordinals. A good question here is: are those properties actually necessary in a framework of rewriting? We might want to do without them, and instead of using sequences, use sets indexed by arbitrary linearly ordered sets. We then retain the linearity of our processes that we lose by using filters or nets, while we still are more general and abstract then rewriting is in its current form. Another question then is: is intuitive/useful/intelligible notion of rewriting based on linearly ordered sets possible?

Considering the "exotic terms" mentioned in Section 4.1.6, the embedding of transfinite TRSs in ToRSs (or in any formalism that models the convergence of reductions of transfinite terms for that matter) is riddled with problems. Some of these problems where noted and discussed in Section 5.2.7, but no embedding was given. The question to what extend such an embedding is possible remains open and interesting. Also, it seems that a full, correct and comprehensive non-topological notion convergence of reductions of transfinite terms is still not available.

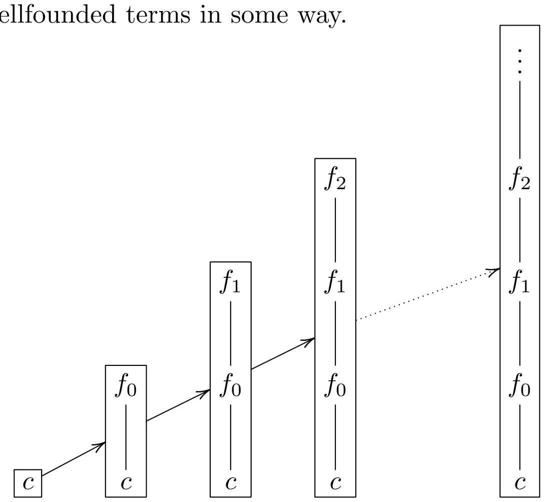
On a more technical level, as written, this thesis does not give an embedding of wide TRSs in PRSs. In Section 5.4.7 some of the problems with giving such an embedding are discussed. Still, such an embedding should be interesting on its own, while comparing ToRSs to PRSs based on how wide TRSs can be embedded into them should also be interesting.

Originally, this thesis also was to include a section on non-wellfounded terms (terms without a root). Such terms seem hard to formalize correctly, but might be very interesting in this setting for the same reason the other 'exotic' terms (wide and transfinite terms) are: seeing how they can be embedded in ToRSs and seeing how such an embedding might compare to possible embeddings in metric or partially ordered formalisms. Also, such terms aren't necessarily less intuitive as generalization of finite terms than infinite terms are. Infinite terms are terms with a beginning, but without an end. Non-wellfounded terms are terms with an end but without a beginning.

Example 6.3.1. An example from infinitary term rewriting that might give rise to a non-wellfounded term is the iTRS with the infinite signature $\{f_n/1 \mid n \in \mathbb{N}\} \cup \{c/0\}$ and the infinite set of reduction rules $\{f_n(x) \to f_{n+1}(f_n(x)) \mid n \in \mathbb{N}\}$. We get the following reduction of length ω:

$$f_0(c) \to_\epsilon f_1(f_0(c)) \to_\epsilon f_2(f_1(f_0(c))) \to_\epsilon f_3(f_2(f_1(f_0(c)))) \to \dots$$

This reduction does not converge to an infinite term in the usual way. The symbol at every position of a term in the reduction changes in the step that is applied to it, the symbols all get 'pushed down'. However if we look at the terms 'from the bottom' instead of from the root, an ever-growing bottom part stays fixed in the reduction. Looked at in this way, the reduction might be said to converge to $\dots(f_2(f_1(f_0(c))$. Such a term is not a regular term though, it is not well-founded. So if we want to express this type of convergence, we need to allow for non-wellfounded terms in some way.



Furthermore, in Appendix B.1 a category theoretical approach to studying rewrite relations, rewrite systems and their relation is given. However this study is far from complete due to the focus of this thesis lying elsewhere. Still, a complete category theoretical account of ‘rewrite systems versus rewrite relation’ should make the issues involved in the comparison clearer and might shed new light on them. This categorical approach could also be extended to the various other fields of rewriting. Specifically, it could be extended to transfinite rewriting. In Chapter 5, embeddings of MRSs in ToRSs and of PRSs in ToRSs were given. The left adjoints of those embeddings, mapping ToRSs back into MRSs and back into PRSs are interesting. As are the adjunctions formed by these back-and-forth mappings and a natural transformation. And more generally, seeing how possible categories of ToRSs, MRSs and PRSs behave with respect to each other, would be interesting.

Also, notions of transfinite conversion, as treated in for instance [11], are not touched upon in this thesis. Transfinite conversion turns out to be interesting but really hard to define correctly. An investigation of that topic, in the topological setting created here, could have been a nice addition to this thesis.

Similarly, it is well known that transfinitary variants of often-used finitary rewriting theorems fail. An example is the failure of a transfinitary version Newman’s lemma [13, p. 7]. In [13] a search for well-behavedness properties under which such theorems do hold is started. Maybe a topological approach can shed some light on that?

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Appendices

Appendix A

Mathematical Details

A.1 Ordinals and Cardinals

A.1.1 Isomorphisms

Both ordinality and cardinality depend on the notion of isomorphism. Isomorphism is studied categorically in category theory (Appendix A.5), but for the specific purpose of studying ordinal and cardinal theory we can independently and a bit more informally define:

Definition A.1.1. A **morphism** is a function from one set to another that is in some sense structure-preserving.

What kind of structure is preserved with the function depends on the morphism

Definition A.1.2. An **isomorphism** is a morphism, $f : X \to Y$, such that there is a morphism $g : Y \to X$ and we have that $g \circ f$ is the identity function on X, id_X, and $f \circ g$ is the identity function on Y, id_Y. For an isomorphism to be called structure preserving (for some kind of structure), both the associated f and g need to preserve that structure.

Isomorphism, taken as a relation between X and Y, is an equivalence relation, that is, it is transitive, symmetric and reflexive.

Lemma A.1.3. *Isomorphism is an equivalence relation.*

Proof.

- Isomorphism is reflexive. For any set X and any kind of structure, there is a structure preserving isomorphism from X to X.

  Let X be any set. For any $x \in X$, id_X maps x to x and hence is structure preserving for any kind of structure. The inverse of id_X is id_X itself, and $id_X \circ id_X = id_X$. Hence id_X is an isomorphism from X to X.

- Isomorphism is symmetric. For any two sets X and Y and any kind of structure, if there is a structure preserving isomorphism from X to Y then there is an isomorphism form Y to X that preserves that structure.

  Assume that $f : X \to Y$ is an isomorphism that preserves some kind of structure. By the definition of isomorphism we have that there exists a function g from Y to X that preserves that kind of structure, $g \circ f = id_X$ and $f \circ g = id_Y$. This is all we need for g to be an isomorphism from Y to X that preserves that same kind of structure.

- Isomorphism is transitive. For any three sets X, Y and Z and any kind of structure, if there is a structure preserving isomorphism from X to Y and isomorphism from Y to Z that preserves the same kind of structure then there is an isomorphism from X to Z that preserves that structure.

  Assume that $f : X \to Y$ is an isomorphism that preserves some kind of structure. By the definition of isomorphism we have that there exists a function f' from Y to X that preserves that kind of structure, $f' \circ f = id_X$ and $f \circ f' = id_Y$. Assume that $g : Y \to Z$ is an isomorphism that preserves the same kind of structure. By the definition of isomorphism we have that there exists a function g' from Z to Y that preserves that kind of structure, $g' \circ g = id_Y$ and $g \circ g' = id_Z$. We have that $g \circ f$ is a function from X to Z and since both f and g preserve our kind of structure, so does $g \circ f$. We have that $f' \circ g'$ is a function from Z to X that for the same reason preserves the same kind of structure. We have $(g \circ f) \circ (f' \circ g') = g \circ (f \circ (f' \circ g')) = g \circ ((f \circ f') \circ g') = g \circ (id_Y \circ g') = g \circ g' = id_Z$ by associativity of function composition and $(f' \circ g') \circ (g \circ f) = id_X$ for the same reason. So we have that $g \circ f$ is an isomorphism between X and Z that preserves our chosen kind of structure. □

This means that we can speak about isomorphisms *between* two sets instead of *from* one set *to* another. We can also say that X and Y are **isomorphic** if there exists an isomorphism $f : X \to Y$. Finally this allows us to talk about isomorphism classes.

Definition A.1.4. Given some isomorphism, the **isomorphism class** of some set is the class of all sets isomorphic to the given set.

By Lemma A.1.3 we can say that all members of such a class are isomorphic to each other and that any set isomorphic to any of the member of the isomorphism class is also a member of the isomorphism class. In other words, an isomorphism class is the maximal class for which all members are isomorphic to each other.

A.1.1.1 Non-structure-preserving Isomorphisms

For cardinality we use isomorphisms that don't preserve any structure at all. First:

Definition A.1.5. A function, $f : X \to Y$, is **injective** if for all $x_1, x_2 \in X$ such that $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$.

So for every $x \in X$ there is a unique $y \in Y$ such that $f(x) = y$.

Definition A.1.6. A function, $f : X \to Y$, is **onto** if for all $y \in Y$ there is a $x \in X$ such that $f(x) = y$.

So the range is equal to the codomain.

Definition A.1.7. A function is a **bijection** if it is injective and onto.

Now we get as alternative characterization for non-structure-preserving isomorphisms:

Lemma A.1.8. *If $f : X \to Y$ is a function then f is a non-structure-preserving isomorphism if and only if it is a bijection*

Proof.

- $\Rightarrow$ Assume that $f : X \to Y$ is non-structure-preserving isomorphism. We have that there exists a function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

  We have that $f \circ g = id_Y$. The range of id_Y is, by definition, Y, so the range of $f \circ g$ is Y and hence the range of f must be Y. So f is onto.

  For all $x \in X$ we have $g(f(x)) = x$. Suppose, for contradiction, that f isn't injective, there are $x_1, x_2 \in X$, such that $x_1 \neq x_2$ for which it holds that $f(x_1) = f(x_2)$. We get $g(f(x_1)) = g(f(x_2)$ and hence $x_1 = x_2$ which is a contradiction. So f is injective.

  So f is a bijection.

- $\Leftarrow$ Assume that $f : X \to Y$ is a bijection, we get that for all $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$. Consider the function that maps all $y \in Y$ to the unique $x \in X$ such that $f(x) = y$. This function is called the inverse of f, it is written as f^{-1} and is a function from Y to X such that $f \circ f^{-1} = id_X$ and $f^{-1} \circ f = id_Y$. This proves that f is an isomorphism. □

For isomorphisms that do actually preserve some kind of structure, only the $\Rightarrow$ direction applies, a bijection might not preserve any structure.

A.1.1.2 Order-preserving Isomorphisms

For ordinality we need morphisms that preserve the order:

Definition A.1.9. A morphism f from one ordered set $(A, \leq_A)$ to another $(B, \leq_B)$ is **order-preserving** if for all $x_1, x_2 \in A$ we have that $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. We can call such a morphism an **order-morphism** and isomorphisms induced by these morphisms are called **order-isomorphisms**.

In the case order-isomorphisms between well-ordered sets we also have an alternative characterization: An order-isomorphism between well-ordered sets is a function that is onto and order-preserving. We can even strengthen the lemma that shows this a bit and talk about linearly ordered sets (which is more general).

Lemma A.1.10. *A linearly ordered set X is order-isomorphic to linearly ordered set Y if and only if there is an order-preserving function $f : X \to Y$ that is onto.*

Proof.

- $\Rightarrow$ By X and Y being isomorphic we have that there is an order-preserving function $f : X \to Y$. We have that $f \circ g = id_Y$. The range of id_Y is, by definition, Y, so the range of $f \circ g$ is Y and hence the range of f must be Y. So f is onto.

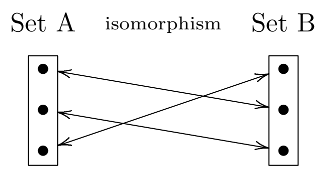
- $\Leftarrow$ First we want to prove that f is injective. We need that for all $x_1, x_2 \in X$ that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. If $x_1 \neq x_2$ we have that $x_1 < x_2$ or $x_2 < x_1$ by linearity of X, and because f is order-preserving we get that $f(x_1) < f(x_2)$ or $f(x_2) < f(x_1)$ and hence $f(x_1) \neq f(x_2)$.

  Because f is injective and onto, it is a bijection and hence it has an inverse and we can define our g as that inverse, f^{-1}. We then get $f \circ f^{-1} = id_X$ and $f^{-1} \circ f = id_Y$ by definition. We still need order-preservingness for f^{-1}, that is, for all $y_1, y_2 \in Y$ it hold that if $y_1 < y_2$ then $f^{-1}(y_1) < f^{-1}(y_2)$. Because f is onto we have that for any y_1 and y_2 there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. So assuming that $y_1 < y_2$ we need that $x_1 < x_2$. Suppose that isn't the case, then by linearity we'd have $x_2 < x_1$ or $x_1 = x_2$. If we have $x_1 = x_2$, then, since f is a function (and hence single-valued), we should also have $y_1 = y_2$, which we don't. Contradiction. If we have $x_2 < x_1$, then, because f is order-preserving, we should also have $y_2 < y_1$, but we already have $y_1 < y_2$. This contradicts the asymmetry of the linear order.

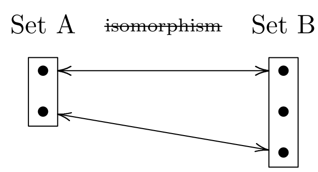
  So we must have order-preservingness for f^{-1} after all. □

A.1.2 Ordinals versus Cardinals

When we want to check if two sets have the same cardinality we only want to know about quantity. Are there as much, more, or less objects in one set when compared to the other? We don't actually care about the aspects of the internal structure of the set other than quantity. So we say that two sets are of the same cardinality when there is a non-structure preserving isomorphism (see Section A.1.1.1) between them. That is, if there is a bijection between them (Lemma A.1.8). Checking for equality of cardinality by checking for bijections conforms to the intuitive method of checking if two sets are of equal size by drawing lines between the objects in both sets. A line then represents an instance of the functions f and g that underlie our isomorphism (f when looked at from left to right, g when looked at from right to left). If in the end all objects in both sets are connected by one (and only one) line to an object in the other set, there is an isomorphism (namely the function represented by the drawn lines) between the sets and they are of equal size. If there are unconnected objects left in one set, then there is no isomorphism and the set with unconnected objects is bigger in size.

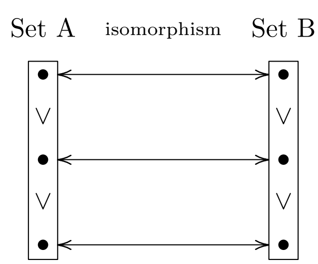


Isomorphism between sets A and B. They are of equal cardinality.

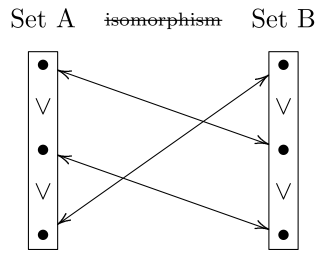


No isomorphism between sets A and B. They are of not equal cardinality (B is larger).

If we want to check if two sets are of the same ordinality, we need something more than just a non-structure-preserving isomorphism between sets. Since ordinality is concerned with the order aspect of numbers, we need to consider an ordering on the elements of the set. We'll be using well-orders and say that two well-ordered sets are of the same ordinality if there is an order-isomorphism between them (see Section A.1.1.2).



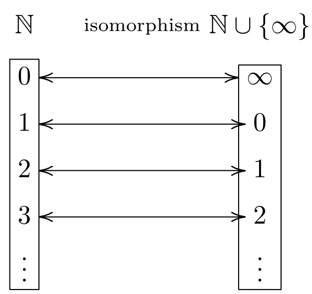
Order-isomorphism between ordered sets A and B (respects the order). Shows that they are of equal ordinality.



Not an order-isomorphism between sets A and B (doesn't respect the order).
Does not show that they are of equal ordinality (but they are).

For finite sets, ordinality and cardinality coincide (no matter what ordering on the set we consider for ordinality). This is the case, because, although there are multiple ways (permutations) to order the elements of a finite set into a sequence ($\{a, b, c\}$ can be ordered as $\langle a, b, c\rangle$, $\langle a, c, b\rangle$, $\langle b, a, c\rangle$, $\langle b, c, a\rangle$, $\langle c, a, b\rangle$, $\langle c, b, a\rangle$), all ordered sets of the same finite cardinality (size) are order-isomorphic and hence so are our permutations. To see this, finite well-ordered sets have a smallest element, a next smallest element, etc. So an order-isomorphism can just map the smallest element (according to the ordering) of the first set to the smallest element of the second set, etc. and hence be order-preserving.

For transfinite purposes though, ordinality and cardinality do not coincide. An example would be the set of natural numbers with the usual ordering and the set of natural number with one element added to it that represents infinity (∞) and is larger than all other elements. These sets are of the same cardinality as shown by the following isomorphism:



This is a perfectly fine bijection, but it is not an order-isomorphism, it does not preserve the order ($0 < 1$ but 0 is mapped to ∞ and 1 is mapped to 0 while $\infty \not< 0$). In fact such an order-isomorphism can not exist since $\mathbb{N} \cup \{\infty\}$ has a greatest element (∞) and $\mathbb{N}$ does not.

A.1.3 Ordinals

Working in ZFC set theory, ordinals can be defined in multiple ways, one such way is the following:

Definition A.1.11. A set S is **transitive** if $y \in S \Rightarrow y \subset S$ (if y is not an ur-element), or equivalently $\forall x, y(x \in y \in S \to x \in S)$. The **class of ordinals**, On, is the class of transitive ZFC sets such that for any $\alpha \in On$, $\in$ is a well-order (Definition A.4.32)) on the members of α. For any $\alpha, \beta \in On$ we write $\alpha < \beta$ if $\alpha \in \beta$.

For transitive sets in general we have:

Lemma A.1.12. *If β and α are transitive sets, $\in$ is linear on β and $\alpha \subset \beta$, then we have that $\alpha \in \beta$.*

Proof. Let β and α be transitive, let $\in$ be linear on β and let $\alpha \subset \beta$. Let ξ be an $\in$-minimal element in $\beta \setminus \alpha$ ($\xi \in (\beta \setminus \alpha)$ and $\xi \cap (\beta \setminus \alpha) = \emptyset$. This $\in$-minimal element exists by the ZFC axiom of foundation.

We prove that $\alpha = \xi$:

$\xi \subseteq \alpha$ Let $\gamma \in \xi$. We get $\gamma \in \beta$ (because $\xi \in \beta$ and β is transitive) and we do have that $\gamma \notin (\beta \setminus \alpha)$, (because $\gamma \in \xi$ and ξ is $\in$-minimal in $\beta \setminus \alpha$). So $\gamma \in \alpha$.

$\alpha \subseteq \xi$ Let $\gamma \in \alpha$.

- Since $\xi \in (\beta \setminus \alpha)$ we get $\xi \notin \alpha$ and hence $\xi \neq \gamma$.
- Suppose for contradiction that $\xi \in \gamma$, we would get $\xi \in \alpha$ by transitivity of α, contradicting $\xi \notin \alpha$. So we have $\xi \notin \gamma$

We have $\gamma \in \alpha \in \beta$ and hence $\gamma \in \beta$ by transitivity of β, can apply $\in$ being linear on β and get $\gamma \in \xi$.

So $\alpha = \xi$ and hence $\alpha \in \beta$. □

So by definition and the result above we get that for all $\alpha, \beta \in On$ we have $\alpha \in \beta \Leftrightarrow \alpha \subset \beta \Leftrightarrow \alpha < \beta$.

To prove that $(On, <)$ (or $(On, \in)$) is a well-ordered class:

Lemma A.1.13. *On is well-ordered by $\in$.*

Proof.

- $\in$ is transitive on On. $\forall \alpha, \beta, \gamma \in On((\alpha \in \beta \wedge \beta \in \gamma) \to \alpha \leq \gamma)$.

  Let $\alpha, \beta, \gamma \in On$ such that $\alpha \in \beta$ and $\beta \in \gamma$. γ is, by definition, a transitive set, so $\alpha \in \gamma$, which proves transitivity.

- $\in$ is linear on On. $\forall \alpha, \beta \in On(\alpha \in \beta \vee \beta \in \alpha \vee \beta = \alpha)$.

  Let $\alpha, \beta \in On$. We have that α and β are transitive sets well-ordered by $\in$ and we need to prove that $\alpha \in \beta$, $\beta \in \alpha$ or $\alpha = \beta$.

  Let $\gamma = \alpha \cap \beta$, we get that $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$ we also get that γ is well-ordered by $\in$ and transitive because α and β are.

  Based on how $\alpha \cap \beta$ compares to α and β we can break the proof down in the following cases:

  - $\alpha \cap \beta = \alpha$ and $\alpha \cap \beta = \beta$. We get $\alpha = \beta$ and we're done immediately.
  - $\alpha \cap \beta \neq \alpha$ but $\alpha \cap \beta = \beta$. We get $\gamma \subset \alpha$ and hence $\beta \subset \alpha$. We now have everything in place to apply Lemma A.1.12 on β and α and get $\beta \in \alpha$ when we do. We're done.
  - $\gamma = \alpha$ but $\gamma \neq \beta$. We get $\gamma \subset \beta$ and hence $\alpha \subset \beta$. We again apply Lemma A.1.12 and get $\alpha \in \beta$. We're done.
  - $\gamma \neq \alpha$ and $\gamma \neq \beta$. We get that $\gamma \subset \beta$, apply Lemma A.1.12 and get $\gamma \in \beta$. We also get that $\gamma \subset \alpha$, apply Lemma A.1.12and get $\gamma \in \alpha$. So because $\gamma \in \beta$ and $\gamma \in \alpha$, by the definition of γ we have $\gamma \in \gamma$. This contradicts the axiom of foundation, so this case is excluded. We're done, this time, really done.

  - $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. We get $\alpha = \beta$ and we're done immediately.
  - $\alpha \nsubseteq \beta$ and $\beta \subseteq \alpha$. We get $\beta \subset \alpha$ and $\beta \in \alpha$ by Lemma A.1.12.
  - $\alpha \subseteq \beta$ and $\beta \nsubseteq \alpha$. We get $\alpha \subset \beta$ and $\beta \in \alpha$ by Lemma A.1.12.
  - $\gamma \neq \alpha$ and $\gamma \neq \beta$. We get $(\alpha \cap \beta) \subset \beta$, apply Lemma A.1.12 and get $\gamma \in \beta$. We also get that $(\alpha \cap \beta) \subset \alpha$, apply Lemma A.1.12 and get $\gamma \in \alpha$. So because $(\alpha \cap \beta) \in \alpha$ and $(\alpha \cap \beta) \in \beta$, we get $(\alpha \cap \beta) \in (\alpha \cap \beta)$. This contradicts the axiom of foundation of ZFC, so this case is excluded. We're done.

- $\in$ is well-founded on On. $\forall X \subseteq On(X \neq \emptyset \rightarrow \exists \xi \in X \forall \alpha \in X(\neg(\alpha \in \xi)))$

  Given the axiom of choice an order is well-founded if and only if there are no infinite descending sequences along the order. Suppose, for contradiction, there would be such a sequence in On: $\alpha \in \beta \in \gamma \in \ldots$. Then there also would be such a sequence in α, namely $\beta \in \gamma \in \ldots$, but α is a set, and according to the axiom of foundation, there are no infinite descending sequences of sets. ☐

Definition A.1.14. If a well-ordered set is order-isomorphic to some ordinal it is said to be of the ordinality of that ordinal.

Lemma A.1.15. *Every well-ordered set is order-isomorphic to some ordinal.*

Proof. Takes loads of space, omitted. ☐

So every well-ordered set can be said to be of some ordinality.

Other properties of the class of ordinals are:

Lemma A.1.16. *On is transitive. If $\beta \in \alpha \in On$ then $\beta \in On$.*

Proof. Let $\beta \in \alpha \in On$. We need to prove that β is a transitive set and β is well-ordered by $\in$.

- β is transitive.

  Let $x \in y \in \beta \in \alpha$, by the transitivity of $\in$ on α we get $x \in \beta \in \alpha$. So $x \in y \in \beta \Rightarrow x \in \beta$. So β is transitive.

- β is well-ordered by $\in$.

  We have $\beta \in \alpha$, so because α is a transitive set we get $\beta \subset \alpha$. By restricting the well-order $\in$ on α to the elements of β we get that $\in$ is a well-order on β. □

Lemma A.1.17. *On is not a set.*

Proof. We have that On is well-ordered by $\in$ (Lemma A.1.13) and transitive (Lemma A.1.16). Suppose it is also a (ZFC) set, then by definition of On we have $On \in On$. This contradicts the axiom of foundation of ZFC. □

This means On is a proper class and not part of the ZFC universe.

Lemma A.1.18. *On is unbounded.* $\forall \alpha \in On \exists \beta \in On(\alpha < \beta)$

Proof. Let $\alpha \in On$. Consider the set $\alpha \cup \{\alpha\}$. This set is transitive because α is and also well-ordered by $\in$ because α is, so $\alpha \cup \{\alpha\} \in On$. And since $\alpha \in (\alpha \cup \{\alpha\})$ we have $\alpha < \beta$, which proves unboundedness of On. □

Even though On is not a set, it is well-ordered and the proof for induction over well-orders (Lemma A.4.33) also holds for well-ordered proper classes. So we have:

Lemma A.1.19. *For any property P it holds that* $\forall \alpha \in On((\forall \beta < \alpha(P(\beta))) \to P(\alpha)) \Rightarrow \forall \alpha \in On P(\alpha)$

Proof. Similar to Lemma A.4.33. □

In the transfinite world of ordinals, we call this scheme **(transfinite) induction**.

The members of On, the ordinals, are also well-ordered themselves. So we also have an induction scheme on any individual ordinal by Lemma A.4.33.

A.1.3.1 The Von Neumann Characterization

Ordinals can be characterized as follows:

Lemma A.1.20. *For every $\alpha \in On$ we have that* $\alpha = \{\beta \in On \mid \beta < \alpha\}$

Proof.

- $\alpha \subseteq \{\beta \in On \mid \beta < \alpha\}$.

  Let $\gamma \in \alpha$. Because On is transitive (Lemma A.1.16) we get $\gamma \in On$. By definition of our order $<$ on ordinals, we also get $\gamma < \beta$. So we have $\gamma \in \{\beta \in On \mid \beta < \alpha\}$.

- $\{\beta \in On \mid \beta < \alpha\} \subseteq \alpha$.

  Let $\gamma \in \{\beta \in On \mid \beta < \alpha\}$. We get $\gamma \in \beta$ by definition of $<$. □

This in turn gives rise to the following explicit characterization (due to John Von Neumann) of the class of ordinals:

Characterization A.1.21. *Every ordinal is the set of its predecessors. When looked at this way, the ordinals are also referred to as* ***Von Neumann ordinals****.*

Because the class of ordinals is well-ordered (Lemma A.1.13) there must be a least ordinal. For the least ordinal, we have that there is no ordinal smaller than it, so by the Von Neumann characterization (or; by Lemma A.1.20) this least ordinal is $\emptyset$.

Definition A.1.22. We'll denote the **least ordinal** by 0. $0 = \emptyset$.

After obtaining 0, it makes sense to speak about a 'next' ordinal, since the class of ordinals is well-ordered and the class of ordinals with 0 removed from it has a least member. This next ordinal is the set having only $\emptyset$ as element, $\{\emptyset\}$. This because $\emptyset$ is the only ordinal smaller than it. In fact for any ordinal, α, we have that the 'next' ordinal (the least ordinal larger than it) is $\alpha \cup \{\alpha\}$.

Lemma A.1.23. $\alpha \cup \{\alpha\}$ *is the least ordinal larger than any* α*.*

Proof. Let $\alpha \in On$. Lemma A.1.18 proves that $\alpha \cup \{\alpha\} \in On$ and $\alpha < \alpha \cup \{\alpha\}$.

Suppose, for contradiction, that there is some $\beta \in On$ such that $\alpha < \beta < (\alpha \cup \{\alpha\})$. By the Von Neumann characterization we have $\beta \in (\alpha \cup \{\alpha\})$ so we either have $\beta \in \alpha$ (and hence $\beta < \alpha$) or $\beta = \alpha$, both form a contradiction with $\alpha < \beta$ and the linearity of $<$. $\square$

Definition A.1.24. If α is an ordinal, then we call the least ordinal larger than α the **successor** of α and write $\alpha\prime$ or $\alpha + 1$. We have $\alpha\prime = \alpha \cup \{\alpha\}$. We denote the class of successor ordinals by SOrd.

Remark. We write $\alpha\prime$ as opposed to α', which we'll use as variable (α' will often denote a variable that is in some way related to the variable denoted by α).

This means that we can make the link with the natural numbers explicit by giving an the following initial segment of the class of (Von Neumann) ordinals.

- $0 := \emptyset = \{\}$
- $1 := 0\prime = \{0\} = \{\emptyset\} = \{\{\}\}$
- $2 := 0\prime\prime = \{0, 1\} = \{\emptyset, \{\emptyset\}\} = \{\{\}, \{\{\}\}\}$
- $3 := 0\prime\prime\prime = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$
- $\ldots$

Definition A.1.25. An ordinal that is not 0 or the successor of another ordinal is called a **limit ordinal**. We denote the class of limit ordinals by LimOrd and all the limit ordinals that are smaller than (and hence a member of) some other ordinal α as LimOrd(α).

Remark. In the literature, LimOrd is sometimes defined as having 0 as member and sometimes as not having 0 as member. Which one we choose might matter while reasoning about ordinals by cases (where we have a case for successor ordinals, a case for limit ordinals and could have a separate case for 0). Defining LimOrd as not containing 0 seems like the safe option. The worst we can do is having a case too many and hence having to do some extra work. This seems better than missing a case and getting in trouble.

The existence of a least one limit ordinal is guaranteed by the infinity axiom of ZFC, which states that there exists a set ω such that $\emptyset \in \omega$ and for all $n \in \omega$ we have $(n \cup \{n\}) \in \omega$. For this one limit ordinal others can be created using the other axioms of set theory. So we have that LimOrd $\neq \emptyset$. We also have that:

Lemma A.1.26. *Limit ordinals are unbounded ($\forall \alpha < \lambda \exists \beta < \lambda.\ \alpha < \beta$)*

Proof. Let $\lambda \in$ LimOrd and $\alpha < \lambda$. We have that $\alpha\prime \neq \lambda$, because limit ordinals are not successor ordinals, by definition. We also have $\lambda \not< \alpha\prime$, because we have $\alpha < \lambda$, $\alpha\prime$ is the least ordinal larger than α and $<$ is linear. So by linearity of On we must have $a\prime < \lambda$, which proves unboundedness for limit ordinals. □

With this classification of ordinals (as either 0, successor ordinals or limit ordinals) we can give a case-by-case version of our transfinite induction scheme (Lemma A.1.19):

Lemma A.1.27. *For any property P it holds that $P(0) \wedge \forall \alpha \in On.\ (P(\alpha) \rightarrow P(\alpha\prime))) \wedge \forall \lambda \in \mathrm{LimOrd}(\forall \alpha < \lambda.\ P(\alpha) \rightarrow P(\lambda)) \Rightarrow \forall \alpha \in On.\ P(\alpha)$*

Proof. We can prove this by adapting the proof of Lemma A.4.33 slightly. Assume $P(0)$, $\forall \alpha \in On.\ (P(\alpha) \rightarrow P(\alpha\prime))$ and $\forall \lambda \in \mathrm{LimOrd}.\ (\forall \alpha < \lambda.\ P(\alpha) \rightarrow P(\lambda))$. Define $F = \{\alpha \in On \mid \neg P(\alpha)\}$. Suppose for contradiction that we do not have $\forall \alpha \in On.\ P(\alpha)$, then F is non-empty and hence has a $<$-least element (because $<$ is well-founded), call this least element ξ, we have $\neg P(\xi)$. By the definitions of the least ordinal, successor ordinals and limit ordinals we have exactly 3 cases for ξ.

- $\xi = 0$. We have $P(0)$ by our first assumption and contradict $\neg P(\xi)$.
- $\xi = \eta\prime$. We have $\eta < \xi$, by the minimality of ξ we have $P(\eta)$, and hence by our second assumption $P(\xi)$. Contradiction.
- $\xi \in$ LimOrd. By the minimality of ξ we get $\forall \eta < \xi.\ (P(\eta))$ and hence by our third assumption $\neg P(\xi)$. Contradiction.

So we get a contradiction in all possible cases, hence we must have $\forall \alpha \in On(P(\alpha))$ after all. □

So, when using this scheme as a proof-technique to prove $\forall \alpha \in On.\ (P(x))$, we need to prove that:

- $P(0)$. We call this the **base case** of our proof by induction.
- $\forall \alpha \in On.\ (P(\alpha) \rightarrow P(\alpha\prime))$. That is, for an arbitrary $\alpha \in On$, we can assume the induction hypothesis $P(\alpha)$ to prove $P(\alpha\prime)$. We call this the **step case** of our induction.
- $\forall \lambda \in \text{LimOrd}.\ (\forall \alpha < \lambda.\ P(\alpha) \rightarrow P(\lambda))$. That is, for an arbitrary limit ordinal λ we can assume the induction hypothesis $\forall \alpha < \lambda.\ P(\alpha)$ to prove $P(\lambda)$. We call this the **limit case** of our induction.

Again, this case-by-case version of transfinite induction not only goes for the class of ordinals, but also goes for all ordinals since their members are either 0, a successor ordinal or a limit ordinal.

A.1.3.2 Natural Numbers

Now we can define a subset of On which we can identify with our **natural numbers**.

Definition A.1.28. $\mathbb{N}$ is the subset of On such that $0 \in \mathbb{N}$ and $\forall n \in \mathbb{N}$ we have $n\prime \in \mathbb{N}$

We get that:

- $\mathbb{N} \cap \text{LimOrd} = \emptyset$
- $\mathbb{N}$ is an initial segment of On $(\forall \alpha \in \mathbb{N}. \forall \beta \in On.\ (\beta < \alpha \rightarrow \beta \in \mathbb{N}))$

We also get:

Lemma A.1.29. *For any property P it holds that $P(0) \wedge \forall \alpha \in \mathbb{N}.\ (P(\alpha) \rightarrow P(\alpha\prime)) \Rightarrow \forall \alpha \in \mathbb{N}.\ P(\alpha)$*

Proof. This follows from the fact that $\mathbb{N} \cap \text{LimOrd} = \emptyset$. Therefore the limit case of Lemma A.1.27 drops out. □

We call this scheme **natural induction**.
We have that:

Lemma A.1.30. *If $M \subset On$ is bounded in On $(\exists \alpha \in On. \forall \beta \in M.\ \alpha \neq \beta \rightarrow \beta < \alpha)$ then there is a least upper bound for M.*

Proof. Consider the class $\{\alpha \in On \mid \forall \beta \in M.\ (\alpha \neq \beta \rightarrow \beta < \alpha\}$, the class of upper bounds of M. Because M is bounded in On this is a non-empty subclass of On and hence by well-foundedness of $<$ is has a least member. This least member is the least upper bound for M by definition. □

Definition A.1.31. For the least upper bound of M we write $\sup M$.

$\mathbb{N}$ is bounded in On by the limit ordinal ω, guaranteed to exist by the axiom of infinity we get:

- $\omega = \sup \mathbb{N}$
- $\omega \notin \mathbb{N}$
- $\mathbb{N}$ is a *proper* initial segment of On
- $\mathbb{N} = \omega$ (although $\mathbb{N}$ is to be thought of as a set of ordinals and ω as an ordinal, from the point of view of set theoretic extensionality, they are equal)

Any ordinal number in $\mathbb{N}$ is called **finite**, any ordinal number not in $\mathbb{N}$ ($\geq \omega$) is called **transfinite**.

A.1.3.3 Ordinal Arithmetic

On the natural numbers we have our everyday-operations, addition and multiplication. These operations can be generalized to ordinals, but commutativity is lost. We do not necessarily have $\alpha + \beta = \beta + \alpha$ or $\alpha * \beta = \beta * \alpha$.

We can define addition on ordinals in an inductive way:

Definition A.1.32.

- For 0 we have $\alpha + 0 = \alpha$
- For successor ordinals we have $\alpha + \beta\prime = (\alpha + \beta)\prime$
- For limit ordinals we have $\alpha + \lambda = \bigsqcup\{\alpha + \gamma \mid \gamma < \lambda\}$

We get:

Proposition A.1.33. For $\alpha, \beta \in On$ such that $\beta \leq \alpha$ we have a unique ordinal γ such that $\alpha = \beta + \gamma$.

And can define:

Definition A.1.34. For any $\alpha, \beta \in On$ such that $\beta \leq \alpha$, if γ is the unique ordinal such that $\alpha = \beta + \gamma$ we write $\alpha - \beta = \gamma$. We can view $-$ as an operation and call it left subtraction.

Right subtraction poses problems, there is no ordinal γ such that $\gamma + 1 = \omega$, so we can't 'right subtract' 1 from ω.

An operation on ordinals that is related to addition and is useful for rewriting is †:

Definition A.1.35. If $\alpha \in On$ then $\alpha^\dagger = \begin{cases} \alpha & \text{if } \alpha \in \text{LimOrd}, \\ \alpha + 1 & \text{otherwise} \end{cases}$

We can also define multiplication on ordinals in an inductive way:

Definition A.1.36.

- For 0 we have $\alpha * 0 = 0$

- For successor ordinals we have $\alpha * \beta\prime = (\alpha * \beta) + \alpha$
- For limit ordinals $\alpha * \lambda = \bigsqcup\{\alpha * \gamma \mid \gamma < \lambda\}$

We get:

Proposition A.1.37. For any $\alpha, \beta \in On$ such that $\beta \neq 0$ we get unique $\gamma, \delta \in On$ such that $\delta < \gamma$ and $\alpha = (\beta * \gamma) + \delta$.

And can define:

Definition A.1.38. For any $\alpha, \beta \in On$ such that $\beta \neq 0$, if γ is the unique ordinal such that there is a $\delta < \gamma$ and $\alpha = (\beta * \gamma) + \delta$ we write $\alpha/\beta = \gamma$ and $\alpha\%\beta = \delta$. We can view / and % as operations and call them respectively **left division** and **left remainder**.

Again, right division does not work.
Using the left remainder operation on ordinals, we can define parity.

Definition A.1.39. An ordinal, α, is either **odd** or **even**.

- If $\alpha\%2 = 0$ then α is **even**.
- If $\alpha\%2 = 1$ then α is **odd**.

We write Even(α) and Odd(α) to denote that α is, respectively odd or even.

We get that 0 and every limit ordinal are even, the successor of every odd ordinal is even and vice versa.

Lemma A.1.40. *Both the even and odd ordinals are unbounded in any limit ordinal.*

Proof. Let $\lambda \in \text{LimOrd}$. For any ordinal $\beta < \lambda$ we have that β is either odd or even, by Lemma A.1.26 λ is unbounded and we have $\beta\prime, \beta\prime\prime < \lambda$. if β is odd, we have $\beta\prime$ is even and $\beta\prime\prime$ is odd, if β is even we have $\beta\prime$ is odd and $\beta\prime\prime$ is even. □

A.1.4 Cardinals

As has been said, cardinality describes quantity. We compared the quantity of objects in a set with quantity the objects in another set by saying that two sets have the same cardinality if there is an isomorphism between those sets.

Definition A.1.41. If there is an isomorphism between two sets, the sets are called **equinumerous**. The isomorphism doesn't have to preserve any structure. Two sets have the same **cardinality** if they are equinumerous.

By Lemma A.1.3 this give rise to isomorphism classes, which we might call **equinumerosity classes**.

So equinumerosity is the notion that formalizes cardinality. As cardinal numbers, the numbers describing cardinality, we might take a representative from each equinumerosity class. Good representatives that come to mind are the ordinals in each class, if we take those, we inherit all the good properties of ordinals (well-orderedness for instance). Each equinumerosity class contains at least one ordinal.

Lemma A.1.42. *Any set is equinumerous to some ordinal.*

Proof. By the axiom of choice we have that any set can be well-ordered. Any well-ordered set is order-isomorphic to some ordinal, so if we don't look at the order of the sets we still have an isomorphism and hence equinumerosity between any set an some ordinal. □

It isn't the case however that any such an equinumerosity class contains only one ordinal. By dropping the order, ordinals that were not order-isomorphic (and hence unequal) might become isomorphic in our 'plain'/cardinal sense. For example, the ordinals $\omega\prime$ and ω are equinumerous, as witnessed by the function $f : \omega\prime \to \omega$ that is such that $f(x) = x\prime$ for each $x < \omega$ while $f(\omega) = 0$. This function is a bijection and hence, by Lemma A.1.8, $\omega\prime$ and ω are isomorphic and hence equinumerous. But since the class of ordinals, On, is well-ordered we might just take the least ordinal in the equivalence class as its representative.

Definition A.1.43. Ca, the class of cardinal numbers, is the class of the least ordinals of the equinumerosity classes. A set is said to be of a certain cardinality if it is equinumerous to the cardinal number that represents that cardinality.

This means that $Ca \subset On$ and that Ca is well-ordered by the same order as the ordinal on ordinals, only restricted on Ca.

An ordinal is not a cardinal if it is equinumerous to a smaller ordinal. So for all $\alpha \in \mathbb{N}$ we have that $\alpha \in Ca$ because all ordinals smaller than α are also in $\mathbb{N}$ ($\mathbb{N}$ is an initial segment of On) and:

Lemma A.1.44. *For all $n, m \in \mathbb{N}$ we have that if m and n are equinumerous then $m = n$.*

Proof. Since $n, m \in \mathbb{N}$ we can apply natural induction. We first do induction over m. We get the following cases:

- If 0 and n are equinumerous then $0 = n$.

  Suppose that 0 and n are equinumerous. This means, by Lemma A.1.8, that there is a bijection between 0 and n. We have that $0 = \emptyset$, so the only function from 0 to n is the empty function. So the empty function is onto n, so $n = \emptyset = 0$.

- We get as induction hypothesis that for any n, if m is equinumerous to n then $m = n$, and need to prove that for any n, if $m\prime$ is equinumerous to n then $m\prime = n$.

  We do induction over n.

– If $m\prime$ and 0 are equinumerous then $m\prime = 0$.

  There is no function from a set with elements (which $m\prime$ is, $m \in m\prime$) to a set without (which 0 is, $0 = \emptyset$). So there can't be an isomorphism between $m\prime$ and 0 and they can never be equinumerous.

– We get as our second induction hypothesis that if $m\prime$ is equinumerous to n then $m\prime = n$ and need to prove that if $m\prime$ is equinumerous to $n\prime$ then $m\prime = n\prime$.

  Suppose that $m\prime$ is equinumerous to $n\prime$. Then by our first induction hypothesis $m = n$ and hence we get $m\prime = n\prime$. □

This formalizes our claim that for the natural numbers, ordinality and cardinality coincide. So $\mathbb{N}$ is not only an initial segment of the ordinals, it is also an initial segment of the cardinals. We can refer to those natural cardinals, by their natural number representation ($0\prime = 1$, $0\prime\prime = 2$, $0\prime\prime\prime = 3$ etc).

The least cardinal that is not in $\mathbb{N}$ is ω, this is because ω is the least ordinal not in $\mathbb{N}$ and it is not equinumerous to an element of $\mathbb{N}$.

Lemma A.1.45. *For all $n \in \mathbb{N}$ we have that n is not equinumerous to ω.*

Proof. We do induction on n. We get the following cases:

- 0 is not equinumerous to ω.

  The only function from 0 to ω is the empty function. This function is not onto since ω contains members, 0 for instance. So there is no bijection between 0 and ω and hence no isomorphism, so 0 and ω are not equinumerous.

- We get as induction hypothesis that n is not equinumerous to ω and need to prove that $n\prime$ is not equinumerous to ω.

  Suppose, for contradiction, that $n\prime$ is equinumerous to ω. By Lemma A.1.8 there is a bijection, $f : \omega \to n\prime$, from ω to $n\prime$. Since this function is onto and $n \in n\prime$ there is an $x \in \omega$ such that $f(x) = n$. We can now construct the function, g, which is such that for all $y < x$ we have $g(y) = f(y)$ and for all y such that $\neg(y < x)$ we have $g(y) = f(y\prime)$. Since f is injective, only one element of ω (x) mapped to n, so n is not in the range of g, but aside from n, g inherits f's range. So range of g is $n\prime \setminus n$, which is n. That means, g is a function from ω to n that is onto. g also inherits f's injectiveness, so g is a bijection between ω and n, which means, by Lemma A.1.8, that n is equinumerous to ω. This contradicts the induction hypothesis, which means f can't exist, so $n\prime$ is not equinumerous to ω. □

When we refer to ω as a cardinal we write $\aleph_0$. A lot of ordinals bigger than ω are non-cardinal ordinals. The first ones are of the same cardinality as ω, $\aleph_0$ ($\omega\prime$ for instance, as mentioned).

Definition A.1.46. We denote the cardinality of a, possibly non-cardinal, set, α, by $\mathrm{card}(\alpha)$.

We can prove that when taking the cardinalities of non-cardinal ordinals, non-strict order is preserved.

We first prove a lemma (which is sometimes used to define the order on cardinals/cardinalities in the first place):

Lemma A.1.47. *For any* $\alpha, \beta \in On$ *we have* $\mathrm{card}(\alpha) \leq \mathrm{card}(\beta)$ *if and only if there is an injective function from* α *to* β*.*

Proof.

- $\Rightarrow$ Assume that $\mathrm{card}(\alpha) \leq \mathrm{card}(\beta)$. We get $\alpha \leq \beta$, which means $\mathrm{card}(\alpha) \subseteq \mathrm{card}(\beta)$ and hence we get an injective function from $\mathrm{card}(\alpha)$ to $\mathrm{card}(\beta)$.

- $\Leftarrow$ Assume that there is an injective function from α to β. Suppose, for contradiction, that we have $\mathrm{card}(\beta) < \mathrm{card}(\alpha)$. We get $\beta < \alpha$ and hence $\beta \subset \alpha$, so there is an injective function from β to α (map all elements of β to themselves). We have, by assumption, an injective function from α to β. These two injective functions, give us a bijection between α and β by the Cantor-Bernstein-Schroeder Theorem. By Lemma A.1.8 this means α and β are isomorphic and hence equinumerous and hence, so $\mathrm{card}(\alpha) = \mathrm{card}(\beta)$, this contradicts $\mathrm{card}(\beta) < \mathrm{card}(\alpha)$. So we can't have $\mathrm{card}(\beta) < \mathrm{card}(\alpha)$, and hence must have $\mathrm{card}(\alpha) \leq \mathrm{card}(\beta)$. □

Now we get:

Lemma A.1.48. *For any* $\alpha, \beta \in On$ *we have* $\alpha \leq \beta$ *implies* $\mathrm{card}(\alpha) \leq \mathrm{card}(\beta)$

Proof. Assume that $\alpha \leq \beta$. We get $\alpha \subseteq \beta$, hence an injective function from α to β (map all elements of α to themselves) and hence by Lemma A.1.47 $\mathrm{card}(\alpha) \leq \mathrm{card}(\beta)$. □

This means that the equinumerosity classes divide On up in intervals. In fact they divide On in left-closed right-open intervals, that is, classes of the form $\{x \in X \mid \alpha \leq x < \beta\}$ for some $\alpha, \beta \in On$.

We also have that:

Lemma A.1.49. *The class of cardinals is unbounded.* $\forall \alpha \in Ca(\exists \beta \in Ca(\alpha < \beta))$

Proof. For any $\alpha \in Ca$ there is a set that is certainly not equinumerous to it, its power set, $\mathcal{P}(\alpha)$.

There is an injection from α to $\mathcal{P}(\alpha)$, namely the function mapping any $x \in \alpha$ to $\{x\}$. So by Lemma A.1.47 so $\alpha \leq \mathrm{card}(\mathcal{P}(\alpha))$.

Let $f : \alpha \to \mathcal{P}(\alpha)$ be any function from α to $\mathcal{P}(\alpha)$ and let $D = \{x \in \alpha \mid x \notin f(x)\}$. Suppose, for contradiction, that there is $a \in \alpha$ such that $f(a) = D$. By the definition of D we have $a \in D \to a \notin D$. Which is a contradiction, so for all $x \in \alpha$ we have $f(x) \neq D$. But $D \in \mathcal{P}(\alpha)$, so no function from α to $\mathcal{P}(\alpha)$ can be onto and hence there can't be a bijection between α and $\mathcal{P}(\alpha)$. There is a bijection between $\mathcal{P}(\alpha)$ and $\mathrm{card}(\mathcal{P}(\alpha))$, so there can't be a bijection between α and $\mathrm{card}(\mathcal{P}(\alpha))$. So $\alpha \neq \mathrm{card}(\mathcal{P}(\alpha))$ and hence $\alpha < \mathrm{card}(\mathcal{P}(\alpha))$. □

This means that there are cardinals larger than $\aleph_0$. Since Ca is well-ordered, there will be a least such cardinal (larger than $\aleph_0$), we call this cardinal $\aleph_1$, after $\aleph_1$ comes $\aleph_2$ etcetera, after all the cardinals indexed by natural numbers, we get $\bigcup_{i<\omega} \aleph_i$, which we can denote by $\aleph_\omega$ and which is not equinumerous to any of the previous cardinals and hence a proper cardinal itself. So we see that we need to index our non-natural cardinals with ordinal numbers.

This brings us to notions of finiteness and the like.

Definition A.1.50.

- A set is said to be **finite** or containing finitely many members, if its cardinality is k, where $k \in \mathbb{N}$.
- A set is said to be **infinite**, **transfinite** or containing infinitely many members if it is not finite.
- A set is said to be **countable** or containing countably many members, if its cardinality is k, where $k \in \mathbb{N}$, or $\aleph_0$.
- A set is said to be **uncountable** or containing uncountably many members, if it is not countable.

A.1.5 Sequences

Now that we have defined ordinals we can formally define the concept of (transfinite) sequences:

Definition A.1.51. A **transfinite sequence** of elements of some set X is a function, s, from an ordinal, α, to X. α is called the **length** of the sequence.

If s is such a sequence of elements from X of length α then for all $n \in \alpha$ we call $s(x)$ the n-th element of the sequence and denote it as s_n. We can denote any finite sequence s of length $\alpha\prime \in \mathbb{N}$ as $\langle s_0, s_1, s_2, \ldots, s_\alpha \rangle$. We can denote a sequence, s of length ω as $\langle s_0, s_1, s_2, \ldots \rangle$. We can denote a sequence s of (arbitrary ordinal) length α as $\langle s_\beta \rangle_{\beta<\alpha}$.

Definition A.1.52. If $\langle s_\beta \rangle_{\beta<\alpha}$ is a sequence and $\gamma < \alpha$ we can call the sequence $\langle s_{\gamma+\beta} \rangle_{\beta \leq \alpha-\gamma}$ the **tail after** γ of the sequence or simply a **tail**.

A.1.5.1 Zipping Sequences

Definition A.1.53. For any two ordinals α_s and α_t we can define a partial function $z_{\alpha_s,\alpha_t} : On \to (\alpha_s \times \{\mathcal{S}\}) \cup (\alpha_t \times \{\mathcal{T}\})$ as follows. For any $\beta \in On$:

- $z_{\alpha_s,\alpha_t}(\beta) = \langle \beta/2, \mathcal{S} \rangle$ if $(\beta < 2\alpha_s, 2\alpha_t) \wedge \text{Even}(\beta)$
- $z_{\alpha_s,\alpha_t}(\beta) = \langle \beta/2, \mathcal{T} \rangle$ if $(\beta < 2\alpha_s, 2\alpha_t) \wedge \text{Odd}(\beta)$
- $z_{\alpha_s,\alpha_t}(\beta) = \langle \alpha_t + (\beta - 2\alpha_t), \mathcal{S} \rangle$ if $(2\alpha_t \leq \beta < 2\alpha_s) \wedge (\beta < 2\alpha_t + (\alpha_s - \alpha_t))$

- $z_{\alpha_s,\alpha_t}(\beta) = \langle \alpha_s + (\beta - 2\alpha_s), \mathcal{T} \rangle$ if $(2\alpha_s \leq \beta < 2\alpha_t) \wedge (\beta < 2\alpha_s + (\alpha_t - \alpha_s))$
- $z_{\alpha_s,\alpha_t}(\beta)$ is undefined otherwise.

We'll call this function the **zip function** of α_s and α_t.

We get:

Lemma A.1.54. *Such a zip function is well-defined.*

Proof. The cases of the definition are well-defined.

- First case: $\beta < 2\alpha_s$, so $\beta/2 < \alpha_s$ and hence $\beta/2 \in \alpha_s$.
- Second case: Like the first case, but for α_t.
- Third case: $2\alpha_t < 2\alpha_s$, so $\alpha_t < \alpha_s$, so $\alpha_s - \alpha_t$ is defined. And:

$$\begin{aligned} \beta &< 2\alpha_t + (\alpha_s - \alpha_t) \\ \beta - 2\alpha_t &< \alpha_s - \alpha_t \\ \alpha_t + (\beta - 2\alpha_t) &< \alpha_s \end{aligned}$$

  So $\alpha_t + (\beta - 2\alpha_t) \in \alpha_s$.

- Fourth case: Like the third case, but for α_t.

Also, the four cases do not overlap, because of the fact that ordinals are linearly ordered and that an ordinal is either odd or even (never both). □

Lemma A.1.55. *A zip function is a total function when we restrict its domain to $2\alpha_t + (\alpha_s - \alpha_t)$ if $\alpha_t \leq \alpha_s$ or $2\alpha_s + (\alpha_t - \alpha_s)$ if $\alpha_s \leq \alpha_t$.*

Proof. Let $\alpha_s, \alpha_t \in On$ and let z_{α_s,α_t} be their zip function.

First; for all $\beta, \gamma \in On$ such that $\gamma < \beta$ we have that if $z_{\alpha_s,\alpha_t}(\beta)$ is defined then, by the conditions of the defining cases the definition, so is $z_{\alpha_s,\alpha_t}(\gamma)$.

Secondly, if $\alpha_t < \alpha_s$ then the least ordinal for which z_{α_s,α_t} is not defined is $2\alpha_t + (\alpha_s - \alpha_t)$. This is because $z_{\alpha_s,\alpha_t}(2\alpha_t + (\alpha_s - \alpha_t))$ is not defined there since cases 1, 2 and 4 of the definition do not apply because $\alpha_t \leq 2\alpha_t \leq 2\alpha_t + (\alpha_s - \alpha_t)$, while 3 trivially does not apply. Also, for any ordinal, $\beta < 2\alpha_t + (\alpha_s - \alpha_t)$ we get that $z_{\alpha_s,\alpha_t}(\beta)$ is defined since we have $\alpha_s - \alpha_t < 2\alpha_s - 2\alpha_t$ (because we have $\alpha_t < \alpha_s$) and hence $2\alpha_t + (\alpha_s - \alpha_t) < 2\alpha_s$, so $\beta < \alpha_s$, so depending on whether $\beta < 2\alpha_t$ or not and on whether β is odd or even either case 1, 2 or 3 applies.

If $\alpha_t < \alpha_s$ we get that the least ordinal for which the z_{α_s,α_t} is not defined is $2\alpha_s + (\alpha_t - \alpha_s)$ in a similar fashion.

If $\alpha_t = \alpha_s$ then we get that the least ordinal for which the z_{α_s,α_t} is not defined is $2\alpha_s + (\alpha_t - \alpha_s) = 2\alpha_s + (\alpha_t - \alpha_s) = 2\alpha_s = 2\alpha_t$. z_{α_s,α_t} is not defined at the ordinal since, trivially, no case applies. It is defined on any smaller ordinal since, depending on whether that ordinal is odd or even, case 1 or 2 applies. □

Lemma A.1.56. *Zip functions are surjective.*

Proof. Let $\alpha_s, \alpha_t \in On$ and let z_{α_s,α_t} be their zip function.

We first prove that for any $\langle \beta, \mathcal{S} \rangle \in (\alpha_s \times \{\mathcal{S}\})$ there is a η such that $z_{\alpha_s,\alpha_t}(\eta) = \langle \beta, \mathcal{S} \rangle$. Let $\beta < \alpha_s$. We get two cases:

- $\beta < \alpha_t$. We get $z_{\alpha_s,\alpha_t}(2\beta) = \langle \beta, \mathcal{S} \rangle$ since case 1 of the definition of zip functions applies for 2β:
  - $2\beta < 2\alpha_s$ (because $\beta < \alpha_s$)
  - $2\beta < 2\alpha_t$ (because $\beta < \alpha_t$)
  - Even(2β) (because $(2\beta/2) + 0 = \beta$)

  and $2\beta/2 = \beta$.

- $\alpha_t \leq \beta$. We get $z_{\alpha_s,\alpha_t}(2\alpha_t + (\beta - \alpha_t)) = \langle \beta, \mathcal{S} \rangle$ since case 3 of the definition applies for $2\alpha_t + (\beta - \alpha_t)$:
  - $2\alpha_t \leq 2\alpha_t + (\beta - \alpha_t)$.
  - $2\alpha_t + (\beta - \alpha_t) < 2\alpha_s$ ($\alpha_t \leq \beta < \alpha_s$, so $\alpha_s - \alpha_t < 2\alpha_s - 2\alpha_t$ and hence $2\alpha_t + (\alpha_s - \alpha_t) < 2\alpha_s$, which with $\beta < \alpha_s$ implies $2\alpha_t + (\beta - \alpha_t) < 2\alpha_s$).

  and $\alpha_t + ((2\alpha_t + (\beta - \alpha_t)) - 2\alpha_t) = \beta$.

Secondly, we prove that for any $\langle \beta, \mathcal{T} \rangle \in (\alpha_t \times \{\mathcal{T}\})$ there is a η such that $z_{\alpha_s,\alpha_t}(\eta) = \langle \beta, \mathcal{T} \rangle$. Let $\beta < \alpha_t$. We get two cases:

- $\beta < \alpha_s$. We get $z_{\alpha_s,\alpha_t}((2\beta)\prime) = \langle \beta, \mathcal{S} \rangle$ since case 2 of the definition applies for 2β:
  - $(2\beta)\prime < 2\alpha_s$ (because $\beta < \alpha_s$)
  - $(2\beta)\prime < 2\alpha_t$ (because $\beta < \alpha_t$)
  - Odd($(2\beta)\prime$) (because $(((2\beta)\prime)/2) + 1 = \beta$)

  and $((2\beta)\prime)/2 = \beta$.

- $\alpha_s \leq \beta$. We get $z_{\alpha_s,\alpha_t}(2\alpha_s + (\beta - \alpha_s)) = \langle \beta, \mathcal{T} \rangle$ since case 4 of the definition applies for $2\alpha_s + (\beta - \alpha_s)$:
  - $2\alpha_s \leq 2\alpha_s + (\beta - \alpha_s)$.
  - $2\alpha_s + (\beta - \alpha_s) < 2\alpha_t$ ($\alpha_s \leq \beta < \alpha_t$, so $\alpha_t - \alpha_s < 2\alpha_t - 2\alpha_s$ and hence $2\alpha_s + (\alpha_t - \alpha_s) < 2\alpha_t$, which with $\beta < \alpha_t$ implies $2\alpha_s + (\beta - \alpha_s) < 2\alpha_t$).

  and $\alpha_s + ((2\alpha_s + (\beta - \alpha_s)) - 2\alpha_s) = \beta$.

□

Lemma A.1.57. *Zip functions are monotonic in their first projection.*

Proof. Let $\alpha_s, \alpha_t \in On$ and let z_{α_s,α_t} be their zip function.

First, if $\alpha_t \leq \alpha_s$, the domain of z_{α_s,α_t} is $2\alpha_t + (\alpha_s - \alpha_t)$. Let $\beta_1, \beta_2 < 2\alpha_t + (\alpha_s - \alpha_t)$ be such that $\beta_1 \leq \beta_2$. β_1 and β_2 do not fall into case 4 in the definition of zip functions since we get $2\alpha_t \leq 2\alpha_s$. We get the following remaining cases:

- β_1 falls into case 1 or 2 in the definition of zip functions ($\beta_1 < 2\alpha_s, 2\alpha_t$).
  - β_2 also falls into case 1 or 2 ($\beta_2 < 2\alpha_s, 2\alpha_t$).
    $\pi_1(z_{\alpha_s,\alpha_t}(\beta_1)) = \beta_1/2 \leq \beta_2/2 = \pi_1(z_{\alpha_s,\alpha_t}(\beta_2))$ by monotonicity of left division in its left argument.
  - β_2 falls into case 3 ($(2\alpha_t \leq \beta_2 < 2\alpha_s) \wedge (\beta_2 < 2\alpha_t + (\alpha_s - \alpha_t))$).
    We have $(\beta_1/2) - \alpha_t \leq 2(\beta_1/2) - 2\alpha_t = \beta_1 - 2\alpha_t$, get $\beta_1/2 \leq \alpha_t + (\beta_1 - 2\alpha_t)$ and hence $\beta_1/2 \leq \alpha_t + (\beta_2 - 2\alpha_t)$ because $\beta_1 \leq \beta_2$, $+$ is monotonic in both its arguments and $-$ is monotonic in its left argument. This means that $z_{\alpha_s,\alpha_t}(\beta_1) \leq z_{\alpha_s,\alpha_t}(\beta_2)$.
- β_1 falls into case 3 ($(2\alpha_t \leq \beta_2 < 2\alpha_s) \wedge (\beta_2 < 2\alpha_t + (\alpha_s - \alpha_t))$).

  Since we have $\beta_1 \leq \beta_2$, β_2 also falls into case 3 and we get $\pi_1(z_{\alpha_s,\alpha_t}(\beta_1)) = \alpha_t + (\beta_1 - 2\alpha_t) \leq \alpha_t + (\beta_2 - 2\alpha_t) = \pi_1(z_{\alpha_s,\alpha_t}(\beta_2))$ because $\alpha_t + (\beta - 2\alpha_t)$ is monotonic in β.

If, on the other hand, $\alpha_s \leq \alpha_t$, we get our result analogously. □

Definition A.1.58. If $\langle s_\beta \rangle_{\beta<\alpha_s}$ and $\langle t_\beta \rangle_{\beta<\alpha_t}$ are two sequences and z_{α_s,α_t} is the zip function of α_s and α_t then the **zip** of $\langle s_\beta \rangle_{\beta<\alpha_s}$ and $\langle t_\beta \rangle_{\beta<\alpha_t}$, denoted by $\text{zip}(\langle s_\beta \rangle_{\beta<\alpha_s}, \langle t_\beta \rangle_{\beta<\alpha_t})$ the sequence such that:

$$\text{zip}(\langle s_\beta \rangle_{\beta<\alpha_s}, \langle t_\beta \rangle_{\beta<\alpha_t})_\beta = \begin{cases} s_{\pi_1(z_{\alpha_s,\alpha_t}(\beta))} & \text{if } \pi_2(z_{\alpha_s,\alpha_t}(\beta)) = \mathcal{S}, \\ t_{\pi_1(z_{\alpha_s,\alpha_t}(\beta))} & \text{if } \pi_2(z_{\alpha_s,\alpha_t}(\beta)) = \mathcal{T} \end{cases}$$

By our previous lemmas about zip functions we get that the zip of two sequences, $\langle s_\beta \rangle_{\beta<\alpha_s}$ and $\langle t_\beta \rangle_{\beta<\alpha_t}$:

- Is a well-defined sequence (Lemma A.1.55).
- Has length $2\alpha_t + (\alpha_s - \alpha_t)$ if $\alpha_t \leq \alpha_s$ and $2\alpha_s + (\alpha_t - \alpha_s)$ if $\alpha_s \leq \alpha_t$ (Lemma A.1.55).
- Contains all elements of $\langle s_\beta \rangle_{\beta<\alpha_s}$ and $\langle t_\beta \rangle_{\beta<\alpha_t}$ (Lemma A.1.56).
- Contains only those elements and each element only once (definition zip and Lemma A.1.57).
- Respects the order of elements in those sequences (Lemma A.1.57).

A.2 General Topology

General topology (or point-set topology) formalizes certain intuitions about the structure of sets in an abstract way. Historically topology developed from geometry, so many of the notions in topology have geometrical connotations and the intuitions behind those notions are based in geometry. Since topology is meant (and used) to also formalize intuitions about structure in far more abstract contexts, the intuitions of nearness, distance or moving are not to be taken in a strictly (euclidean) geometrical way.

The main concept in general topology is that of a subset of a set being open. One intuitive perspective on open sets and hence on topology is that an open set is a set whose members are all objects with a certain finitely observable property, where finitely observable means that this property can be observed in a finite amount of times. From an extensional perspective we might identify the set with that property ([1, p. 642]). So according to this perspective, general topology formalizes the structure of a set by expressing what properties its members have.

A more geometrical intuition is that you can 'move' a 'bit', in any direction, from every point in the open set without getting out. So as a global property this means that an open set does not include its boundary. Looked at from this perspective, general topology formalizes nearness of points in the set. For any point in the set there is a certain degree of nearness such that any point that is near the first point to that degree is still in the open set.

A.2.1 Defining Topology

First we need to define the following:

Definition A.2.1. A set S is **closed under** an operation $\star$ if for any subset X of S we have $\star(X) \in S$. If S is a set and $S^\star$ is the smallest set superset of S that is closed under $\star$ then $S^\star$ is the **closure** of S under $\star$.

And then we can define topological spaces:

Definition A.2.2. A **topology** T on a set S is a collection of subsets of S such that:

- It is closed under finite intersection. (So for any finite subset A of T we have that $\bigcap A \in T$)
- It is closed under arbitrary union. (So for any subset A of T we have that $\bigcup A \in T$)

We call the structure (S, T) a **topological space**. The sets in T are the **open sets** of (S, T). An open set $O \in T$ is also called T-open. The elements of S are referred to as **points**. In the light of some topology T on S we might also refer to S as a **topological space** (totum pro parte) or simply as **space**.

Any topology T at least has $\emptyset$, the nullary union ($\cup\emptyset$), and the full space S, the nullary intersection ($\cap\emptyset$), as open sets.

Topologies on a set can be defined by giving a base or subbase of the topology.

Definition A.2.3. A **base** B of a topology T is a subset of T such that every T-open set is the arbitrary union of sets in B.

A topology can have more than one base. For instance, for any topology we have that topology itself is trivially a base for it and the topology minus $\emptyset$ is also a base for it ($\emptyset$ is the nullary union). Since a topology T is closed under arbitrary union, we get that, if B is a base of T, T is the closure of B under arbitrary union. This means that a base uniquely defines the topology that it is a base of and because of this we can define the following:

Definition A.2.4. If B is a base of topology T then we also say that T is the topology **generated by** B.

Given a space, S, we might be interested in which sets of subsets of S are the base of some topology on S. That is, which sets of subsets of a space actually generate a topology. Sets of subsets which are closed under finite intersection do this, and we can show that.

As a pure set theoretical result, we have that:

Lemma A.2.5. *If B is a family of sets in S with index sets I and J, then there is another family of sets in S, C, with index sets K and J such that $\bigcap_{i\in I}(\bigcup_{j\in J} B_{i,j}) = \bigcup_{k\in K}(\bigcap_{i\in I} C_{k,i})$.*

Proof. Define K to be the set of all total functions from I to J. Define C such that for all $k \in K$ and $i \in I$ we have $C_{k,i} = B_{i,k(i)}$. We'll prove these are the C and K that we're looking for, that is, we have $\bigcap_{i\in I}(\bigcup_{j\in J} B_{i,j}) = \bigcup_{k\in K}(\bigcap_{i\in I} C_{k,i})$.

Let $x \in S$ be arbitrary, we prove $\forall i \in I \exists j \in J(x \in B_{i,j}) \Leftrightarrow \exists k \in K \forall i \in I(x \in C_{k,i})$.

- $\Leftarrow$ Assume $\forall i \in I \exists j \in J(x \in B_{i,j})$, let $k^* \in K$ be such that for each i, $k^*(i)$ is some j such that we have that $x \in B_{i,j}$. There must be such a $k^* \in K$ since we have $\forall i \in I \exists j \in J(x \in B_{i,j})$. Now for any $i \in I$ we have $x \in B_{i,k^*(i)}$ and because $B_{i,k^*(i)} = C_{k^*,i}$ we have $x \in C_{k^*,i}$. So k^* is a witness to $\exists k \in K \forall i \in I(x \in C_{k,i})$.

- $\Rightarrow$ Assume $\exists k \in K \forall i \in I(x \in C_{k,i})$, let $k^* \in K$ be a witness to this, we have $\forall i \in I(x \in C_{k^*,i})$. Now for all $i \in I$ we also have $C_{k^*,i} = B_{i,k^*(i)}$ by the definition of C and hence $x \in B_{i,k^*(i)}$. So for all $i \in I$, $k^*(i)$ is a witness to $\exists j \in J(x \in B_{i,j})$, so we have $\forall i \in I \exists j \in J(x \in B_{i,j})$.

So, for any $x \in S$, we get $\forall i \in I \exists j \in J(x \in B_{i,j}) \Leftrightarrow \exists k \in K \forall i \in I(x \in C_{k,i})$, and hence $\bigcap_{i\in I}(\bigcup_{j\in J} B_{i,j}) = \bigcup_{k\in K}(\bigcap_{i\in I} C_{k,i})$. □

From this, it follows that:

Lemma A.2.6. *If a set is closed under finite intersection, then its closure under arbitrary union is still closed under finite intersection.*

Proof. Let B be a set that is closed under finite intersection and let B^* be the closure of B under arbitrary union. For all members of B^* we have that they are the arbitrary union of some sets in B. Any finite intersection of sets in B^* then is the finite intersection of arbitrary unions of sets in B. By Lemma A.2.5, such an intersection can be written as the arbitrary union of finite intersections of sets in B. B is closed under finite intersection, so these are merely unions of sets in B and hence, by definition, members of B^*. So we have that any finite intersection of sets in B^* is in B^* and that means that B^* is closed under finite intersection. □

Which, as a direct result back in topology, gives:

Proposition A.2.7. If S is some set, B is a set of subsets of S and B is closed under finite intersection, then B is a base of some topology T on S.

The converse does not hold. If B is a base of T then it need not be closed under finite intersection.

Example A.2.8. Consider the set $\{a, b\}$ and on it, the discrete, or full, topology: $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$. $\{\{\}, \{a\}, \{b\}\}$ is a base of this topology since every open set in the topology is the union of these sets ($\{a, b\}$ being the union of $\{a\}$ and $\{b\}$). But this base is not closed under finite intersection, since the intersection of nothing (the whole set, $\{a, b\}$) is not in it.

An example that may be a bit less contrived is the following:

Example A.2.9. Consider the set $\{a, b, c, d\}$ with as topology on it:

$$\{\{\}, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$$

We have that $\{\{\}, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}$ is not closed under finite intersection ($\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$). But it is a base of the topology since $\{a\} \cup \{b\} = \{a, b\}$.

So being closed under finite intersection is a sufficient criterion for being a base, but it is not a necessary criterion. A criterion that is both sufficient and necessary is the following.

Definition A.2.10. The **base-criterion** states, for a set B, that if for any $X, Y \in B$ we have that $p \in X \cap Y$ then there is a $Z \in B$ such that $Z \subseteq X \cap Y$ and $p \in Z$.

Lemma A.2.11. *If the base criterion holds for B then the finite (non-nullary) intersection of any set of sets in B is the arbitrary union of some set of sets in B.*

Proof. Assume that the base criterion holds for B. We need to proof that the finite (non-nullary) intersection of any set of sets in B is the union of some set of sets in B. Because the intersection is finite we can do natural induction over the number of sets that we're intersecting. We start at one since we're excluding nullary intersection. We get the following cases:

- Base case: The lemma trivially holds for the intersection of only one set in B. The intersection of one set is the set itself, it is also the union itself.

- Step case: We get as induction hypothesis that the intersection of any set of n sets in B is the arbitrary union of some set of sets in B and need to prove that the intersection of any set of $n+1$ sets in B is the union of some set of sets in B.

  Let X be our set of $n+1$ sets in B. Let $x \in X$. $X \setminus \{x\}$ is of size n, so by induction hypothesis, there is some set of sets in B, call it Y, such that $\bigcap(X \setminus \{x\}) = \bigcup Y$.

  Let $p \in (\bigcup Y \cap x)$, by definition we get that there is a $y \in Y$ such that $p \in (y \cap x)$. Because $x, y \in B$ we can use the base criterion to construct a $z_p \in B$ such that $z_p \subseteq x$, $z_p \subseteq y$ and $p \in z_p$. The set $\{z_p \mid p \in (\bigcup Y \cap x)\}$ is a set of sets in B and we will prove that $\bigcup\{z_p \mid p \in (\bigcup Y \cap x))\} = \bigcap X$.

  For any $p \in (\bigcup Y \cap x)$ there is some $y \in Y$ such that $p \in y$ and hence $z_p \subseteq y$, which means $z_p \subseteq \bigcup Y$, which in turn means $z_p \subseteq \bigcap(X \setminus \{x\})$, while we also have $z_p \subset x$. This means we have $z_p \subset \bigcap X$. So we have $\bigcup\{z_p \mid p \in (\bigcup Y \cap x)\} \subseteq \bigcap X$.

  For all $p \in \bigcap X$ we have $p \in x$ (because $x \in X$) and $p \in \bigcap(X \setminus x)$ which means $p \in \bigcup Y$ and $p \in (x \cap \bigcup Y)$, we have $p \in z_p$ and hence $p \in \bigcup\{z_p \mid p \in (\bigcup Y \cap x)\}$. So we have $\bigcap X \subseteq \bigcup\{z_p \mid p \in (\bigcup Y \cap x)\}$.

  So we must have $\bigcup\{z_p \mid p \in (\bigcup Y \cap x)\} = \bigcap X$. □

Lemma A.2.12. *If S is a set and B is a set of subsets of S, then the base criterion holds for B if and only if it is a base of some topology on S.*

Proof.

⇒ If the base criterion holds for B then B is a base of a topology on S. (The base criterion is sufficient.)

  Assume that the base criterion holds for B. We want to prove that the closure of B under arbitrary union is closed under finite intersection (and hence an actual topology). Let T be the closure of B under arbitrary union. We need to prove that, for any finite subset of T, we have that its intersection is also in T. Let X be a subset of size $n \in \mathbb{N}$ of T. If we view X as a family indexed by n then we can write the intersection of sets in X, $\bigcap X$, as $\bigcap_{i<n} X_i$. For any $i < n$ we have that $X_i \in X$ and hence $X_i \in T$. Because the sets in T are the arbitrary unions of sets in B, we can view X_i as a family write it as $\bigcup_{j \in J} X_{i,j}$ for some appropriate index set J. So we can write $\bigcap X$ as $\bigcap_{i<n} \bigcup j \in J X_{i,j}$. By Lemma A.2.5,

there is another family of sets in B, Y, with index sets K and n, such that $\bigcap_{i<n}(\bigcup_{j\in J} X_{i,j}) = \bigcup_{k\in K}(\bigcap_{i<n} Y_{k,i})$, so $\bigcap X = \bigcup_{k\in K}(\bigcap_{i<n} Y_{k,i})$. By Lemma A.2.11 we get that, because the base criterion holds for B, the finite intersection of every subset of the sets in B is the union of some sets in B. So we get $\bigcap X = \bigcup_{k\in K}(\bigcup_{i\in I} Y_{k,i})$, for some index set I. We can collapse the unions and get that $\bigcap X = \bigcup_{x\in K\times I} Y_x$, so $\bigcap X$ is the union of sets in B and because T is the closure of B under arbitrary union, we have $\bigcap X \in T$. This means T is closed under finite intersection and hence T is a topology and B is a base for it. So the base criterion is a sufficient condition for B being a base.

$\Leftarrow$ If B is a base of T, then the base criterion holds for B.

Assume that B is a base of T. Let $X, Y \in B$ and $p \in X \cap Y$. We have $X \in T$ and $Y \in T$ because B is a base of T. We get $X \cap Y \in T$ because T is closed under finite intersection. Because B is a base of T we have that $X \cap Y$ is the union of sets in B, so because we have $p \in X \cap Y$ we have that there is a $Z \in B$ such that $Z \subseteq X \cap Y$ and $p \in Z$. This means that the base criterion holds for B and hence is a necessary condition for B being a base. □

So the base criterion is indeed sufficient and necessary.

Definition A.2.13. A **subbase** A of a topology T is a subset of T such that every T-open set is the union of finite intersections of sets in A

This means that T is the closure of A under arbitrary union and finite intersection. The closure of A under finite intersection is a base of T (by Proposition A.2.7). T is the smallest topology containing A.

We also have that:

Lemma A.2.14. *Any set of subsets, A, of a set S is a subbase of some topology T on S.*

Proof. Let B be the closure of A under finite intersection and let T be the closure of B under arbitrary union. T is a topology on S because it is closed under arbitrary union by definition and closed under finite intersection by Lemma A.2.6. A is a subbase of T since we have that every set in T is the arbitrary union of finite intersections of sets in A by definition of T. □

We can lift the notion of a base generating a topology to subbases:

Definition A.2.15. If A is a subbase of topology T then we say that T is the topology **generated by** A.

So if T is the topology generated by subbase A, then it is the closure of A under finite intersection and arbitrary union. Because every set of subsets of a space S is the subbase of a (unique) topology on the space, we can give any set of subsets of S as subbase to define a topology; the topology generated by it.

A.2.2 Topological Notions

A.2.2.1 Neighborhoods

An important notion in topology is that of neighborhood.

Definition A.2.16. If (S, T) is a topological space and we have a point $x \in S$ and an open set $O \in T$ then a $N \subseteq S$ such that $x \in O \subseteq N$ is called a **neighborhood of** x.

We have:

Lemma A.2.17. *A set is open if and only if it is a neighborhood for all of its points.*

Proof. Let (S, T) be a topological space and let $X \subseteq S$.

$\Rightarrow$ Assume that X is open. We get that X is a neighborhood for all of its points by the definition of neighborhoods.

$\Leftarrow$ Assume that X is a neighborhood for all of its points. For all $x \in X$ there is some open set $O \in T$ such that we have $x \in O \subseteq X$. Call the set of these sets A. $\bigcup A$ is open by the definition of open sets (arbitrary unions of open sets are open). $\bigcup A$ is a subset of X since all $O \in A$ are. X is a subset of $\bigcup A$ since for every $x \in X$ we have $x \in O$ for some $O \in A$. So we have $X = \bigcup A$ and hence X is open. □

Due to this tight link between open sets and neighborhoods in a topology, many topological notions can be expressed both in terms of neighborhoods and in terms of open sets. For some purposes definitions in terms of neighborhoods are more convenient, for other purposes definitions in terms of open sets are. We will tend towards definitions in terms of open sets.

The notion of neighborhood can be lifted to sets;

Definition A.2.18. In a topological space (S, T) a **neighborhood of a set** $X \subseteq S$ is a set $N \subseteq S$ such that for all $x \in X$ we have that N is a neighborhood of x.

A.2.2.2 Continuity

Another important notion that general topology allows to express is that of function continuity.

Definition A.2.19. If (X, T_X) and (Y, T_Y) are topological spaces and $f : X \to Y$ is a function between the two, then f is **continuous** if for every T_Y-open set we have that its inverse image is T_X-open.

The notion of function continuity also has a localized version.

Definition A.2.20. If (X, T_X) and (Y, T_Y) are topological spaces, $f : X \to Y$ is a function between them and $x \in X$, then f is **continuous at** x if every open set that $f(x)$ is in contains the image of an open set that x is in.

Remark. Continuity, and especially continuity at x is one of the notions that is usually expressed in terms of neighborhoods instead of open sets. Still, I think that, for our purposes and for the sake of consistency, expressing in terms of open sets is the better choice.

We get that:

Lemma A.2.21. *A function is continuous if and only if it is continuous at all of the points in its domain.*

Proof. Let (X, T_X) and (Y, T_Y) be topological spaces and let $f : X \to Y$ be a function between them.

$\Rightarrow$ Assume that f is continuous. Let $x \in X$ be arbitrary and let O be a T_Y-open set that $f(x)$ is in. By assumption we get that its inverse image $f^{-1}(O)$ is T_X-open. Since $f(x) \in O$ we get $x \in f^{-1}(O)$. So f is continuous at x.

$\Leftarrow$ Assume that f is continuous at all $x \in X$. Let O_Y be a T_Y-open set. For any $x \in f^{-1}(O_Y)$ we get that f is continuous at x by assumption and since $f(x) \in O_Y$ we get a T_X-open set O_x such that $x \in O_x$ and $f(O_x) \subseteq O_Y$. We get $\bigcup\{f(O_x) \mid x \in f^{-1}(O_Y)\} \subseteq O_Y$ and hence $\bigcup\{O_x \mid x \in f^{-1}(O_Y)\} \subseteq f^{-1}(O_Y)$. We also have $f^{-1}(O_Y) \subseteq \bigcup\{O_x \mid x \in f^{-1}(O_Y)\}$ since for every $x \in f^{-1}(O_Y)$ we have $x \in O_x$. So $\bigcup\{O_x \mid x \in f^{-1}(O_Y)\} = f^{-1}(O_Y)$. Also, $\bigcup\{O_x \mid x \in f^{-1}(O_Y)\}$ is T_X-open since for all $x \in f^{-1}(O_Y)$ we have that O_x is T_X-open (arbitrary unions of open sets are open). So $f^{-1}(O_Y)$ is T_X-open and f is continuous. □

A.2.2.3 Filters and Convergence

Definition A.2.22. A **filter** in a topological space (S, T) is a set, F, of subsets of S such that:

- $S \in F$ and $\emptyset \notin F$
- $X, Y \in F \Rightarrow X \cap Y \in F$
- $X \subseteq Y \subseteq S \wedge X \in F \Rightarrow Y \in F$

Definition A.2.23. A **filter base** in a topological space (S, T) is a set, B, of subsets of S such that:

- $B \neq \emptyset$ and $\emptyset \notin B$
- $X, Y \in B \Rightarrow \exists Z \in B.\ Z \subseteq (X \cap Y)$

The closure of a filter base under supersets is said to be the filter **generated by** the filter base.

So if B is a filter base in a space S the set $\{Y \subseteq S \mid X \subseteq Y \wedge X \in B\}$ is the filter generated by B.

Lemma A.2.24. *Any filter base generates an actual filter.*

Proof. Let B be a filter base and let F the set of all supersets of sets in B. To prove that F is a filter:

- $S \in F$ and $\emptyset \notin F$

  We have $S \in F$ because B is non-empty and S is a superset of any subset of S. We have $\emptyset \notin F$ because $\emptyset \notin B$ and $\emptyset$ is not a superset of any set that is not $\emptyset$.

- $X, Y \in F \Rightarrow X \cap Y \in F$

  If we have $X, Y \in F$ the we have that there are $X', Y' \in B$ such that $X' \subseteq X$ and $Y' \subseteq Y$. We get a $Z' \in B$ such that $Z' \subseteq (X' \cap Y')$ by the definition of filter bases. We have $X' \cap Y' \subseteq (X \cap Y)$, so $Z \subseteq (X \cap Y)$ and hence $(X \cap Y) \in F$.

- $X \subseteq Y \subseteq S \wedge X \in F \Rightarrow Y \in F$.

  If we have $X \subseteq Y \subseteq S$ and $X \in F$ then we have a $X' \in B$ such that $X' \subseteq X$, so we get $X' \subseteq Y$ and hence $Y \in F$. □

A lot of notions that are usually expressed in terms of filters can also be expressed using filter bases. Since definitions using filter bases are usually less involved, I will immediately give the definitions in their 'filter base-form'.

Definition A.2.25. A filter base F in a topological space (S, T) is said to **converge to** a point $x \in S$. If for every T-open set O such that $x \in O$ there is a $X \in F$ such that $X \subseteq O$. We call x is a **limit** of F. The set of limits of F is denoted by $\mathrm{Lim}(F)$.

A filter converging to a limit is to be interpreted as a collection of approximations of the limit. A filter can converge to multiple limits. If we have a base for our topological space we can express the notion of convergence in terms of this base.

Lemma A.2.26. *A filter base converges to a point if and only if for every base set that it is in we have a filter base set that is contained in the base set.*

Proof. Let (S, T) be a topological space, let B be a base of T, let $x \in S$ and let F be a filter base in (S, T).

$\Rightarrow$ Assume that F converges to x. Let $A \in B$ such that $x \in A$, we get that A is T-open. So, by assumption, there is a $X \in F$ such that $X \in A$.

$\Leftarrow$ Assume that for every $A \in B$ such that $x \in A$ there is an $X \in F$ such that $X \subseteq A$. Let O be a T-open set such that $x \in O$. O is the union of some set of base sets, so we get a base set A such that $x \in A$. We get some $X \in F$ such that $X \subseteq A$ by assumption. We have $A \subseteq O$ and hence $X \subseteq O$. □

For sequences the following can be defined:

Definition A.2.27. If $\langle s_\beta \rangle_{\beta<\alpha}$ is a transfinite sequence, such that $\alpha \neq 0$, then its **filter base of tails** is $\{\{s_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$.

So the filter base of tails of a sequence is the set of all tails of the sequence. To prove that this is an actual filter base.

Lemma A.2.28. *The filter base of tails of a non-empty sequence is a filter base.*

Proof. Let $\langle s_\beta \rangle_{\beta<\alpha}$ such that $0 \neq \alpha$ be a transfinite sequence.

- The intersection of any two tails is contained in another tail.

  Let $\{s_\gamma \mid \beta_1 \leq \gamma < \alpha\}$ and $\{s_\gamma \mid \beta_2 \leq \gamma < \alpha\}$ be two tails of $\langle s_\beta \rangle_{\beta<\alpha}$. By linearity of $<$ we get either $\beta_1 < \beta_2$, $\beta_2 < \beta_1$ or $\beta_1 = \beta_2$.

  - $\beta_1 < \beta_2$. We get $\{s_\gamma \mid \beta_1 \leq \gamma < \alpha\} \cap \{s_\gamma \mid \beta_2 \leq \gamma < \alpha\} = \{s_\gamma \mid \beta_1 \leq \gamma < \alpha\}$.
  - $\beta_2 < \beta_1$. We get $\{s_\gamma \mid \beta_1 \leq \gamma < \alpha\} \cap \{s_\gamma \mid \beta_2 \leq \gamma < \alpha\} = \{s_\gamma \mid \beta_1 \leq \gamma < \alpha\}$.
  - $\beta_1 = \beta_2$. We get $\{s_\gamma \mid \beta_1 \leq \gamma < \alpha\} \cap \{s_\gamma \mid \beta_2 \leq \gamma < \alpha\} = \{s_\gamma \mid \beta_1 \leq \gamma < \alpha\} = \{s_\gamma \mid \beta_2 \leq \gamma < \alpha\}$.

- The set of tails of the sequence is non empty.

  Follows from $\alpha \neq 0$.

- No tail is empty.

  For any tail $\{s_\gamma \mid \beta \leq \gamma < \alpha\}$ we have that at least s_β is in it. □

Definition A.2.29. We say that a non-empty sequence **converges** to a limit if its filter base of tails does so. We denote the set of limits of a sequence $\langle s_\beta \rangle_{\beta<\alpha}$ by $\mathrm{Lim}(\langle s_\beta \rangle_{\beta<\alpha})$.

Remark. Defined sequence convergence this way, we do not have convergence behaviour for the empty sequence (the sequence with length 0). This seems strange, but the filter base of tails of the empty sequence would be empty and hence not a filter base at all. In this thesis it shouldn't matter much, but for completeness sake (and a nice fit-in with the proposition below) we might say that the empty sequence does not converges to any limit.

Unfolding definitions we get:

Proposition A.2.30. A sequence $\langle s_\beta \rangle_{\beta<\alpha}$ in a topological space (S, T) converges to a limit $x \in S$ if and only if, for every T-open set O such that $x \in O$, there is a $\beta < \alpha$ such that for every γ with $\beta \leq \gamma < \alpha$ we have $s_\gamma \in O$.

So a sequence converges to a limit if and only, for any open set that the limit is in, the sequence is **eventually in** that open set.

It easily follows that:

Lemma A.2.31. *Any sequence with a successor ordinal as length converges to its last member.*

Proof. Let (S, T) be a topological space, let $\langle s_\beta \rangle_{\beta < \alpha\prime}$ a sequence in it. Let $O \in T$ be such that $s_\alpha \in O$. For all γ such that $\alpha \leq \gamma < \alpha\prime$ we have $s_\gamma \in O$ since the only such γ is α. □

For zipped sequences we have:

Lemma A.2.32. *If two sequences are not of the same length, the zip of the two sequences converges to a point if and only if the longer sequence converges to that point.*

Proof. Let (S, T) be a topological space, let $x \in S$, let $\alpha_s, \alpha_t \in On$, let z_{α_s,α_t} be the zip function for α_s and α_t and let $\langle s_\beta \rangle_{\beta < \alpha_s}$ and $\langle t_\beta \rangle_{\beta < \alpha_t}$ be two sequences in that space such that $\text{zip}(\langle s_\beta \rangle_{\beta < \alpha_s}, \langle t_{\alpha_t} \rangle_{\beta < \lambda}) = \langle r_\beta \rangle_{\beta < \alpha_r}$. By Lemma A.1.55 we get that $\alpha_r = 2\alpha_t + \alpha_s - \alpha_t$ if $\alpha_t < \alpha_s$ and $\alpha_r = 2\alpha_s + \alpha_t - \alpha_s$ if $\alpha_s < \alpha_t$.

⇒ Assume that $\langle r_\beta \rangle_{\beta < \alpha_r}$ converges to x. Let $O \in T$ be such that $x \in O$. By assumption we get a $\beta < \alpha_r$ such that for all γ with $\beta \leq \gamma < \alpha_r$ we have that $r_\gamma \in O$. We get two cases:

- $\alpha_t < \alpha_s$. Let γ be such that $\max(\alpha_t, \pi_1(z_{\alpha_s,\alpha_t}(\beta))) \leq \gamma < \alpha_s$. By Lemma A.1.56 there is an η such that $\pi_1(z_{\alpha_s,\alpha_t}(\eta)) = \gamma$, because we have $\alpha_t \leq \gamma$ we must have $\pi_2(z_{\alpha_s,\alpha_t}(\eta)) = \mathcal{S}$, and hence $r_\eta = s_\gamma$. Because we have $\pi_1(z_{\alpha_s,\alpha_t}(\beta)) \leq \gamma = \pi_1(z_{\alpha_s,\alpha_t}(\eta))$ we get $\beta \leq \eta$ by Lemma A.1.57 and hence $s_\gamma \in O$.
- If $\alpha_s < \alpha_t$ we get the same result analogously.

⇐ We get two cases:

- $\alpha_t < \alpha_s$. Assume that $\langle s_\beta \rangle_{\beta < \alpha_s}$ converges to x. Let $O \in T$ be such that $x \in O$. By assumption we get a $\beta < \alpha_s$ such that for all γ with $\beta \leq \gamma < \alpha_s$ we have that $s_\gamma \in O$. By Lemma A.1.56 we get a η such that $z_{\alpha_s,\alpha_t}(\eta) = \langle \beta, \mathcal{S} \rangle$.

  Let γ be such that $\max(\alpha_t, \eta) \leq \gamma < \alpha_r$. Because we have $\alpha_t \leq \gamma$ we get $\pi_2(z_{\alpha_s,\alpha_t}(\gamma)) = \mathcal{S}$, so we get $r_\gamma = s_{z_{\alpha_s,\alpha_t}(\gamma)}$. We have $\eta \leq \gamma$, get $\pi_1(z_{\alpha_s,\alpha_t}(\eta)) \leq \pi_1(z_{\alpha_s,\alpha_t}(\gamma))$ by Lemma A.1.57. So $\beta \leq \pi_1(z_{\alpha_s,\alpha_t}(\gamma))$ and hence $r_\gamma \in O$.
- If $\alpha_s < \alpha_t$ we get the same result analogously. □

Lemma A.2.33. *Two sequences of the same limit ordinal length converge to a point if and only if the zip of those sequences converges to that point.*

Proof. Let (S, T) be a topological space, let $x \in S$, let $\lambda \in \text{LimOrd}$, let $z_{\lambda,\lambda}$ be the zip function for λ and λ and let $\langle s_\beta \rangle_{\beta < \lambda}$ and $\langle t_\beta \rangle_{\beta < \lambda}$ be two sequences in the space such that $\text{zip}(\langle s_\beta \rangle_{\beta < \lambda}, \langle t_\beta \rangle_{\beta < \lambda}) = \langle r_\beta \rangle_{\beta < \lambda}$. (The zip of the sequences is of the same length as the sequences themselves by Lemma A.1.55)

$\Rightarrow$ Assume that $\langle s_\beta \rangle_{\beta<\lambda}$ and $\langle t_\beta \rangle_{\beta<\lambda}$ converge to x. Let $O \in T$ be such that $x \in O$. By assumption we get a $\beta_s < \lambda$ such that for all γ with $\beta_s \leq \gamma < \lambda$ we have that $s_\gamma \in O$. We also get such a β_t such that for all γ with $\beta_t \leq \gamma < \lambda$ we have that $t_\gamma \in O$. Let $\beta = \max(\beta_s, \beta_t)$, we get that for all γ with $\beta \leq \gamma < \lambda$ we have that $s_\gamma \in O$ and $t_\gamma \in O$. By Lemma A.1.56 we get an $\eta < \lambda$ such that $\pi_1(z_{\lambda,\lambda}(\eta)) = \beta$. Let γ be such that $\eta \leq \gamma < \lambda$. By Lemma A.1.57 we have $\pi_1(z_{\lambda,\lambda}(\eta)) \leq \pi_1(z_{\lambda,\lambda}(\gamma))$. Depending on whether we have $\pi_2(z_{\lambda,\lambda}(\gamma)) = \mathcal{S}$ or $\pi_2(z_{\lambda,\lambda}(\gamma)) = \mathcal{T}$ we get $r_\gamma = s_{\pi_1(z_{\lambda,\lambda}(\gamma))}$ or $r_\gamma = t_{\pi_1(z_{\lambda,\lambda}(\gamma))}$. But since we have $\beta = \pi_1(z_{\lambda,\lambda}(\eta)) \leq \pi_1(z_{\lambda,\lambda}(\gamma))$ we get $r_\gamma \in O$ in both cases. So $\langle r_\beta \rangle_{\beta<\lambda}$ converges to x.

$\Leftarrow$ Assume that $\langle r_\beta \rangle_{\beta<\lambda}$ converges to x. Let $O \in T$ be such that $x \in O$. By assumption we get a $\beta < \lambda$ such that for all γ with $\beta \leq \gamma < \lambda$ we have that $r_\gamma \in O$. Let γ be such that $\pi_1(z_{\lambda,\lambda}(\beta)) \leq \gamma < \lambda$. By Lemma A.1.56 we get an η such that $\pi_1(z_{\lambda,\lambda}(\eta)) = \gamma$ and $\pi_2(z_{\lambda,\lambda}(\eta)) = \mathcal{S}$, by Lemma A.1.57 and the fact that the ordinals are totally ordered we get that $\beta \leq \eta$. So we get $r_\eta \in O$ and because $\pi_2(z_{\lambda,\lambda}(\eta)) = \mathcal{S}$ we get $r_\eta = s_{\pi_1(z_{\lambda,\lambda}(\eta))} = s_\gamma \in O$ and $\langle s_\beta \rangle_{\beta<\lambda}$ converges to x. We can show that $\langle t_\beta \rangle_{\beta<\lambda}$ converges to x in the same way. □

A.2.2.4 Other Notions

Definition A.2.34. In a topological space, a **limit point** is a point in the space for which it holds that every open set that it is in also has another member. A member of the space that is not a limit point is called an **isolated point**.

Filters converging to a limit point need not have the singleton set of the limit point as member. For every isolated point, there is an open set that only has the isolated point as member, so filters converging to an isolated point do need to have the singleton set of the isolated point as member. If we define a **proper approximation** of a point as an approximation that does not straight-up gives the point, then limit points are points which can be properly approximated as limits of converging filters, where isolated points can only be improperly approximated.

For isolated points we have that:

Lemma A.2.35. *Any function between topological spaces is continuous at any isolated point in its domain.*

Proof. Let (X, T_X) and (Y, T_Y) be topological spaces, let f be a function between them and let $x \in X$ be isolated. Every open set that $f(x)$ is in trivially contains the image of $\{x\}$, which is open because x is isolated, witnessing the continuity of f at x. □

Definition A.2.36. The **specialization quasi-order** of a topological space (S, T) is a quasi-order $\leq_T$ on S that is such that for any $x, y \in S$ we have $x \leq_T y$ if and only if y is a member of all open set that x is a member of.

The intuition behind calling this quasi-order the specialization quasi-order is that, because y is at least in any open set that x is in, y has all finitely observable properties that x has and possibly more. So y is a (non-strictly) more specified (or special) version of x.

Every quasi-order is the specialization quasi-order of some topology, but different topologies might have the same specialization quasi-order.

Example A.2.37. A simple example is given by the following two topologies on $\mathbb{N} \cup \{\omega\}$: $T_A = \{O_A(\alpha) \mid \alpha \in (\mathbb{N} \cup \{\omega\})\}$ where $O_A(\alpha) = \{\beta \in (\mathbb{N} \cup \{\omega\}) \mid \alpha \leq \beta\}$ and $T_S = \{\beta \in (\mathbb{N} \cup \{\omega\}) \mid \alpha < \beta\} = T_A \setminus \{\{\omega\}\}$. T_A is the Alexandroff topology on $\mathbb{N} \cup \{\omega\}$ and T_S is the Scott topology on $\mathbb{N} \cup \{\omega\}$ (Definition A.2.57). They are different but the omission of $\{\omega\}$ as open set in T_S does not matter for the specialization quasi-order. We still get that:

- For all $\alpha \in \mathbb{N} \cup \{\omega\}$ we have $\alpha \leq_{T_S} \omega$ because for any open set $O \in T_S$ with $\alpha \in O$ we still have $\omega \in O$.
- For no $n \in \mathbb{N}$ we have $\omega \leq_{T_S} n$ since $\omega \in O_A(n+1)$ and $n \notin O_A(n+1)$.

Lastly:

Definition A.2.38. If (S, T) is a topological space then $X \subset S$ is **closed** if $S \setminus X$ is open.

A.2.3 Properties of Topologies

A.2.3.1 Separation

We can classify topologies based on whether they satisfy certain so-called **separation axioms**. Those are axioms expressing how we can distinguish distinct points in the space using the topology.

Definition A.2.39. Let (S, T) be a topological space:

- (S, T) is said to be T_0 or **Kolmogorov** if distinct points are **topologically distinguishable**. That is, if for all $x, y \in S$ such that $x \neq y$ there is a $O \in T$ such that $x \in O \wedge y \notin O$ or $y \in O \wedge x \notin O$.
- (S, T) is said to be T_1 or **Fréchet** if distinct points can be **separated**. That is, if for all $x, y \in S$ such that $x \neq y$ there is a $O \in T$ such that $x \in O \wedge y \notin O$ and there is an $O \in T$ such that $y \in O \wedge x \notin O$.
- (S, T) is said to be T_2 or **Hausdorff** if distinct points can be **separated by open sets**. That is, if for all $x, y \in S$ such that $x \neq y$ there are $O_x, O_y \in T$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$.

We get that:

Proposition A.2.40. Any T_2-space is T_1. Any T_1-space is T_0.

And:

Theorem A.2.41. *A space is Hausdorff if and only if every filter base has at most one limit.*

Proof. Let (S, T) be a topological space.

⇒ Assume (S, T) is Hausdorff. Let B be a filter base in (S, T) converging to limits x and y. Suppose, for contradiction, that $x \neq y$. By (S, T) being Hausdorff we get open sets $O_x \in T$ with $x \in O_x$ and $O_y \in T$ with $y \in O_y$ such that $O_x \cap O_y = \emptyset$. Because B converges to x we get some $B_x \in B$ such that $B_x \subseteq O_x$, because B converges to y we get some $B_y \in B$ such that $B_y \subseteq O_y$. Now, because B is a filter base, we get some subset of $B_x \cap B_y$ in B. This subset is non-empty because B is a filter base, so $B_x \cap B_y$ is non-empty, so $O_x \cap O_y$ is non-empty. Contradiction, so we must have $x = y$ after all.

⇐ Assume that every sequence in S has at most one limit. Now suppose for contradiction that (S, T) is not Hausdorff, that is, there are $x, y \in S$ such that $x \neq y$ that can not be separated by open sets. Now consider $B = \{O_x \cap O_y \mid O_x \in T \wedge x \in O_x \wedge O_y \in T \wedge y \in O_y\}$ we can show that this is a filter base:

- $\emptyset \notin B$ because x and y are not separable by open sets
- Let $B_1, B_2 \in B$, we get that $B_1 = O_{1,x} \cap O_{1,y}$ and $B_2 = O_{2,x} \cap O_{2,y}$, where $O_{1,x}, O_{2,x}, O_{1,y}, O_{2,y} \in T$ such that $x \in O_{1,x}, O_{2,x}$ and $y \in O_{1,y}, O_{2,y}$. Now $B_1 \cap B_2 = (O_{1,x} \cap O_{1,y}) \cap (O_{2,x} \cap O_{2,y}) = (O_{1,x} \cap O_{2,x}) \cap (O_{1,y} \cap O_{2,y})$. We get $(O_{1,x} \cap O_{2,x}) \in T$ because T is closed under finite intersection and $x \in (O_{1,x} \cap O_{2,x})$ because $x \in O_{1,x}, O_{2,x}$. We also get $(O_{1,y} \cap O_{2,y}) \in T$ because T is closed under finite intersection and $y \in (O_{1,y} \cap O_{2,y})$ because $y \in O_{1,y}, O_{2,y}$. We get $(O_{1,x} \cap O_{2,x}) \cap (O_{1,y} \cap O_{2,y}) \in B$ by construction of B, and hence $B_1 \cap B_2 \in B$.

For any $O_x \in T$ with $x \in O_x$, we get that $O_x \cap S \in B$ and $O_x = O_x \cap S$, so $O_x \in B$ and B converges to x. For any $O_y \in T$ with $y \in O_y$ we get that $S \cap O_y \in B$ and $O_y = S \cap O_y$, so B converges to y. This contradicts our assumption of every filter having at most one limit, so we must conclude that (S, T) is Hausdorff. □

From this, we directly get that:

Proposition A.2.42. In a Hausdorff space every sequence converges to a unique limit.

However, due to statements about filter convergence being strictly stronger than statements about sequence convergence, the converse need not hold. In fact, it does not ([19]).

A statement more specific than Proposition A.2.42 that will be useful is:

Lemma A.2.43. *If two points in a topology are separable by open sets, then a sequence that converges to one of those points does not converge to the other.*

Proof. Let (S, T) be a topological space, let $x, y \in S$, let $O_x, O_y \in T$ be such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$ and let $\langle s_\beta \rangle_{\beta<\alpha}$ be a sequence in S converging to x.

We have a $\beta < \alpha$ such that for all γ with $\beta \leq \gamma < \alpha$ it holds that $s_\gamma \in O_x$, so we get $\{s_\gamma \mid \beta \leq \gamma < \alpha\} \subseteq O_x$. Now by assumption we get $\{s_\gamma \mid \beta \leq \gamma < \alpha\} \cap O_y = \emptyset$. So O_y witnesses that $\langle s_\beta \rangle_{\beta<\alpha}$ doesn't converge to y. □

We also have that:

Definition A.2.44. Let (S, T) be a topological space:

- (S, T) is said to be **regular** if any point $x \in S$ and any non-empty closed set $C \subseteq S$ such that $x \notin C$ can be separated by neighborhoods. That is, if there is a neighborhood N_x of x and a neighborhood N_C of C such that $N_x \cap N_C = \emptyset$.

- (S, T) is said to be **normal** if disjoint closed sets can be separated by neighborhoods. That is, if for any two closed sets C_1 and C_2 such that $C_1 \cap C_2 = \emptyset$ we have neighborhoods N_{C_1} of C_1 and N_{C_2} of C_2 that $N_{C_1} \cap N_{C_2} = \emptyset$.

Remark. In T_0, T_1 and T_2 spaces, points are separated (to varying degrees) and hence those notions can be expressed using open sets. In regular and normal spaces, separation of (closed) sets is required, therefore we need neighborhoods, instead of open sets, to express the notions.

Now:

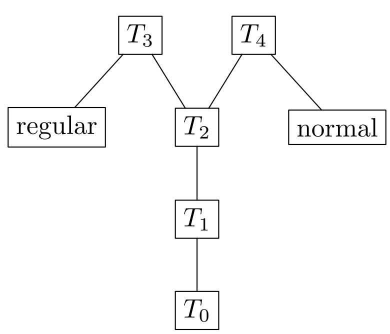
Definition A.2.45. Let (S, T) be a topological space:

- (S, T) is said to be T_3 if it is T_2 and regular.

- (S, T) is said to be T_4 if it is T_2 and normal.

Remark. Technically, for T_4 spaces it is enough to require that they are regular and T_0 and for T_4 spaces it is enough to require that they are normal and T_1.

So we get:



A.2.3.2 Comparison

On any set or space there might be multiple different topologies. These different topologies on the same space are to a certain extent comparable.

Definition A.2.46. If S is some space and T_1 and T_2 are topologies on S then we have $T_1 \leq_{top_S} T_2$ if we have $T_1 \subseteq T_2$. We say that T_1 is **coarser** than T_2 and T_2 is **finer** than T_1.

This order $\leq_{top_S}$ is a complete lattice.

Lemma A.2.47. *If S is a space then the order on its topologies $\leq_{top_S}$ is a complete lattice*

Proof.

- $\leq_{top_S}$ is reflexive because $\subseteq$ is.
- $\leq_{top_S}$ is transitive because $\subseteq$ is.
- $\leq_{top_S}$ is antisymmetric because $\subseteq$ is.
- Each collection X of topologies on S, has as greatest lower bound $\bigcap X$. $\bigcap X$ is a topology on S because:
  - $\bigcap X$ is closed under finite intersection.
    Let M be a finite subset of $\bigcap X$. For all $O \in M$ and $T \in X$ we have $O \in T$ (by construction). Any $T \in X$ is a topology, so because for all $O \in M$ we have $O \in T$ we also have $\bigcap M \in T$ (topologies are closed under finite intersection). This means that we also have $\bigcap M \in \bigcap X$.
  - $\bigcap X$ is closed under arbitrary union.
    Let M be any subset of $\bigcap X$. For all $O \in M$ and $T \in X$ we have $O \in T$ (by construction). Any $T \in X$ is a topology, so because for all $O \in M$ we have $O \in T$ we also have $\bigcup M \in T$ (topologies are closed under arbitrary union). This means that we also have $\bigcup M \in \bigcap X$.

  Clearly we have for all $T \in X$ that $\bigcap X <_{top_S} T$ since we have that $\bigcap X \subseteq T$ (by construction).
- Each collection X of topologies on S, has as least upper bound the topology generated by $\bigcup X$. ($\bigcup X$ itself might not be a topology)

  Lets denote the topology generated by $\bigcup X$ by X^*. For all $T \in X$ we have $T \subseteq \bigcup X \subseteq X^*$ and $T \leq_{top_S} X^*$. □

We have:

Lemma A.2.48. *Every sequence that converges to a point in a topological space converges to that point in the space endowed with a coarser topology.*

Proof. Let (S, T_1) and (S, T_2) be a topological spaces such that $T_1 \leq_{Top_S} T_2$, let $\langle s_\beta \rangle_{\beta<\alpha}$ be a sequence in S and let $x \in S$.

Assume that $x \in \mathrm{Lim}(\langle s_\beta \rangle_{\beta<\alpha})$ in (S, T_2). Let $O \in T_1$ be such that $x \in O$. Because $T_1 \leq_{Top_S} T_2$ we get $O \in T_2$ and because $x \in \mathrm{Lim}(\langle s_\beta \rangle_{\beta<\alpha})$ in (S, T_2) we get that there is some $\beta < \alpha$ such that for all γ with $\beta \leq \gamma < \alpha$ we have $s_\beta \in O$. So $x \in \mathrm{Lim}(\langle s_\beta \rangle_{\beta<\alpha})$ in (S, T_1) □

A.2.3.3 Countability

Definition A.2.49. In a topological space (S, T), $X \subseteq T$ is said to be a **local basis** for $x \in S$ if for any $X_i \in X$ we have $x \in X_i$ and for any $O \in T$ with $x \in O$ there is a $X_i \in X$ such that $X_i \subseteq O$. T is said to be **first-countable** if every $x \in S$ has a countable local basis.

Lemma A.2.50. *First-countability can be expressed using base sets instead of open sets.*

Proof. Let (S, T) be a topological space with B as base. We prove that (S, T) is first-countable if and only if for every $x \in S$ there is a countable $Y \subseteq B$ such that for any $Y_i \in Y$ we have $x \in Y_i$ and for any $A \in B$ with $x \in A$ there is a $Y_i \in Y$ such that $Y_i \subseteq A$

⇒ Assume that (S, T) is first-countable.

Let $x \in S$, we get a countable local basis $X \subseteq T$. For any $X_i \in X$ we have $x \in X_i$ and for any $O \in T$ with $x \in O$ there is a $X_i \in X$ such that $X_i \subseteq O$. Since B is a base for (S, T) we get for every $X_i \in X$ that it is the union of sets in B. Since $x \in X_i$, at least one such base set must have x as member, call one such base set Y_i and let Y be the set of all such sets Y_i. Let $A \in B$ be such that $x \in A$, we get $A \in T$ and hence get some $X_i \subseteq A$ by assumption. Now because $Y_i \subseteq X_i$, we get $Y_i \subseteq A$.

⇐ Assume that for every $x \in S$ there is a countable $Y \subseteq B$ such that for any $Y_i \in Y$ we have $x \in Y_i$ and for any $A \in B$ with $x \in A$ there is a $Y_i \in Y$ such that $Y_i \subseteq A$.

Let $x \in S$, we get the assumed $Y \subseteq B$. We have $Y \subseteq T$ since B is a base for T. Now for any $O \in T$ such that $x \in O$ we get that it is the union of base sets in B. At least one such base set must have x as member, call one such base set A. By assumption we get some $Y_i \in Y$ such that $Y_i \subseteq A$ and since $A \subseteq O$ we get $Y_i \subseteq O$. That means Y is a local basis for x, proving first-countability □

Definition A.2.51. A topological space (S, T) is **second-countable** if it has a countable base.

It follows from Lemma A.2.50 that:

Proposition A.2.52. Every second-countable space is first-countable

A.2.4 Specific Topologies

A.2.4.1 Order Topologies and Ordinal Topologies

An important topology on an ordered set that is easily defined using the notions of base and subbase is the order topology.

Definition A.2.53. The **order topology** on a linearly ordered set $(A, \leq)$ is the topology generated by the following subbase:

$$\{\{x \in X \mid a < x\} \mid a \in A\} \cup \{\{x \in X \mid x < a\} \mid a \in A\}$$

By Lemma A.2.14 this definition by subbase generates an actual topology (as does any definition by subbase).

The sets $\{x \mid a < x\}$ and $\{x \mid x < a\}$ are called **open rays**. So the above subbase for the order topology on $(A, \leq)$ is the set of all open rays in $(A, \leq)$. The sets $\{x \mid a < x < b\}$ are called **open intervals**. The closure of the subbase of the order topology under finite intersection yields a base of the order topology on $(A, \leq)$, this base is the set of all open rays and open intervals in $(A, \leq)$.

Any ordinal is linearly orders, so it can be endowed with the order topology. The order topology is in fact the standard topology on ordinals and is, in the context of ordinals, also called the **ordinal topology**.

Lemma A.2.54. *For any ordinal, the limit points in its ordinal-topological space are exactly the limit ordinals below the ordinal.*

Proof. Let $\alpha \in On$ and let T be the ordinal topology on it.

- Assume that $\lambda \in \alpha$ is a limit point in (α, T).

  We have that the open ray $\{\gamma \in \alpha \mid \gamma < 1\}$ is $\{0\}$. So 0 is isolated and hence $\lambda \neq 0$. For any β, the open interval $\{\gamma \in \alpha \mid \beta\prime < \gamma < \beta\prime\prime\}$ is $\{\beta\prime\}$. So any successor ordinal $\beta\prime$ is isolated and hence λ is not a successor ordinal. That means that λ must be a limit ordinal.

- Assume that $\lambda < \alpha$ such that $\lambda \in \text{LimOrd}$.

  - Let $\{\gamma \in \alpha \mid \gamma < \beta\}$ be a (downwards) open ray with λ as member. We get $0 < \lambda$ since $\lambda \in \text{LimOrd}$. We also get $\lambda < \beta$, so $0 < \beta$ and hence $0 \in \{\gamma \in \alpha \mid \gamma < \beta\}$.
  - Let $\{\gamma \in \alpha \mid \beta < \gamma\}$ be an (upwards) open ray with λ as member. We get $\beta < \lambda$ and by Lemma A.1.26 we get some η such that $\beta < \eta < \lambda$. That means that $\eta \in \{\gamma \in \alpha \mid \beta < \gamma\}$.
  - Let $\{\gamma \in \alpha \mid \beta_1 < \gamma < \beta_2\}$ be an open interval with λ as member. We get $\beta_1 < \lambda$ and by Lemma A.1.26 we get some η such that $\beta < \eta < \lambda$. That means that $\eta \in \{\gamma \in \alpha \mid \beta_1 < \gamma < \beta_2\}$.

  So for any base set that λ is in, we get that it has at least one other member (in fact, it has infinitely many other members). So every open set that λ is in also has at least one other member. That means that λ is a limit point. □

The contraposition of this result is that the isolated points in the ordinal topology on α are 0 and the successor ordinals below α.

We can view sequences as functions with the ordinal that expresses its length as domain. This means that we can express the continuity of sequences using the ordinal topology on this length ordinal.

Definition A.2.55. A sequence $\langle s_\beta \rangle_{\beta<\alpha}$ in a topological space (S, T) is continuous if the underlying function, s from the ordinal-topological space of α to (S, T), is continuous.

We get that:

Lemma A.2.56. *A sequence in a topological space is continuous if and only if, for any limit ordinal index, the sequence up to that index converges to the member at that index.*

Proof. Let $\lambda < \alpha$ be a topological space and let $\langle s_\beta \rangle_{\beta<\alpha}$ a sequence in it.

⇒ Assume that $\langle s_\beta \rangle_{\beta<\alpha}$ is continuous as a function from the ordinal topological space of α to (S, T).

Let $\lambda < \alpha$ be a limit ordinal. By the continuity of the sequence at λ we get that for any open set that contains s_λ there is a $\beta < \lambda$ such that for all γ with $\beta \le \gamma < \lambda$ we have that s_γ is a member of that open set. This means that $\langle s_\beta \rangle_{\beta<\lambda}$ converges to s_λ by Proposition A.2.30.

⇐ Assume that for every limit ordinal $\lambda < \alpha$ we have that $\langle s_\beta \rangle_{\beta<\lambda}$ converges to s_λ.

Let $\lambda < \alpha$ be a limit ordinal. We get that for any open set that contains s_λ there is a $\beta < \lambda$ such that for all γ with $\beta \le \gamma < \lambda$ we have that s_γ is a member of that open set. This means that $\langle s_\beta \rangle_{\beta<\alpha}$ is continuous at λ. For any $\eta < \alpha$ such that η is not a limit ordinal we get that η is isolated in (S, T) by Lemma A.2.54. Now by Lemma A.2.35, $\langle s_\beta \rangle_{\beta<\alpha}$ is continuous at β. So $\langle s_\beta \rangle_{\beta<\alpha}$ is continuous at all $\eta < \alpha$ and hence is continuous by Lemma A.2.21. □

A.2.4.2 Scott Topologies

On any dcpo we can define the Scott topology:

Definition A.2.57. If $(X, \le)$ is a dcpo, then the **Scott topology** on X induced by $\le$ is the topology where the open sets $O \subseteq X$ are sets for which it holds that:

- O is upper.
- If D is a directed subset of X and $\bigsqcup D \in O$ then $\exists d \in D\ d \in O$. (Every directed set that has a least upper bound in O is **eventually** in O)

This defines an actual topology.

Lemma A.2.58. *The Scott topology is an actual topology.*

Proof. Let $(X, \leq)$ is a dcpo and let T be the Scott topology on X.

- T is closed under finite intersection.

  Let $\{O_i \mid i \in I\}$ be a finite subset of T.

  - Let $x \in \bigcap\{O_i \mid i \in I\}$ be arbitrary and let $y \in X$ be such that $x \leq y$. For all $i \in I$ we have $x \in O_i$ and hence $y \in O_i$ because O_i is upper. So we get $y \in \bigcap\bigcap\{O_i \mid i \in I\}$ and hence $\bigcap\bigcap\{O_i \mid i \in I\}$ is an upper set.
  - Let D be a directed subset of X and $\bigsqcup D \in \bigcap\{O_i \mid i \in I\}$. For all $i \in I$ we have $\bigsqcup D \in O_i$, so we get an $x_i \in D$ such that $x_i \in O_i$. Because D is directed and I is finite we get an upper bound, $d \in D$, for $\{x_i \mid i \in I\}$. For any $i \in I$ we get $d \in O_i$ because $x_i \leq d$, $x_i \in O_i$ and O_i is upper. So we have $d \in \bigcap\{O_i \mid i \in I\}$ and D is eventually in $\bigcap\{O_i \mid i \in I\}$.

- T is closed under arbitrary union.

  Let A be some subset of T.

  - Let $x \in \bigcup A$ be arbitrary and $y \in X$ such that $x \leq y$. For some $O \in A$ we must have $x \in O$ because $x \in \bigcup A$, so we get $y \in O$ because O is upper. This means $y \in \bigcup A$ and $\bigcup A$ is an upper set.
  - Let D be a directed subset of X and $\bigsqcup D \in A$. For some $O \in A$ we must have $\bigsqcup D \in O$ because $\bigsqcup D \in \bigcup A$, we get a $d \in D$ such that $d \in O$ because d is eventually in O. So we get $d \in \bigcup A$ and hence D is eventually in $\bigcup A$. □

For dcpos with infinite elements (elements that are above infinitely many other elements), the Scott Topology is the preferred topology ([1, p. 648],[4, p. 22]).

To inhabit some of the open sets in the Scott topology we can prove that:

Lemma A.2.59. *For any x we have that $\{y \mid y \not\leq x\}$ is open in the Scott topology.*

Proof. Let $(X, \leq)$ be a dcpo, let T be the Scott topology induced by it and let $x \in X$.

- $\{y \mid y \not\leq x\}$ is an upper set.

  Let $z_1 \in \{y \mid y \not\leq x\}$, we get $z_1 \not\leq x$. Let z_2 be such that $z_1 \leq z_2$. Suppose for contradiction that we have $z_2 \notin \{y \mid y \not\leq x\}$, we get $z_2 \leq x$ and hence by transitivity $z_1 \leq x$ which is a contradiction. So we much have $z_2 \in \{y \mid y \not\leq x\}$.

- Every directed set that has a least upper bound in $\{y \mid y \not\leq x\}$ is eventually in $\{y \mid y \not\leq x\}$.

  Let $D \subseteq S$ be a directed set such that $\bigsqcup D \in \{y \mid y \not\leq x\}$, we get $\bigsqcup D \not\leq x$. Suppose for contradiction that for all $d \in D$ we have $d \notin \{y \mid y \not\leq x\}$. For any $d \in D$ we get $d \leq x$, so x is an upper bound of D, so $\bigsqcup D \leq x$, which is a contradiction. So we must have some $d \in D$ such that $d \in \{y \mid y \not\leq x\}$. □

Using this, it can be proven that:

Lemma A.2.60. *The specialization quasi-order induced by a Scott topology induced by a dcpo is that dcpo.*

Proof. Let $(S, \leq)$ be a dcpo, let T be the Scott topology that it induces and let $\leq_T$ be the specialization quasi-order that (S, T) induces.

We prove that for all $a, b \in S$ we have $a \leq b \Leftrightarrow a \leq_T b$. Let $a, b \in S$ be arbitrary:

- $\Leftarrow$ Assume $a \leq b$.

  For any $O \in T$ with $a \in O$ we have $b \in O$ because O is an upper set, so we get $a \leq_T b$.

- $\Rightarrow$ Assume $a \leq_T b$.

  For any $O \in T$ with $a \in O$ we get $b \in O$. Suppose for contradiction that we have $a \not\leq b$, we get $a \in \{y \mid y \not\leq b\}$. Because $\{y \mid y \not\leq b\}$ is open by Lemma A.2.59 we get $b \in \{y \mid y \not\leq b\}$ and hence $b \not\leq b$, which contradicts reflexivity of our order. So we must have $a \leq b$. □

We also have that:

Lemma A.2.61. *Different dcpos have different Scott topologies associated with them.*

Proof. Let $(A, \leq_1)$ and (A, leq_2) be different dcpos and let resp. T_1 and T_2 be the topologies associated with them. Without loss of generality we can assume that there are $a, b \in A$ such that $a \leq_1 b$ and $a \not\leq_2 b$. By Lemma A.2.59 we get that $\{y \mid y \not\leq_2 b\} \in T_2$. We also get $a \in \{y \mid y \not\leq_2 b\}$ by assumption and $b \notin \{y \mid y \not\leq_2 b\}$ by reflexivity of $\leq_2$. But for every $O \in T_1$ we get $a \in O \to B \in O$ since $a \leq_1 b$ and O is an upper set. So $\{y \mid y \not\leq_2 b\} \notin T_1$ and hence $T_1 \neq T_2$. □

There is a connection between the $\mathscr{S}$-limits of a sequence in a dcpo and the limits of the sequence in the Scott topology induced by that dcpo. First; we'll refer to the limits of a sequence in Scott topology induced by a dcpo as the **topological limits** of the sequence in the dcpo. We have:

Lemma A.2.62. *Every $\mathscr{S}$-limit of a sequence in a dcpo is a topological limit.*

Proof. Let $(X, \leq)$ be a dcpo, let T be the Scott topology induced by it and let $\langle x_\beta \rangle_{\beta < \alpha}$ be a sequence in it.

First; if $\beta_1 \leq \beta_2 < \alpha$ then $\bigsqcap\{x_\gamma \mid \beta_1 \leq \gamma < \alpha\} \leq \bigsqcap\{x_\gamma \mid \beta_2 \leq \gamma < \alpha\}$ since the former is simply the greatest lower bound of a superset of the set the latter is the greatest lower bound for. This means that $\{\bigsqcap\{a_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$ is directed.

Now, let x be a $\mathscr{S}$-limit of $\langle x_\beta \rangle_{\beta<\alpha}$ and let $O \in T$ be such that $x \in O$. We get $\liminf(\langle x_\beta \rangle_{\beta<\alpha}) \in O$ because $x \leq \liminf(\langle x_\beta \rangle_{\beta<\alpha})$ and O is an upper set. We get a $\beta < \alpha$ such that for all γ with $\beta \leq \gamma < \alpha$ we have $\bigsqcap\{x_\gamma \mid \beta \leq \gamma < \alpha\} \in O$ because $\{\bigsqcap\{x_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$ is directed and $\liminf(\langle x_\beta \rangle_{\beta<\alpha})$ is its upper bound. Then, for all γ with $\beta \leq \gamma < \alpha$ we get $x_\gamma \in O$ because O is upper. That means that we get that $x \in \mathrm{Lim}(\langle a_\beta \rangle_{\beta<\alpha})$ in (X, T). □

We would also like to have the converse, that every topological limit of a sequence in of a dcpo is an $\mathscr{S}$-limit.

In [3] it is proven that the classes of $\mathscr{S}$-limits and topological limits of a net in a partially ordered set coincide if and only if the partially ordered set is continuous. Since we are concerned with sequences and dcpos instead of nets and partially ordered sets, we can use a weakened version of the 'if'-direction of this theorem to say that the sets of $\mathscr{S}$-limits and topological limits of a sequence in a dcpo coincide if the dcpo is continuous.

Theorem A.2.63. *If a dcpo is continuous then every topological limit of a sequence in it is an $\mathscr{S}$-limit.*

Proof. By [3]. □

The inverse of this would state that: if the dcpo is non-continuous then there is a sequence in it that has a topological limit that is not a $\mathscr{S}$-limit. This does not seem to hold. This is probably because statements about nets are strictly stronger than statements about sequences. Every sequence gives rise to a net, but not vice versa. It turns out to be hard to get the machinery used in the proof rolling when using just sequences. However if we just restrict ourselves to continuous dcpos, we are in the clear and get the nice result that:

Proposition A.2.64. In a continuous dcpo, the $\mathscr{S}$-limits of a sequence are exactly the topological limits of this sequence.

A.2.4.3 Discrete Topologies

Definition A.2.65. The **discrete topology** on a set S is the set of all subsets, $\mathcal{P}(S)$, of S. $(S, \mathcal{P}(S))$ is called a discrete topological space.

The set $\{\{p\} \mid p \in S\}$ generates the discrete topology and is a base of it. This is because the closure of $\{\{p\} \mid p \in S\}$ under arbitrary union yields $\mathcal{P}(S)$. By definition, every point in a discrete topological space is isolated. The discrete topology is finer than any other topology and hence is the top of the complete lattice of topologies on a space.

A.2.4.4 Trivial Topologies

Definition A.2.66. The **trivial topology** on a set S is $\{\emptyset, S\}$. $(S, \{\emptyset, S\})$ is called a trivial topological space.

The empty set generates the trivial topology and is a base of it. This is because the intersection of $\emptyset$ is S and the union of $\emptyset$ is $\emptyset$. By definition, every point in a trivial topological space is a limit point. The trivial topology is coarser than any other topology and hence is the bottom of the complete lattice of topologies on a space.

A.2.4.5 Subspace Topologies

Given a topological space we can define the topology on a subset of the space in a natural way:

Definition A.2.67. If (S,T) is a topological space and $X \subseteq S$ then the **subspace topology** on X is $\{O \cap X \mid O \in T\}$. We say that $\{O \cap X \mid O \in T\}$ is the **restriction of** T **to** X, that $(X, \{O \cap X \mid O \in T\})$ is a **subspace** of (S,T) and that (S,T) is a **superspace** of $(X, \{O \cap X \mid O \in T\})$.

Lemma A.2.68. *The subspace topology is an actual topology.*

Proof. Let (S,T) be a topological space and $X \subseteq S$.

- $\{O \cap X \mid O \in T\}$ is closed under finite intersection.

  Let $\{O_i \cap X : i \in I\}$ be a finite indexed family of open sets in $\{O \cap X \mid O \in T\}$. We have $\bigcap\{O_i \cap X : i \in I\} = \bigcap\{O_i : i \in I\} \cap X$, also $\bigcap\{O_i : i \in I\} \in T$ because T is closed under finite intersection, so $(\bigcap\{O_i : i \in I\} \cap X) \in (\{O \cap X \mid O \in T\})$.

- $\{O \cap X \mid O \in T\}$ is closed under arbitrary intersection.

  Let $\{O_i \cap X : i \in I\}$ be an indexed family of open sets in $\{O \cap X \mid O \in T\}$. We have $\bigcup\{O_i \cap X : i \in I\} = \bigcup\{O_i : i \in I\} \cap X$, also $\bigcup\{O_i : i \in I\} \in T$ because T is closed under arbitrary union, so $(\bigcup\{O_i : i \in I\} \cap X) \in (\{O \cap X \mid O \in T\})$. □

We get:

Lemma A.2.69. *A sequence in a subspace of a topological space converges to a point in the subspace if and only if it converges to that point in the superspace.*

Proof. Let (S,T) be a topological space, let $X \subset S$, let $\langle x_\beta \rangle_{\beta<\alpha}$ be a sequence in X and let $x \in X$.

$\Rightarrow$ Assume that $\langle x_\beta \rangle_{\beta<\alpha}$ converges to x in $(X, \{O \cap X \mid O \in T\})$.

Let $O \in T$ be such that $x \in O$, we get that $x \in (O \cap X)$ and hence there is a $\beta < \alpha$ such that for all γ with $\beta \le \gamma < \alpha$ we have $x_\gamma \in (O \cap X) \subseteq O$. So $\langle x_\beta \rangle_{\beta<\alpha}$ converges to x in (X,T).

$\Leftarrow$ Assume that $\langle x_\beta \rangle_{\beta<\alpha}$ converges to x in (X,T).

Let $(O \cap X) \in \{O \cap X \mid O \in T\}$ be such that $x \in (O \cap X)$, we get that $x \in O$ and hence there is a $\beta < \alpha$ such that for all γ with $\beta \le \gamma < \alpha$ we have $x_\gamma \in O$ and since $\langle x_\beta \rangle_{\beta<\alpha}$ is a sequence in X we get $x_\gamma \in (O \cap X)$. So $\langle x_\beta \rangle_{\beta<\alpha}$ converges to x in $(X, \{O \cap X \mid O \in T\})$. □

A.3 Metric Spaces

Metric spaces formalize the structure of a set by defining a distance measure on the set. This formalizes the intuition that some points in the set are alike or close to each other.

Definition A.3.1. A **metric** on a set S is a function $d : (S \times S) \to \mathbb{R}^{0+}$ such that for all $x, y, z \in S$ we have:

- $d(x, x) = 0$ — REFLEXIVITY
- $d(x, y) = 0 \wedge d(y, x) = 0 \to x = y$ — IDENTITY OF INDISCERNIBLES
- $d(x, y) = d(y, x)$ — SYMMETRY
- $d(x, z) \leq d(x, y) + d(y, z)$ — TRIANGLE INEQUALITY

The structure (S, d) is called a **metric space**. The elements of S are referred to as **points**. In the light of some metric d on S we might also refer to S as a **metric space**, totum pro parte, or simply as **space**.

The metric is of course the formalization of the concept of **distance**. $d(x, y) = \epsilon$ has as intended meaning that the distance between x and y is ϵ. These requirements can be strengthened to obtain stronger notion of metricity:

Definition A.3.2. A **ultrametric** on a set S is a function $d : (S \times S) \to \mathbb{R}^{0+}$ that obeys REFLEXIVITY, IDENTITY OF INDISCERNIBLES, SYMMETRY and such that for all $x, y, z \in S$ we have:

- $d(x, z) \leq \max(d(x, y), d(y, z))$ — STRONG TRIANGLE INEQUALITY

The structure (S, d) is called an **ultrametric space**.

Since $d(x, y), d(y, z) \in R^{0+}$ implies $\max(d(x, y), d(y, z)) \leq d(x, y) + d(y, z)$, STRONG TRIANGLE INEQUALITY implies TRIANGLE INEQUALITY. So any ultrametric is a metric and ultrametricity is a strictly stronger notion than metricity.

The requirements can also be weakened:

Definition A.3.3. A distance function that satisfies REFLEXIVITY, IDENTITY OF INDISCERNIBLES and TRIANGLE INEQUALITY (but not necessarily SYMMETRY) is called a **quasi-metric**. A distance function that satisfies REFLEXIVITY, SYMMETRY and TRIANGLE INEQUALITY (but not necessarily IDENTITY OF INDISCERNIBLES) is called a **pseudometric**.

A.3.1 Useful Examples

An often used example of a metric space is the real line:

Definition A.3.4. The **real line** $(\mathbb{R}, d_{\mathbb{R}})$ is the metric space where $\mathbb{R}$ is the set of real numbers and $d_{\mathbb{R}}$ is the absolute difference between those numbers, $d_{\mathbb{R}}(x, y) = |x - y|$.

Another often used metric is the discrete metric which can easily be defined on any set to obtain a metric space:

Definition A.3.5. The **discrete metric** ρ on any set S is the metric such that for all $x, y \in S$ we have:

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

The topology underlying the discrete metric (in a sense that will be explained in the section on induced topologies) is the discrete topology on S.

For any given metric space, a metric on the sets of points of the space can be given as follows:

Definition A.3.6. If (S, d) is a metric space, the Hausdorff metric $d_H : \mathcal{P}(S) \times \mathcal{P}(S)$ is the metric such that:

$$d_H(X,Y) = \max(\bigsqcup\{\bigsqcap\{d(x,y) \mid x \in X\} \mid y \in Y\}, \bigsqcup\{\bigsqcap\{d(x,y) \mid y \in Y\} \mid x \in X\})$$

Lemma A.3.7. *The Hausdorff metric is an actual metric.*

Proof. ‘ Let (S, d) be a metric space.

- $d_H(X,X) = 0$ REFLEXIVITY

  For any $x \in X$ we get $d(x,x) = 0$ (by REFLEXIVITY of d), so $\bigsqcap\{d(x,y) \mid y \in X\} = 0$, hence means that $\bigsqcup\{\bigsqcap\{d(x,y) \mid y \in X\} \mid x \in X\} = 0$ and $\max(\bigsqcup\{\bigsqcap\{d(x,y) \mid y \in X\} \mid x \in X\}, \bigsqcup\{\bigsqcap\{d(x,y) \mid x \in X\} \mid y \in X\}) = d_H(X,X) = 0$.

- $d_H(X,Y) = d_H(Y,X)$ SYMMETRY.

  $$d_H(X,Y) = \max(\bigsqcup\{\bigsqcap\{d(x,y) \mid x \in X\} \mid y \in Y\}, \bigsqcup\{\bigsqcap\{d(x,y) \mid y \in Y\} \mid x \in X\} = \max(\bigsqcup\{\bigsqcap\{d(x,y) \mid y \in Y\} \mid x \in X\}, \bigsqcup\{\bigsqcap\{d(x,y) \mid x \in X\} \mid y \in Y\}) = d_H(Y,X)$$

  (By the commutativity of max)

- $d_H(X,Y) = 0 \wedge d(Y,X) = 0 \rightarrow Y = X$ IDENTITY OF INDISCERNABLES

  Assume that $d_H(X,Y) = 0$. Suppose for contradiction that $X \neq Y$. By SYMMETRY we can assume, without loss of generality, that there is an $x \in X$ such that $x \notin Y$. We get that there is no $y \in Y$ with $d(x,y) = 0$ by IDENTITY OF INDISCERNABLES of d. So $\bigsqcap\{d(x,y) \mid y \in Y\} > 0$ and hence $\bigsqcup\{\bigsqcap\{d(x,y) \mid y \in Y\} \mid x \in X\} > 0$, contradicting that $d_H(X,Y) = 0$. We must have $X = Y$ after all.

- $d_H(X,Z) \leq d_H(X,Y) + d_H(Y,Z)$ TRIANGLE INEQUALITY

  We get:

$$\begin{aligned}
&\bigsqcup\{\bigsqcap\{d(x,y) \mid x \in X\} \mid y \in Y\} + \bigsqcup\{\bigsqcap\{d(y,z) \mid y \in Y\} \mid z \in Z\} &=\\
&\bigsqcup\{\bigsqcap\{d(x,y_1) \mid x \in X\} + \bigsqcap\{d(y,z) \mid y \in Y\} \mid y_1 \in Y, z \in Z\} &=\\
&\bigsqcup\{\bigsqcap\{d(x,y_1) + d(y_2,z) \mid x \in X, y_2 \in Y\} \mid y_1 \in Y, z \in Z\} &\leq\\
&\bigsqcup\{\bigsqcap\{d(x,y_1) + d(y_1,z) \mid x \in X\} \mid y_1 \in Y, z \in Z\} &\leq\\
&\bigsqcup\{\bigsqcap\{d(x,z) \mid x \in X\} \mid y_1 \in Y, z \in Z\} &=\\
&\bigsqcup\{\bigsqcap\{d(x,z) \mid x \in X\} \mid z \in Z\}
\end{aligned}$$

  And similarly:

$$\begin{aligned}
&\bigsqcup\{\bigsqcap\{d(x,y) \mid y \in Y\} \mid x \in X\} + \bigsqcup\{\bigsqcap\{d(y,z) \mid z \in Z\} \mid y \in Y\} &=\\
&\bigsqcup\{\bigsqcap\{d(x,z) \mid z \in Z\} \mid x \in X\}
\end{aligned}$$

  So:

$$\begin{aligned}
&d_H(X,Y) + d_H(Y,Z) &=\\
&\max(\bigsqcup\{\bigsqcap\{d(x,y) \mid x \in X\} \mid y \in Y\}, \bigsqcup\{\bigsqcap\{d(x,y) \mid y \in Y\} \mid x \in X\} &+\\
&\max(\bigsqcup\{\bigsqcap\{d(y,z) \mid y \in Y\} \mid z \in Z\}, \bigsqcup\{\bigsqcap\{d(y,z) \mid z \in Z\} \mid y \in Y\} &=\\
&\max(\bigsqcup\{\bigsqcap\{d(x,z) \mid x \in X\} \mid z \in Z\}, \bigsqcup\{\bigsqcap\{d(x,z) \mid z \in Z\} \mid x \in X\} &=\\
&d_H(X,Z)
\end{aligned}$$

So d_H is a metric on $\mathcal{P}(S)$. □

And similarly:

Proposition A.3.8. The Hausdorff metric induced by an ultrametric is ultrametric.

A.3.2 Induced Topologies

A metric d on a space S induces a notion open sets on S and hence induces a topology associated with the metric.

Definition A.3.9. For any point x in a metric space (S,d), the set $\{y \in S \mid d(x,y) < \epsilon\}$ for some $\epsilon \in \mathbb{R}^+$ is called an **open (ϵ-)ball** around x. We denote this ball by $B_\epsilon(x)$.

So an open ϵ-ball around x is just the set of all points within a certain distance (ϵ) of x. Now:

Definition A.3.10. The topology **induced by** a metric d on a space S is the topology that is generated by the set of all open balls in (S, d). The resulting to topology T is called the **metric topology** or **ϵ-ball topology**.

So the topology induced by a metric is the one of which the set of all open balls in the metric space is a subbase. In fact the set of all open balls is also a base for this topology.

Lemma A.3.11. *The set of all open balls in a metric space, (S, d), is the base of the topology on S induced by d.*

Proof. By Lemma A.2.12, we need to prove that the base-criterion holds for the set of all open balls.

Suppose we have two open balls, $B_{\epsilon_1}(x)$ and $B_{\epsilon_2}(y)$ in S (we have $x, y \in S$, $\epsilon, \epsilon_2 \in \mathbb{R}^+$). Let $p \in B_{\epsilon_1}(x)$ such that $p \in B_{\epsilon_2}(y)$. We have that $d(x,p) < \epsilon_1$ and $d(y,p) < \epsilon_2$. We have $\tau_1, \tau_2 \in \mathbb{R}^+$ such that $d(x,p)+\tau_1 = \epsilon_1$ and $d(y,p)+\tau_1 = \epsilon_2$. Let τ be $\min(\tau_1, \tau_2)/2$. $B_\tau(p)$ is an open ball.

For any $q \in B_\tau(p)$ we have that $d(p,q) < \tau$, and hence $d(p,q) < \tau_1$ and $d(p,q) < \tau_2$. So we have $d(x,p) + d(p,q) < \epsilon_1$ and hence, by TRIANGLE INEQUALITY, $d(x,q) < \epsilon_1$, so we have $q \in B_{\epsilon_1}(x)$, so $B_\tau(p) \subseteq B_{\epsilon_1}(x)$. Likewise, we have $d(y,p)+d(p,q) < \epsilon_2$ and hence, by TRIANGLE INEQUALITY, $d(y,q) < \epsilon_2$, so we have $q \in B_{\epsilon_2}(y)$, so $B_\tau(p) \subseteq B_{\epsilon_2}(y)$.

This means that the base-criterion holds for the set of all open balls in (S, d) and hence the set of open balls is a base for some topology on S. By definition this topology is the one (S, d) induces. □

This means that the topology induced by a metric space is one where the open sets are unions of open balls.

The fact that metric spaces induce topologies begs the question which topologies are induced by a metric space. First we define:

Definition A.3.12. A topological space for which there is a metric that induces it is called **metrizable**.

We get that not all topological spaces are metrizable. A simple example is the already mentioned Sierpiński space: the set $\{0, 1\}$ with on it the topology $\{\emptyset, \{1\}, \{0, 1\}\}$. Theorems that give topological properties that are sufficient for a space to be metrizable are called metrization theorems. An early and important metrization theorem is Urysohn's metrization theorem.

Theorem A.3.13. *A topological space is metrizable if it is T_1, regular and second-countable.*

Proof. [12, p. 125] □

A necessary condition for a topological space to be metrizable arises from the following:

Lemma A.3.14. *Any metric topology is Hausdorff.*

Proof. Let (S, d) be a metric space and let T be the topology induced by it, let $x, y \in S$ such that $x \neq y$. By IDENTITY OF INDISCERNIBLES we get that $d(x, y) > 0$, so $d(x, y)/2 > 0$. We get:

- $\{z \in S \mid d(x, z) < d(x, y)/2\}$ is open in T.
- $x \in \{z \in S \mid d(x, z) < d(x, y)/2\}$.
- $\{z \in S \mid d(y, z) < d(x, y)/2\}$ is open in T.
- $y \in \{z \in S \mid d(y, z) < d(x, y)/2\}$.
- Now, suppose for contradiction that there is a c such that we have $c \in \{z \in S \mid d(x, z) < d(x, y)/2\}$ and $c \in \{z \in S \mid d(y, z) < d(x, y)/2\}$. We get $d(x, c) < d(x, y)/2$ and $d(y, c) < d(x, y)/2$. By SYMMETRY we also get $d(c, y) < d(x, y)/2$ and then by TRIANGLE INEQUALITY we get $d(x, y) < d(x, y)$, contradiction. So we must conclude that there is no such c. Which means that $\{z \in S \mid d(x, z) < d(x, y)/2\} \cap \{z \in S \mid d(y, z) < d(x, y)/2\} = \emptyset$. So T is Hausdorff.

So we get that x and y are separated by open sets and hence T is Hausdorff. □

So a topological space being Hausdorff is a necessary condition for it being metrizable. The example of the Sierpiński space not being metrizable can be derived from this (in the Sierpiński space, the two points can not be separated by open sets and hence the space is not Hausdorff).

There are stronger necessary conditions, for instance, any metrizable space is T_4, but the condition of being Hausdorff is enough for our purposes.

Also:

Lemma A.3.15. *Any metric space is first-countable.*

Proof. Let (S, d) be a metric space, let $x \in S$ and consider $\{B_{1/n}(x) \mid n \in \mathbb{N}\}$ as local basis for x. Let O such that $x \in O$ be open in the topology induced by d, we get that O is the union of open balls in (S, d). Since $x \in O$ we get some $\epsilon \in \mathbb{R}^+$ such that $B_\epsilon(x) \subseteq O$. We get $\epsilon \leq 1/\lceil 1/\epsilon \rceil$, so $B_{1/\lceil 1/\epsilon \rceil}(x) \subseteq B_\epsilon(x)$ and hence $B_{1/\lceil 1/\epsilon \rceil}(x) \subseteq O$. We also have $\lceil 1/\epsilon \rceil \in \mathbb{N}$, so $\{B_{1/n}(x) \mid n \in \mathbb{N}\}$ is a local basis for x. $\{B_{1/n}(x) \mid n \in \mathbb{N}\}$ is countable. □

This gives another necessary condition for metrizability.

A.3.3 Sequences, Convergence and Completeness

We can also specify the notions and the associated notions of convergence and limits of transfinite sequences (via filter bases) to metric spaces. From the topological definition we get that a sequence of elements of a metric space converges to a limit x if for every base set B there is a set X in its filter base of tails such that $X \subseteq B$. Unfolding this, we get that:

Proposition A.3.16. A sequence s of length α of elements of a metric space converges to a limit x, if for every $\epsilon \in \mathbb{R}^+$ there is a $\beta < \alpha$ such that for all γ with $\beta \leq \gamma < \alpha$ we have $d(s_\gamma, x) < \epsilon$.

Or; for every open ball around the limit we get that, after some index, the sequence stops getting outside that ball. Or more intuitively; the points in the sequence eventually get arbitrary close to the limit.

We get:

Lemma A.3.17. *In a metric space every sequence has at most one limit.*

Proof. Let (S, d) be a metric space, let $\langle s_\beta \rangle_{\beta<\alpha}$ be a sequence in it, let x and y be limits of this sequence and let $\epsilon \in \mathbb{R}^+$. By x and y being limits, we must have some $\gamma < \alpha$ such that $d(s_\gamma, x) < \epsilon/2$ and $d(s_\gamma, y) < \epsilon/2$. By SYMMETRY and TRIANGLE INEQUALITY we get $d(x, y) < \epsilon$. Since this holds for any $\epsilon \in \mathbb{R}^+$ we must have $d(x, y) = 0$ and hence x and y are the same point after all by SYMMETRY and IDENTITY OF INDISCERNIBLES. □

This means that we can use the following notation:

Definition A.3.18. If $\langle s_\beta \rangle_{\beta<\alpha}$ is a sequence in a metric space and x is its limit, then we can write $x = \lim(\langle s_\beta \rangle_{\beta<\alpha})$

A notion related to convergence of a sequence is that of a sequence being Cauchy.

Definition A.3.19. If (S, d) is a metric space and $A \subseteq S$ then we write $|A|_d$ for the **diameter** of A; $|A|_d = \sup \{d(x, y) \mid x, y \in A\}$. A filter base B is **Cauchy** if for every $\epsilon \in \mathbb{R}^+$ there is a set $A \in B$ such that $|A|_< \epsilon$. A sequence is **Cauchy** if its filter base of tails is Cauchy.

By unfolding this definition we get:

Proposition A.3.20. A sequence $\langle s_\beta \rangle_{\beta<\alpha}$ is Cauchy if and only if for every $\epsilon \in \mathbb{R}^+$ there is a $\beta < \alpha$ such that for all γ_1, γ_2 with $\beta \leq \gamma_1, \gamma_2$ we have $d(s_{\gamma_1}, s_{\gamma_2}) < \epsilon$.

We have:

Lemma A.3.21. *Any convergent sequence is Cauchy.*

Proof. Let (S, d) be a metric space and let $\langle s_\beta \rangle_{\beta<\alpha}$ be a sequence converging to $x \in S$ in this space. Let $\epsilon \in \mathbb{R}^+$ be arbitrary. Because $\langle s_\beta \rangle_{\beta<\alpha}$ converges to x we get a $\beta < \alpha$ such that for all γ with $\beta \leq \gamma < \alpha$ we have $d(s_\gamma, x) < \frac{\epsilon}{2}$. Now for any $\gamma_1, \gamma_2 < \alpha$ such that $\beta \leq \gamma_1$ and $\beta \leq \gamma_2$ we have:

$$\begin{aligned} d(s_{\gamma_1}, s_{\gamma_2}) &\leq d(s_{\gamma_1}, x) + d(x, s_{\gamma_2}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Which proves Cauchyness. □

So, in a Cauchy sequence, we have that for every distance ϵ, there is some index in the sequence after which distances between sequence members never becomes more than ϵ. So sequence members get closer and closer to each other as the sequence progresses. It might be said, these sequences should also converge, were it not for the fact that there might not a be limit in the space for the sequence to converge to. One might imagine that in a superset of the space that does include a point that would function a limit for the sequence, the Cauchy sequence would actually converge.

Definition A.3.22. A metric space (S, d) is said to be **complete** if every Cauchy sequence in the space also converges to a limit in the space.

By the above intuitions about sequences being Cauchy, the intuition of a sequence being complete is clear. There are no sequence limits 'missing' in the space.

The notions of a sequence being Cauchy or a space being complete are explicitly a metric notions. They hinge on the notion of distance (the metric d) and diameter which can not be generalized to a general topological context. So these concepts are not definable for general topological spaces.

Remark. Above I defined notions of Cauchyness and completeness on sequences that are transfinite in the broadest sense (possibly larger than ω), I couldn't find such a definition anywhere in the literature. The standard notion is one of Cauchyness of sequences of length ω. To differentiate this notion with my true transfinite notion explicit I'll call this notion ω-Cauchyness and the associated completeness-notion ω-completeness. I do this because they are actually different notions and because my notion is in fact more general.

Definition A.3.23. A sequence s, of length ω, in a metric space is ω**-Cauchy** if for every $\epsilon \in \mathbb{R}^+$ there is an $n < \omega$ such that for all k, l with $n \leq k, l < \omega$ $d(s_k, s_l) < \epsilon$.

It can easily be seen that this is a special case of the previous Cauchy definition. All ω-Cauchy sequences are Cauchy, but the notion of ω-Cauchyness doesn't allow for sequence with a length $> \omega$. The notion of ω-Cauchyness gives rise to a definition of completeness that is technically different.

Definition A.3.24. A metric space (S, d) is said to be ω**-complete** if every ω-Cauchy sequence in the space converges to a limit in the space.

This definition turns out to be indeed only technically different.

Lemma A.3.25. *A metric space (S, d) is ω-complete if and only it is complete.*

Proof.

- $\Rightarrow$ Assume that a (S, d) is ω-complete, all ω-Cauchy sequences in the space converge to a limit. Let s of length $\alpha \in On$ be any Cauchy sequence in the space.

  Let λ be a function from ω to α that is recursively defined in the following way:

– $\lambda(0) = 0$
– $\lambda(n) = \beta \Rightarrow \lambda(n+1) = \gamma$ such that $\beta \leq \gamma$ and for all $\eta, \sigma < \alpha$ such that $\gamma \leq \eta$ and $\gamma \leq \sigma$ we have $d(s_\eta, s_\sigma) < 1/(n+1)$

The existence of such γs is guaranteed by the Cauchyness of s.

Now consider the sequence $\langle s_{\lambda(n)} \rangle_{n<\omega}$. This sequence is ω-Cauchy, by the way we defined λ. Because our space is ω-complete we have that this sequence converges to a limit, call it x.

Now we can prove that s also converges to x.

Let $\epsilon \in \mathbb{R}^+$ be arbitrary. Let $n < \omega$ be such that $1/n < \epsilon/2$ and $d(s_{\lambda(n)}, x) < \epsilon/2$, such an n must exist since x is the limit of $\langle s_{\lambda(n)} \rangle_{n<\omega}$ and $\langle 1/n \rangle_{n<\omega}$ converges to 0. Let $\gamma < \alpha$ be such that $\lambda(n) \leq \gamma$. We have $d(s_\gamma, s_{\lambda(n)}) < 1/n$ by the way we defined λ and $d(s_\gamma, x) \leq d(s_\gamma, s_{\lambda(n)}) + d(s_{\lambda(n)}, x)$ by TRIANGLE INEQUALITY. So we get $d(s_\gamma, x) < 1/n + d(s_{\lambda(n)}, x)$ and, by the way we defined n, $d(s_\gamma, x) < \epsilon$.

So for any $\epsilon \in \mathbb{R}^+$ there is a n such that for any $\gamma < \alpha$ for which it holds that $\lambda(n) \leq \gamma$ we have $d(s_\gamma, x) < \epsilon$. So s converges to x.

$\Leftarrow$ Assume that a (S, d) is complete, all Cauchy sequences in the space converge to a limit. Any ω-Cauchy sequence is a Cauchy sequence, so we also have that also every ω-Cauchy also converges to a limit and hence (S, d) is also ω-complete. □

Remark. Another pair of Cauchy and completeness notions that is used in the literature is "net-Cauchyness" and "net-completeness". Since every transfinite sequence is a net (but not every net is a sequence) we directly have that net-completeness implies completeness. Net-theory has a theorem that says that a metric space is net-complete if and only if it is ω-complete. So in a metric space we have that: net-completeness $\Leftrightarrow$ completeness $\Leftrightarrow$ ω-completeness. Since this thesis is not concerned with nets (filters are used as generalizations of sequences), net-completeness won't be discussed. It might be good to know however, that in the net community, net-completeness is known as completeness and ω-completeness is known as sequential completeness. For obvious reasons, these are not good naming conventions for the purposes of this thesis, hence the non-standard terminology.

A metric space that is not complete contains (ω)-Cauchy sequences that do not converge to a limit in the space. Intuitively, we would say that we could just add limits for every one of those sequences to make the space complete. We can formalize that idea.

Definition A.3.26. The **completion** of a metric space (S, d) is the structure (S^*, d^*) such that

- d^* is the distance measure on ω-Cauchy sequences in (S, d) such that $d^*(\langle x_n \rangle_{n<\omega}, \langle y_n \rangle_{n<\omega}) = \lim(\langle d(x_n, y_n) \rangle_{n<\omega})$.

- S^* is the set of equivalence classes of ω-Cauchy sequence in (S, d), where two ω-Cauchy sequences are equivalent if the distance between them is 0.

To show that such a completion is a metric space itself, we should first prove d^* is a total function by showing that the defined limit exist.

Lemma A.3.27. *If (S, d) is a metric space and $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega})$ are ω-Cauchy sequences in this space then we have that $\langle d(x_n, y_n) \rangle_{n<\omega}$ has a limit in $\mathbb{R}^{0+}$.*

Proof. We first prove that $\langle d(x_n, y_n) \rangle_{n<\omega}$ is an ω-Cauchy sequence in $\mathbb{R}^{0+}$.

All values of $\langle d(x_n, y_n) \rangle_{n<\omega}$ are in $\mathbb{R}^{0+}$. Let $\epsilon \in \mathbb{R}^+$, because $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ are ω-Cauchy there is a $n < \omega$ such that for all $k, k' > n$ we have $d(x_k, x_{k'}) < \epsilon/2$ and $d(y_k, y'_k) < \epsilon/2$. So we also have that the absolute difference (which is the standard metric on $\mathbb{R}$ and subsets of it) between $d(x_k, y_k)$ and $d(x_{k'}, y_{k'})$ is less then ϵ. This proves the ω-Cauchyness of $\langle d(x_n, y_n) \rangle_{n<\omega}$.

Because $\mathbb{R}^{0+}$ is complete this means that $\langle d(x_n, y_n) \rangle_{n<\omega}$ has a limit in $\mathbb{R}^{0+}$. □

Now we can show that d^* is an actual metric on S^*:

Lemma A.3.28. *If (S, d) is a metric space then d^* is a pseudometric on the set of ω-Cauchy sequences in S and hence a metric on S^*.*

Proof. Let (S, d) be a metric space and let (S^*, d^*) be its completion. We first show that d^* is a pseudometric on the set of ω-Cauchy-sequences in S. A pseudometric space is set with a binary relation on it for which REFLEXIVITY, SYMMETRY and TRIANGLE INEQUALITY hold. That is, it is (almost) a metric space where IDENTITY OF INDISCERNIBLES does not necessarily hold.

- REFLEXIVITY

  This follows from REFLEXIVITY of d.

- SYMMETRY

  This follows from SYMMETRY of d.

- TRIANGLE INEQUALITY

  Let $\langle x_n \rangle_{n<\omega}$, $\langle y_n \rangle_{n<\omega}$ and $\langle z_n \rangle_{n<\omega}$ be ω-Cauchy sequences in (S, d). By the Cauchyness of the sequences we get that for any $\epsilon \in R^+$ there is a $k < \omega$ such that:

  - $|\{x_n \mid k \leq n < \omega\}|_{d^*} < \epsilon$
  - $|\{y_n \mid k \leq n < \omega\}|_{d^*} < \epsilon$
  - $|\{z_n \mid k \leq n < \omega\}|_{d^*} < \epsilon$

  We get:

  - $|d^*(\langle x_n \rangle_{n<\omega}, \langle y_n \rangle_{n<\omega}) - d(x_k, y_k)| < 2\epsilon$

- $|d^*(\langle y_n \rangle_{n<\omega}, \langle z_n \rangle_{n<\omega}) - d(y_k, z_k)| < 2\epsilon$
- $|d^*(\langle x_n \rangle_{n<\omega}, \langle z_n \rangle_{n<\omega}) - d(x_k, z_k)| < 2\epsilon$

And hence:

$$\begin{aligned} d^*(\langle x_n \rangle_{n<\omega}, \langle y_n \rangle_{n<\omega}) + d^*(\langle y_n \rangle_{n<\omega}, \langle z_n \rangle_{n<\omega}) &< d(x_k, y_k) + d(y_k, z_k) + 4\epsilon \\ &\leq d(x_k, z_k) + 4\epsilon \\ &< d^*(\langle x_n \rangle_{n<\omega}, \langle z_n \rangle_{n<\omega}) + 6\epsilon \end{aligned}$$

In $(\mathbb{R}, d_{\mathbb{R}})$ we have $(\forall \epsilon.\ x \leq (y + \epsilon)) \rightarrow x \leq y$. So applying that:

$$d^*(\langle x_n \rangle_{n<\omega}, \langle y_n \rangle_{n<\omega}) + d^*(\langle y_n \rangle_{n<\omega}, \langle z_n \rangle_{n<\omega}) \leq d^*(\langle x_n \rangle_{n<\omega}, \langle z_n \rangle_{n<\omega})$$

So d^* is a pseudometric on the set of ω-Cauchy sequences in S, but since S^* is the set of (d^*-)equivalence classes of ω-Cauchy sequences in S, d^* is a true metric on S^*. □

Finally, we can prove that (S^*, d^*) is complete.

Lemma A.3.29. *If (S, d) is a metric space then its completion (S^*, d^*) is a complete metric space.*

Proof. Let (S, d) be a metric space, by Lemma A.3.28 its completion (S^*, d^*) is a metric. Remains to prove that it is complete.

Let f be a function, that, given a ω-Cauchy sequence, $\langle x_i \rangle_{i<\omega}$ and a $n \in \mathbb{N}$, gives the maximum of n and least m such that for all $k \geq m$ we have $d(x_k, x_m) < 1/n$ (such an m must exist since the sequence is ω-Cauchy).

Let $\langle\langle x_{i,j} \rangle_{j<\omega}\rangle_{i<\omega}$ be an arbitrary ω-Cauchy sequence of ω-Cauchy sequences in S, the limit of this sequence is $\langle x_{i,f(\langle x_{i,j} \rangle_{j<\omega}, i)} \rangle_{i<\omega}$ (lets denote it by l for readabilities' sake).

First we prove that l is a ω-Cauchy sequence. Let $\epsilon \in \mathbb{R}^+$ be arbitrary. Let n_1 be such that for all $i, i' > n_1$ we have $d^*(\langle x_{i,j} \rangle_{j<\omega}, \langle x_{i',j} \rangle_{j<\omega}) < \epsilon/3$, such an n_1 must exist because $\langle\langle x_{i,j} \rangle_{j<\omega}\rangle_{i<\omega}$ is ω-Cauchy. Let n_2 be such that $1/n_2 < \epsilon/3$. Let $n = \max(n_1, n_2)$. Let $a, b \geq n$. Let $c \geq f(\langle x_{a,j} \rangle_{j<\omega}, a)$ be such that $d(x_{a,c}, x_{b,c}) < \epsilon/3$. Such a c must exist since $a, b \geq n \geq n_1$ which means $d^*(\langle x_{a,j} \rangle_{j<\omega}, \langle x_{b,j} \rangle_{j<\omega}) < 1/(3 * \epsilon)$, so, unfolding the definition of d^*, we have that $\langle d(x_{a,j}, x_{b,j}) \rangle_{j<\omega}$ tends to some limit below $\epsilon/3$ and hence eventually (from index c) stays below $\epsilon/3$.

We get:

- $d(l_a, x_{a,c}) < \epsilon/3$; We have $d(x_{a,f(\langle x_{a,j} \rangle_{j<\omega}, a)}, x_{a,c}) < 1/a$ by the definition of f. We also have $a \geq n \geq n_2$ which means $1/a \leq 1/n_2 < \epsilon/3$, so we get $d(x_{a,f(\langle x_{a,j} \rangle_{j<\omega}, a)}, x_{a,c}) < \epsilon/3$. We have $l_a = x_{a,f(\langle x_{a,j} \rangle_{j<\omega}, a)}$ by the definition of l, so $d(l_a, x_{a,c}) < \epsilon/3$ follows.
- $d(x_{a,c}, x_{b,c}) < \epsilon/3$; This follows from the way we defined c.

- $d(x_{b,c}, l_b) < \epsilon/3$; We have $d(x_{b,c}, x_{b,f(\langle x_{b,j}\rangle_{j<\omega},b)}) < 1/b$ by the definition of f. We also have $b \geq n \geq n_2$ which means $1/b \leq 1/n_2 < \epsilon/3$, so we get $d(x_{b,f(\langle x_{b,j}\rangle_{j<\omega},b)}, x_{b,c}) < \epsilon/3$. We have $l_b = x_{b,f(\langle x_{b,j}\rangle_{j<\omega},b)}$ by the definition of l, so $d(l_b, x_{b,c}) < \epsilon/3$ follows.

Now we can apply that d is a metric and by TRIANGLE INEQUALITY we get that $d(l_a, l_b) < \epsilon$. We chose $a, b \geq n$ arbitrarily, so for all $i, i' \geq n$ we have $d(l_i, l_{i'}) < \epsilon$. We chose ϵ arbitrarily so for all $\epsilon \in \mathbb{R}^+$ we have that there is an $n < \omega$ such that for all $i, i' < n$ we have $d(l_i, l_{i'}) < \epsilon$. This proves that l is ω-Cauchy.

Now we prove that l is the limit of $\langle\langle x_{i,j}\rangle_{j<\omega}\rangle_{i<\omega}$. Let $\epsilon \in \mathbb{R}^+$ be arbitrary. Let n be such that for all $i, i' > n$ we have $d^*(\langle x_{i,j}\rangle_{j<\omega}, \langle x_{i',j}\rangle_{j<\omega}) < \epsilon/3$, such an n must exist because our sequence is ω-Cauchy. Let $a \geq n$ be arbitrary. Let m_1 be such that for all $j, j' \geq m_1$ we have $d(x_{a,j}, x_{a,j'}) < \epsilon/3$, since $\langle x_{a,j}\rangle_{j<\omega}$ is ω-Cauchy, such an m_1 exists. Let m_2 be such that $1/m_2 < \epsilon/3$). Let $m = \max(m_1, m_2, n)$. Let $b \geq m$ be arbitrary. Let $c \geq f(\langle x_{b,j}\rangle_{j<\omega}, b)$ be such that $d(x_{a,c}, xb, c) < \epsilon/3$. Such a c must exist since $a, b \geq n$ which means $d^*(\langle x_{a,j}\rangle_{j<\omega}, \langle x_{b,j}\rangle_{j<\omega}) < \epsilon/3)$, so, unfolding the definition of d^*, we have that $\langle d(x_{a,j}, x_{b,j})\rangle_{j<\omega}$ tends to some limit below $\epsilon/3$ and hence eventually (from index c) stays below $\epsilon/3$.

We have:

- $d(x_{a,b}, x_{a,c}) < \epsilon/3$; This follows because we have $c \geq f(\langle x_{b,j}\rangle_{j<\omega}, b) \geq b \geq m \geq m_1$ and for all $j, j' \geq m_1$ it holds that $d(x_{a,j}, x_{a,j'}) < \epsilon/3$.
- $d(x_{a,c}, x_{b,c}) < \epsilon/3$; This follows from the way we defined c.
- $d(x_{b,c}, x_{b,f(\langle x_{b,j}\rangle_{j<\omega},b)}) < \epsilon/3$; This holds because $d(x_{b,c}, x_{b,f(\langle x_{b,j}\rangle_{j<\omega},b)}) < 1/b$ by the definition f and we have $b \geq m \geq m_2$ which means $1/b \leq 1/m_2 < \epsilon/3$.

Now we can apply that d is a metric and by TRIANGLE INEQUALITY we get that $d(x_{a,b}, x_{b,c}) < \epsilon$. By the definitions of c and l (which both hinge on f) we have that $x_{b,c} = l_b$, so we have $d(x_{a,b}, l_b) < \epsilon$. We chose $b \geq m$ arbitrarily, so we have that for all $j \geq m$ it holds that $d(x_{a,j}, l_j) < \epsilon$. Because l is ω-Cauchy, this means that $\langle d(x_{a,j}, l_j)\rangle_{j<\omega}$ tends to some limit below ϵ, so $d^*(\langle x_{a,j}\rangle_{j<\omega}, l) < \epsilon$. We chose $a \geq n$ arbitrarily, so we have that for all $i \geq n$ it holds that $d^*(\langle x_{i,j}\rangle_{j<\omega}, l) < \epsilon$. We chose ϵ arbitrarily, so for all $\epsilon \in \mathbb{R}^+$ we have that there is an $n < \omega$ such that for all $i \geq n$ it holds that $d^*(\langle x_{i,j}\rangle_{j<\omega}, l) < \epsilon$. This means that l is a limit of $\langle\langle x_{i,j}\rangle_{j<\omega}\rangle_{i<\omega}$.

We chose $\langle\langle x_{i,j}\rangle_{j<\omega}\rangle_{i<\omega}$ arbitrarily so every ω-Cauchy sequence in (S^*, d^*) has a limit in (S^*, d^*) and hence (S^*, d^*) is complete. □

We have a good embedding of a space in its completion.

Proposition A.3.30. If (S, d) is a metric space and (S^*, d^*) is its completion, (S, d) can be embedded in (S^*, d^*) by the function, $f : S \to S^*$, that maps $x \in S$ to the (d^*-)equivalence class of ω-Cauchy sequences in (S, d) that have x as limit.

It is easy to see that this function is an isomorphism with respect to the metrics d and d^* (and hence called an **isometry**), so the embedding has the necessary properties. We might speak about members of S^* modulo this embedding and hence talk about $x \in S^*$ instead of $f(x) \in S^*$.

A.4 Order

A.4.1 Non-strictly Ordered Sets

Definition A.4.1. A **quasi-ordered set** is a structure $(A, \leq)$ such that A is a set and $\leq \subseteq A \times A$ is a binary relation on A such that:

- $\forall x \in A(x \leq x)$ REFLEXIVITY
- $\forall x, y, z \in A((x \leq y \wedge y \leq z) \rightarrow x \leq z)$ TRANSITIVITY

Here $\leq$ is called a **quasi-order** on A. If $b \leq a$ we say the b is smaller or equal to a or simply that b is below a.

Remark. Quasi-ordered sets are sometimes also called pre-ordered sets, however, according to [15] the notion of a pre-ordered set is a strictly weaker one.

For quasi-ordered sets we have:

Definition A.4.2. If $(A, \leq)$ is a quasi-ordered set and $X \subseteq A$ then:

- $a \in A$ is an **upper bound** of X if $\forall x \in X(x \leq a)$
- $a \in A$ is a **lower bound** of X if $\forall x \in X(a \leq x)$
- X is an **upper set** if $\forall x \in X(\forall a \in A(x \leq a \rightarrow a \in X))$
- X is an **lower set** if $\forall x \in X(\forall a \in A(a \leq x \rightarrow a \in X))$

Definition A.4.3. A **directed set** is a quasi-ordered set such that:

- Every pair of elements has an upper bound. DIRECTEDNESS

The order is called a **directed order**. A subset of any ordered set for which this property holds is called a **directed subset**.

Definition A.4.4. A **partially ordered set** or **poset** is an quasi-ordered set $(A, \leq)$ such that:

- $\forall x, y \in A(x \leq y \wedge y \leq x \rightarrow x = y)$ ANTISYMMETRY

Here $\leq$ is said to be a **partial order** on A.

For partially ordered sets we have:

Definition A.4.5. If $(A, \leq)$ is a partially ordered set and $X \subseteq A$ then:

- $a \in X$ is the **greatest element** of X if $\forall x \in X(x \leq a)$
- $a \in X$ is the **least element** of X if $\forall x \in X(a \leq x)$
- $a \in A$ is the **least upper bound** or **supremum** or **lub** of X if $\forall x \in X(x \leq a) \wedge \forall a' \in A((\forall x \in X(x \leq a')) \rightarrow (a \leq a'))$ (a is the least element of the set of upper bounds of X in A)
- $a \in A$ is the **greatest lower bound** or **infimum** or **glb** of X if $\forall x \in X(a \leq x) \wedge \forall a' \in A((\forall x \in X(a' \leq x)) \rightarrow (a' \leq a))$ (a is the greatest element of the set of lower bounds of X in A)

By the antisymmetry of our partial order, least and greatest elements, if they exist, are unique.

A given subset of a partially ordered set might not have a least upper bound or greatest lower bound, but if it does, this least upper bound or greatest lower bound is, again, unique and can be denoted as respectively $\bigsqcup X$ or $\bigsqcap X$. We can characterize the least upper bound $a \in A$ of $X \subseteq A$ by $\forall a' \in A((\forall x \in X(x \leq a')) \leftrightarrow (a \leq a'))$ and the greatest lower bound $a \in A$ of $X \subseteq A$ by $\forall a' \in A((\forall x \in X(a' \leq x)) \leftrightarrow (a' \leq a))$.

Definition A.4.6. A **directed complete partial order (dcpo)** is a partially ordered set such that:

- Every subset of the set that is directed has least upper bound in the set

Definition A.4.7. A **complete partial order (cpo)** is a dcpo such that:

- It has a least element

Definition A.4.8. A partially ordered set is **bounded complete** if:

- Every subset that has an upper bound in the set has a least upper bound in the set

Lemma A.4.9. *Bounded completeness of a partially ordered set implies having a least element.*

Proof. Let $(A, \leq)$ be a bounded complete partially ordered set. Any element of A is an upper bound of $\emptyset$, so by bounded completeness, $\emptyset$ has a least upper bound. This least upper bound is the least of all upper bounds of $\emptyset$, which are all the elements of A, so this least upper bound is the least element of A. □

So the notion of a bounded complete dcpo is actually equal to that of a bounded complete cpo.

Lemma A.4.10. *Every non-empty subset in a bounded complete dcpo set has a greatest lower bound.*

Lemma A.4.11. *Let* $(A, \leq)$ *be a bounded complete dcpo and let* $X \subseteq A$ *such that* X *is not empty. Denote the set of lower bounds of* X *by* B*. For any* $b \in B$ *and* $x \in X$ *we get* $b \leq x$*, so for any* $x \in X$ *we have that* x *is an upper bound of* B*. Since* X *is non-empty we also get that there is an* $x \in X$ *such that* x *is an upper bound of* B*. Now by bounded completeness, we also get a least upper bound of* B*,* $\bigsqcup B$*. For any* $x \in X$ *we have that it is an upper bound of* B *and by the* $\bigsqcup B$ *being the least upper bound we get* $\bigsqcup B \leq x$*, so* $\bigsqcup B$ *is a lower bound of* X*. Since* B *is the set of lower bounds of* X *and for all* $b \in B$ *we have* $b \leq \bigsqcup B$ *we get that* $\bigsqcup B$ *is the greatest lower bound of* X*.*

Definition A.4.12. A **join-semilattice** is a partially ordered set for which it holds that:

- Every pair of elements in has a least upper bound

Definition A.4.13. A **meet-semilattice** is a partially ordered set for which it holds that:

- Every pair of elements in has a greatest lower bound

By Lemma A.4.10 every bounded complete dcpo set is meet-semilattice. This is why a bounded complete dcpo is sometimes also called a **complete semilattice** or **complete meet-semilattice**. We wont use those terms since they are in some contexts also use for other concepts.

Definition A.4.14. A **lattice** is a partially ordered set for which it holds that:

- Every pair of elements has a least upper bound
- Every pair of elements has a greatest lower bound

So an ordered set is a lattice if and only if it is a meet-semilattice and a join-semilattice.

Definition A.4.15. A **complete lattice** is a partially ordered set for which it holds that:

- Every subset of the set has a least upper bound in the set
- Every subset of the set has a greatest lower bound in the set

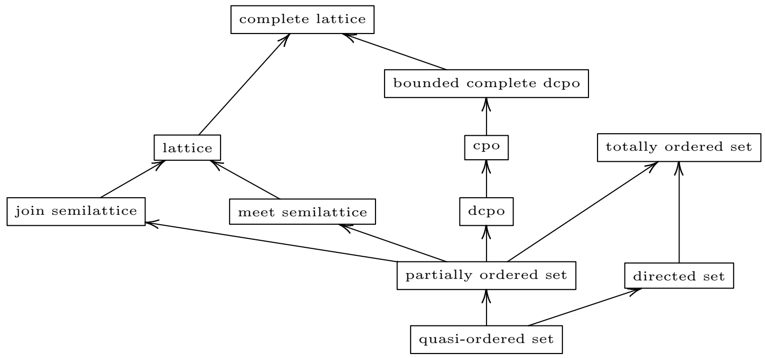
So every complete lattice is a lattice, a dcpo, cpo and a bounded complete dcpo (but not vice versa).

Definition A.4.16. A **totally ordered set** is a partially ordered set $(A, \leq)$ such that:

- $\forall x, y \in A(x \leq y \vee y \leq x)$ TOTALITY

Here $\leq$ is called a **total order** on A.

Every totally ordered set is directed. Subsets of any partial ordered set might be totally ordered, we refer to these subsets as **chains**. Every chain in a partially ordered set is a directed subset of that set.



A.4.1.1 Approximation

On partially ordered sets we can define the following relation:

Definition A.4.17. If $(A, \leq)$ is a partially ordered set and $a, b \in A$ then b is **way below** a, denoted by $b \ll a$, if for every directed subset D of A such that $\bigsqcup D$ exists and $a \leq \bigsqcup D$ there is a $d \in D$ such that $b \leq d$.

And using that:

Definition A.4.18. If $(A, \leq)$ is a partially ordered set then $c \in A$ is said to be **compact** or **finite** if $c \ll c$.

Definition A.4.19. A partially ordered set, $(A, \leq)$, is **continuous** if for all $a \in A$ we have:

- $\{x \mid x \ll a\}$ is directed
- $\bigsqcup\{x \mid x \ll a\} = a$

Definition A.4.20. A partially ordered set, $(A, \leq)$, is **algebraic** if for all $a \in A$ we have:

- $\{x \mid x \leq a \land x \ll x\}$ is directed
- $\bigsqcup\{x \mid x \leq a \land x \ll x\} = a$

Here, $\{x \mid x \leq a \land x \ll x\}$ is the set of all compact elements below a.

We have that:

Lemma A.4.21. *In an algebraic poset we have that every compact element below another element is also way below that element.*

Proof. Let $(A, \leq)$ be an algebraic poset and let $a \in A$ and let $c \leq a$ be compact. Let $D \subseteq A$ be a directed set such that $a \leq \bigsqcup D$, we get $c \leq \bigsqcup D$ and by compactness of c a $d \in D$ such that $c \leq d$. That proves $c \ll a$. □

Lemma A.4.22. *In an algebraic poset we have that $b \ll a \Leftrightarrow b \leq c \leq a$ for some compact element c.*

Proof. Let $(A, \leq)$ be an algebraic poset and let $a, b \in A$.

$\Rightarrow$ Assume that $b \ll a$. By algebraicity of $(A, \leq)$ we get that $a \leq \bigsqcup\{x \mid x \leq a \wedge x \ll x\}$ so because $b \ll a$ there is a $c \in \{x \mid x \leq a \wedge x \ll x\}$ such that $b \leq c$.

$\Leftarrow$ Assume that $b \leq c \leq a$ for some compact element c. Let $D \subseteq A$ be a directed set such that $a \leq \bigsqcup D$, we get $c \leq \bigsqcup D$, by compactness of c we get a $d \in D$ such that $c \leq d$ and hence $b \leq d$, which proves $b \ll a$. □

So we get:

Lemma A.4.23. *Every algebraic poset is continuous.*

Proof. Let $(A, \leq)$ be an algebraic poset and let $a \in A$.

- $\{x \mid x \ll a\}$ is directed

  Let $b_1, b_2 \in \{x \mid x \ll a\}$. By Lemma A.4.22 we get compact elements c_1 and c_2 such that $b_1 \leq c_1 \leq a$ and $b_2 \leq c_2 \leq a$, so $c_1, c_2 \in \{x \mid x \leq a \wedge x \ll x\}$. By algebraicity of $(A, \leq)$ we get that $\{x \mid x \leq a \wedge x \ll x\}$ is directed and hence get a $c \in \{x \mid x \leq a \wedge x \ll x\}$ such that it is an upper bound of c_1 and c_2. By Lemma A.4.21 we get that $c \ll a$ and hence $c \in \{x \mid x \ll a\}$.

- $\bigsqcup\{x \mid x \ll a\} = a$

  For all $x \ll a$ we get some compact c such that $x \leq c \leq a$ by Lemma A.4.22. So every member of $\{x \mid x \ll a\}$ is below a member of $\{x \mid x \leq a \wedge x \ll x\}$. $\bigsqcup\{x \mid x \leq a \wedge x \ll x\} = a$ (because $(A, \leq)$ is algebraic), so a is an upper bound for $\{x \mid x \ll a\}$.

  By Lemma A.4.21 we get that $\{x \mid x \leq a \wedge x \ll x\} \subseteq \{x \mid x \ll a\}$. That means that any upper bound of $\{x \mid x \ll a\}$ is also an upper bound of $\{x \mid x \leq a \wedge x \ll x\}$. Now since a is the least upper bound of $\{x \mid x \leq a \wedge x \ll x\}$ it is also the least upper bound of $\{x \mid x \leq a \wedge x \ll x\}$. □

A.4.1.2 Sequences

Definition A.4.24. If $(A, \leq)$ is a partially ordered set and $\langle a_\beta \rangle_{\beta < \alpha}$ is a sequence in this set then:

- The **limit inferior** of this sequence is $\bigsqcup\{\bigsqcap\{a_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$ and is denoted by $\liminf(\langle a_\beta \rangle_{\beta < \alpha})$

- The **limit superior** of this sequence is $\bigsqcap\{\bigsqcup\{a_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$ and is denoted by $\liminf(\langle a_\beta\rangle_{\beta<\alpha})$.

When the greatest lower bounds or least upper bounds involved in these definitions do not exist, the limit inferior/limit superior is said not to exist.

In a complete lattice every sequence has a limit inferior and a limit superior since any subset of the complete lattice has a least upper bound and greatest lower bound.

Lemma A.4.25. *In a bounded complete dcpo every nonempty-sequence has a limit inferior.*

Proof. Let $(A, \leq)$ be a bounded complete dcpo and let $\langle a_\beta\rangle_{\beta<\alpha}$ be a sequence such that $\alpha > 0$.

For any $\beta < \alpha$ we have that $\{a_\gamma \mid \beta \leq \gamma < \alpha\}$ is not empty because $\alpha > 0$ and hence has a greatest lower bound by Lemma A.4.10.

Furthermore, for any $\beta_1, \beta_2 < \alpha$ we have that $\{a_\gamma \mid \beta_1 \leq \gamma < \alpha\}$ and $\{a_\gamma \mid \beta_2 \leq \gamma < \alpha\}$ both have $a_{\max(\beta_1,\beta_2)}$ as a member. This means that $\bigsqcap\{s_\gamma \mid \beta_1 \leq \gamma < \alpha\} \leq a_{\max(\beta_1,\beta_2)}$ and $\bigsqcap\{a_\gamma \mid \beta_2 \leq \gamma < \alpha\} \leq a_{\max(\beta_1,\beta_2)}$, so $a_{\max(\beta_1,\beta_2)}$ is an upper bound $\bigsqcap\{a_\gamma \mid \beta_1 \leq \gamma < \alpha\}$ and $\bigsqcap\{a_\gamma \mid \beta_2 \leq \gamma < \alpha\}$. This means that $\{\bigsqcap\{a_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$ is directed and because $(A, \leq)$ is a dcpo that means that $\{\bigsqcap\{a_\gamma \mid \beta \leq \gamma < \alpha\} \mid \beta < \alpha\}$ has a least upper bound, the limit inferior of $\langle a_\beta\rangle_{\beta<\alpha}$. □

For sequences with successor ordinal length things are simple.

Lemma A.4.26. *If the limit inferior of a sequence with successor ordinal length exists then is its last member.*

Proof. Let $(A, \leq)$ be a partially ordered set and let $\langle a_\beta\rangle_{\beta<\alpha}$ be a sequence such that $\alpha = \eta\prime$ and $\liminf(\langle a_\beta\rangle_{\beta<\alpha})$ exists.

We have $\{a_\gamma \mid \eta \leq \gamma < \alpha\} = \{a_\eta\}$, so we get $a_\eta = \bigsqcap_{\eta\leq\gamma<\alpha} a_\gamma$, and hence $a_\eta \leq \bigsqcup_{\beta<\alpha} \bigsqcap_{\beta\leq\gamma<\alpha} a_\gamma$.

For every $\beta < \alpha$ we have that $\beta \leq \eta$, and hence $a_\eta \in \{a_\gamma \mid \beta \leq \gamma < \alpha\}$. So for every $\beta < \alpha$ we get $a_\eta \geq \bigsqcap_{\beta\leq\gamma<\alpha} a_\gamma$, and we must have $a_\eta \geq \bigsqcup_{\beta<\alpha} \bigsqcap_{\beta\leq\gamma<\alpha} a_\gamma$.

So we can conclude $a_\eta = \bigsqcup_{\beta<\alpha} \bigsqcap_{\beta\leq\gamma<\alpha} a_\gamma$. □

Definition A.4.27. If $(A, \leq)$ is a partially ordered set and $\langle a_\beta\rangle_{\beta<\alpha}$ a sequence then every $a \in A$ such that $a \leq \liminf(\langle a_\beta\rangle_{\beta<\alpha})$ is called an **$\mathscr{S}$-limit** of $\langle a_\beta\rangle_{\beta<\alpha}$.

A.4.2 Strictly Ordered Sets

All the orders in the previous section are reflexive and are therefor called **non-strict**. Some of their concepts have irreflexive counterparts, these are called **strict** orders and are more useful in some contexts. Quasi-orders, dcpos, lattices, bounds, least and greatest elements can not be axiomatized as nicely in a strict way. We follow the literature by defining them non-strictly.

Where non-strict orders are denoted by $\leq$, strict orders are denoted by $<$. The most basic order-theoretic structure that can be nicely axiomatized in a strict way is simply called a strictly ordered set.

Definition A.4.28. A **strictly ordered set** is a structure $(A, <)$ such that A is a set and $< \subseteq A \times A$ is a binary relation on A such that:

- $\forall x \in A(\neg(x < x))$ IRREFLEXIVITY
- $\forall x, y, z \in A((x < y \wedge y < z) \rightarrow x < z)$ TRANSITIVITY

Here $<$ is called a **strict order** on A.

Remark. This definition implies:

- $\forall x, y \in A(x < y \rightarrow \neg(y < x))$ ASYMMETRY

A back-and-forth association between strictly ordered sets and partially (non-strict) ordered sets is easily defined.

Definition A.4.29. If $(A, \leq)$ is a partially ordered set, then the **associated strictly ordered set** is $(A, <)$ where $\forall x, y \in A(x < y \leftrightarrow x \leq y \wedge x \neq y)$. If $(A, <)$ is a strictly ordered set then the **associated partial ordered set** is $(A, \leq)$ where $\forall x, y \in A(x \leq y \leftrightarrow x < y \vee x = y)$.

Via this association we can use notions defined on partially ordered sets on strictly ordered set and vice versa.

Definition A.4.30. If $(A, <)$ is a strictly ordered set and $X \subseteq A$ then:

- $m \in X$ is a **maximal element** of X if $\neg\exists x \in X(m < x)$
- $m \in X$ is a **minimal element** of X if $\neg\exists x \in X(x < m)$

Any greatest element of a partially ordered set is maximal in the associated strictly ordered set, but necessarily not vice versa. Any least element of a partially ordered set is minimal in the associated strictly ordered set, but necessarily not vice versa. The maximal elements in a partially ordered set $(A, \leq)$ are precisely the elements $m \in A$ such that for all $x \in A$ we have $m \leq x \rightarrow m = x$. The minimal elements in a partially ordered set $(A, \leq)$ are precisely the elements $m \in A$ such that for all $x \in A$ we have $x \leq m \rightarrow m = x$.

Definition A.4.31. A **linearly ordered set** is a strictly ordered set $(A, <)$ such that:

- $\forall x, y \in A(x < y \vee y < x \vee y = x)$ LINEARITY

Here $<$ is called a **linear order** on A.

The non-strict order associated with any linear order is a total order. The strict order associated with any total order is a linear order.

In a linearly or totally ordered set any maximal element is also greatest and any minimal element is least.

Definition A.4.32. A **well-ordered set** is a linearly ordered set $(A, <)$ such that:

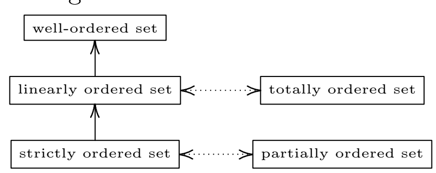
- $\forall X \subseteq A(X \neq \emptyset \to \exists x \in X \neg \exists y \in X(y < x)))$ WELL-FOUNDEDNESS

Here $<$ is called a **well-order** on A. A set with an order or relation on it for which well-foundedness holds, but which is not necessarily well-ordered is said to be a **well-founded set**.

Well-foundedness states that every non-empty subset of our set has a minimal element. Because well-ordered sets are also linear, in a well-ordered set this minimal element is also a least element ($\forall y \in X(x = y \vee x < y)$ or $\forall y \in X(x \neq y \leftrightarrow x < y)$). Well-foundedness implies irreflexivity.

An ordered set being well-founded implies that there are no infinite (strictly) descending chains in the set. If we assume the axiom of choice, the converse also holds (for an ordered set, not containing any infinite descending chains implies that it is well-founded).

So for strict orders we get:



A.4.2.1 Induction

Every well-founded set has an **induction scheme** associated with it. This scheme states that it is true that, if some property holding for all elements smaller than a certain element implies that the property holds for that element then the property holds for all elements of the well-ordered set.

Lemma A.4.33. *For any well-founded set,* $(A, <)$*, and any property,* P*, we have* $(\forall x \in A(\forall y < xP(y)) \to P(x)) \Rightarrow (\forall x \in AP(x))$.

Proof. Let $(A, <)$ be a well-founded set, P a property on the elements of A and assume that $\forall x \in A(\forall y < xP(y)) \to P(x)$.

Let $F = \{x \in A \mid \neg P(x)\}$. Suppose, for contradiction, that we do not have $\forall x \in A(P(x))$, then F is non-empty and hence has a $<$-minimal element (because $<$ is well-founded), call one such minimal element ξ, we have $\neg P(\xi)$. For all $y \in A$ with $y < \xi$ we have that $P(y)$ because ξ is a minimal element of the set of elements for which P does not hold. But then by our assumption we do get $P(\xi)$. Contradiction. So we must have $\forall x \in A(P(x))$ after all. □

We can use such an induction scheme as a proof-technique to prove statements of the form $\forall x \in A(P(x))$ where A is any well-founded set. For an arbitrary x, we assume $\forall y < xP(y)$, prove $P(x)$ from it and by application of the induction scheme, we have proven $\forall x \in AP(x)$. Here we can say we have proven by **induction on** x or by **induction over** A. We call $\forall x(\forall y < xP(y))$ our **induction hypothesis**.

A.5 Category Theory

The main construct in category theory is that of a category:

Definition A.5.1. A **category C** is a structure consisting of **objects** for which we write $\text{ob}_{\mathbf{C}}$ and **morphisms** for which we write $\text{mor}_{\mathbf{C}}$.

- Every morphism $m \in \text{mor}_{\mathbf{C}}$ has a **source** and a **target** which are objects of the category, we write, respectively, $\text{src}(m) \in \text{ob}_{\mathbf{C}}$ and $\text{tgt}(m) \in \text{ob}_{\mathbf{C}}$. If $\text{src}(()m) = X$ and $\text{tgt}(()m) = Y$, we write $m : X \to Y$ for m. If $X, Y \in \text{ob}_{\mathbf{C}}$, we write $\text{mor}_{\mathbf{C}}(X, Y)$ for the class $\{m \mid m : X \to Y \in \text{mor}_{\mathbf{C}}\}$.

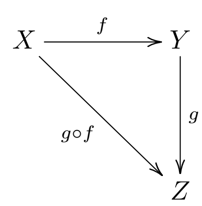
- For all $X, Y, Z \in \text{ob}_{\mathbf{C}}$ we have a binary operation, $\circ : \text{mor}_{\mathbf{C}}(X, Y) \times \text{mor}_{\mathbf{C}}(Y, Z) \to \text{mor}_{\mathbf{C}}(X, Z)$, called **composition**. It is associative, that is: for any $f, g, h \in \text{mor}_{\mathbf{C}}$ we have $f \circ (g \circ h) = (f \circ g) \circ h$.

- For any $X \in \text{ob}_{\mathbf{C}}$ there exists a $\text{id}_X \in \text{mor}_{\mathbf{C}}(X, X)$ such that for any other morphism $f \in \text{mor}_{\mathbf{C}}(A, B)$ we have $\text{id}_B \circ f = f = f \circ \text{id}_A$. id_X is called the **identity morphism** on X. By definition we have that id_X is unique.

We can turn an arbitrary system of mathematical structures into a category by letting the structures be objects in the category, abstracting away from their internal structure, and letting the mappings between the structures that in some sense preserve this internal structure be morphisms in the category. The morphisms then express the interrelations of the mathematical-structures-turned-into-objects, that their internal structure imposed on them, in some abstract sense. An example of a category is **Set**:

Definition A.5.2. **Set** is the category where the objects are sets and the morphisms are total functions between sets.

In category theory it is common practice to express properties like composition with **commutative diagrams**. The following diagram expresses composition of morphisms:



There are several types of morphisms:

Definition A.5.3. If **C** is a category an $X, Y, Z \in \text{ob}_{\mathbf{C}}$, $f \in \text{mor}_{\mathbf{C}}(X, Y)$ then:

- f is an **epimorphism** if for any $g_1, g_2 \in \text{mor}_{\mathbf{C}}(Y, Z)$ it holds that:

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

- f is an **monomorphism** if for any $g_1, g_2 \in \text{mor}_{\mathbf{C}}(Z, X)$ it holds that:

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

If $\mathbf{C}$ is a category, $X, Y \in \text{ob}_{\mathbf{C}}$, $f \in \text{mor}_{\mathbf{C}}(X, Y)$ and $g \in \text{mor}_{\mathbf{C}}(Y, X)$ such that $f \circ g = \text{id}_Y$ then g is a **section** of f, and f is a **retraction** of g. If it also holds that $g \circ f = \text{id}_X$ then f and g are **isomorphisms**, X and Y are said to be **isomorphic** and g can be written as f^{-1}, while f can be written as g^{-1}.

In a concrete category (defined below) a morphism of which the underlying function is injective is a monomorphism and a function of which the underlying function is surjective is an epimorphism. The notions of morphism and isomorphism are native to category, these definitions should generalize the more informal definitions in Section A.1.1.

Since categories themselves are mathematical structures we can abstract over them and express how they relate by morphisms. Such morphisms between categories are called functors.

Definition A.5.4. If $\mathbf{C}$ and $\mathbf{D}$ are categories then $U = \langle U^{ob}, U^{mor} \rangle$ is a **functor** from $\mathbf{C}$ to $\mathbf{D}$, written as $U : \mathbf{C} \to \mathbf{D}$, if:

- $U^{ob} : \text{ob}_{\mathbf{C}} \to \text{ob}_{\mathbf{D}}$ and $U^{mor} : \text{mor}_{\mathbf{C}} \to \text{mor}_{\mathbf{D}}$ are such that for any $f \in \text{mor}_{\mathbf{C}}(X, Y)$ we have $U^{mor}(f) \in \text{mor}_{\mathbf{D}}(U^{ob}(X), U^{ob}(Y))$. For U^{ob} and U^{mor} we may write just U (totum pro parte).

- It preserves identity, that is: for all identity morphisms $\text{id}_X \in \text{mor}_{\mathbf{C}}(X, X)$ we have that $U^{mor}(\text{id}_X) = \text{id}_{U^{ob}(X)}$ where $\text{id}_{U^{ob}(X)}$ is the identity morphism on $U^{ob}(X) \in \text{ob}_{\mathbf{D}}$.

- It preserves composition, that is: For all $X, Y, Z \in \text{ob}_{\mathbf{C}}$, $f \in \text{mor}_{\mathbf{C}}(X, Y)$ and $g \in \text{mor}_{\mathbf{C}}(Y, Z)$ we have $U^{mor}(g) \circ U^{mor}(f) = U^{mor}(g \circ f)$.

These requirements ensure that, when applying a functor, the category structure is preserved to a certain extent.

A trivial example of a functor is the (unique) **identity functor**, $\text{ID} : C \to C$, on some category C, mapping every object and every morphism to itself.

Functors between categories can have certain properties that express how the categories relate.

Definition A.5.5. If $\mathbf{C}$ and $\mathbf{D}$ are categories and $F : \mathbf{C} \to \mathbf{D}$ a functor between them, then for any $X, Y \in \text{ob}_{\mathbf{C}}$, F, induces a function $F_{X,Y} : \text{mor}_{\mathbf{C}}(X, Y) \to \text{mor}_{\mathbf{D}}(F(X), F(Y))$. Now:

- F is said to be **faithful** if for all $X, Y \in \text{ob}_{\mathbf{C}}$ we have that $F_{X,Y}$ is injective.

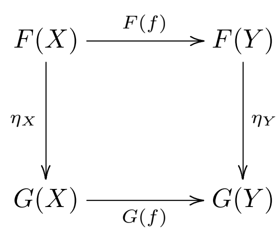
- F is said to be **full** if for all $X, Y \in \text{ob}_{\mathbf{C}}$ we have that $F_{X,Y}$ is surjective.

Definition A.5.6. A category is said to be **concrete** if there is a faithful functor mapping it into **Set**

Again our functors from some category **C** to some other category **D** are mathematical structures and we can once again do the same abstraction and view them as objects of some category. If we do, the morphisms of that category are called natural transformations.

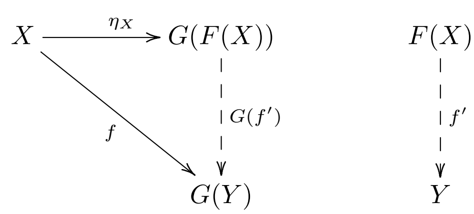
Definition A.5.7. If $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ are functors then a **natural transformation** $\eta : F \to G$ is such that to every object $X \in \text{ob}_{\mathbf{C}}$ it associates a morphism $\eta_X \in \text{mor}_{\mathbf{D}}(F(X), G(X))$ such that for any $f \in \text{mor}_{\mathbf{C}}(X, Y)$ we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. Here η_X is called the **component** of η at X.

We get the following commutative diagram:



Definition A.5.8. Two categories **C** and **D**, two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and a natural transformation $\eta : \text{ID}_{\mathbf{C}} \to G \circ F$ form an **adjunction** if for every $X \in \text{ob}_{\mathbf{C}}$, $Y \in \text{ob}_{\mathbf{D}}$ and $f \in \text{mor}_{\mathbf{C}}(X, G(Y))$ there is a unique $f' \in \text{mor}_{\mathbf{D}}(F(X), Y)$ such that $G(f') \circ \eta_X = f$. F is said to be **left adjoint** to G and G is said to be **right adjoint** to F.

We get the following commutative diagram:



Appendix B

Assorted material

B.1 Abstract Rewriting: Category Theory

As mentioned, the comparison of rewrite relations to rewrite systems, and, with that, the comparison of reduction sequence to reductions, as done in Section 3.3 can also be done using category theoretic tools. Perhaps the comparison can even be said to be better off being done using category theoretic tools. The following is my attempt to do so.

B.1.1 Rwr

Definition B.1.1. If $\mathcal{A} = (A, \to_A)$ and $\mathcal{B} = (B, \to_B)$ are rewrite relations then $f^{ob} : A \to B$ **preserves the rewrite structure** of $\mathcal{A}$ if for all $x, y \in A$ we have $x \to_A y \Rightarrow f^{ob}(x) \to_B f^{ob}(y)$. A **rewrite relation morphism** is a structure $\langle f^{ob} \rangle : \mathcal{A} \to \mathcal{B}$, such that $f^{ob} : A \to B$ preserves the rewrite structure of $\mathcal{A}$. The **composition operator** on rewrite relation morphisms, $\circ$, is such that if $\langle f^{ob} \rangle : \mathcal{A} \to \mathcal{B}$ and $\langle g^{ob} \rangle : \mathcal{B} \to \mathcal{C}$ are rewrite relation morphisms, then $\langle g^{ob} \rangle \circ \langle f^{ob} \rangle = \langle g^{ob} \circ f^{ob} \rangle : \mathcal{A} \to \mathcal{C}$.

We have that:

Lemma B.1.2.

- *Composition of rewrite relation morphisms yields a proper rewrite relation morphism.*
- *For every rewrite relation we have an identity morphism.*
- *Composition of rewrite relation morphisms is associative.*

Proof.

- Let $\mathcal{A} = (A, \to_A)$, $\mathcal{B} = (B, \to_B)$ and $\mathcal{C} = (C, \to_C)$ be rewrite relations and let $\langle f^{ob} \rangle : \mathcal{A} \to \mathcal{B}$ and $\langle g^{ob} \rangle : \mathcal{B} \to \mathcal{C}$ be rewrite relation morphisms.

For all $x, y \in A$ we have $x \to_A y \Rightarrow f^{ob}(x) \to_B f^{ob}(y)$ and for all $x, y \in B$ we have $x \to_B y \Rightarrow g^{ob}(x) \to_C g^{ob}(y)$ by preservation of rewrite structure. So for any $x, y \in A$ such that $x \to_A y$ we get $g^{ob}(f^{ob}(x)) \to_C g^{ob}(f^{ob}(y))$, so $g^{ob} \circ f^{ob}$ preserves the rewrite structure and hence $\langle g^{ob} \circ f^{ob} \rangle : \mathcal{A} \to \mathcal{C}$ is a rewrite relation morphism.

- Let $\mathcal{A} = (A, \to)$ be a rewrite relation. $\langle \mathrm{id}_A \rangle$, where id_A is the identity function on A, is an identity morphism on $\mathcal{A}$ because:
  - id_A preserves the rewrite structure of $\mathcal{A}$ and hence $\langle \mathrm{id}_A \rangle$ is a morphism from $\mathcal{A}$ to $\mathcal{A}$
  - For any other rewrite relation morphism $\langle f^{ob} \rangle$ we have $\langle f^{ob} \rangle \circ \langle \mathrm{id}_A \rangle = \langle f^{ob} \circ \mathrm{id}_A \rangle = \langle f^{ob} \rangle$ and $\langle \mathrm{id}_A \rangle \circ \langle f^{ob} \rangle = \langle \mathrm{id}_A \circ f^{ob} \rangle = \langle f^{ob} \rangle$.
- Let $\langle f^{ob} \rangle$, $\langle g^{ob} \rangle$ and $\langle h^{ob} \rangle$ be rewrite relation morphisms.

$$\begin{aligned}
\langle h^{ob} \rangle \circ (\langle g^{ob} \rangle \circ \langle f^{ob} \rangle) &= \langle h^{ob} \rangle \circ \langle g^{ob} \circ f^{ob} \rangle \\
&= \langle h^{ob} \circ (g^{ob} \circ f^{ob}) \rangle \\
&= \langle (h^{ob} \circ g^{ob}) \circ f^{ob} \rangle \\
&= \langle h^{ob} \circ g^{ob} \rangle \circ \langle f^{ob} \rangle \\
&= (\langle h^{ob} \rangle \circ \langle g^{ob} \rangle) \circ \langle f^{ob} \rangle
\end{aligned}$$ □

So the class of rewrite relations together with the class of rewrite relation morphisms forms a category, we'll refer to it as the category of rewrite relations.

Definition B.1.3. We denote the category of rewrite relations with their morphisms as **Rwr**

Because morphisms in the category of rewrite relations preserve the rewrite structure, they also preserve reduction sequences, conversion sequences and hence $\twoheadrightarrow$ and $\leftrightarrow^*$.

Lemma B.1.4. **Rwr** *is a concrete category.*

Proof. Consider the functor $F : \mathbf{Rwr} \to \mathbf{Set}$ that is such that:

- $F^{ob}((A, \to)) = A$.
- $F^{mor}(\langle f^{ob} \rangle) = \langle f^{ob} \rangle$

We show that F is faithful and hence **Rwr** concrete. Let $\mathcal{A} = (A, \to_A)$ and $\mathcal{B} = (B, \to_B)$, F induces the functions $F_{\mathcal{A},\mathcal{B}} : \mathrm{mor}_{\mathbf{Rwr}}(\mathcal{A}, \mathcal{B}) \to \mathrm{mor}_{\mathbf{Set}}(F(\mathcal{A}), F(\mathcal{B}))$ (functions from the class of rewrite relation morphisms from $\mathcal{A}$ to $\mathcal{B}$ to the class of set morphisms, total functions, from $F(\mathcal{A})$ to $F(\mathcal{B})$). $F_{\mathcal{A},\mathcal{B}}$ is injective because F^{mor} is the identity mapping. □

B.1.2 Rws

Definition B.1.5. If $\mathbf{\Phi} = (\Phi, A, \text{src}_\Phi, \text{tgt}_\Phi)$ and **Psi** $= (\Psi, B, \text{src}_\Psi, \text{tgt}_\Psi)$ are rewrite systems then $f^{st} : \Phi \to \Psi$ and $f^{ob} : A \to B$ **preserve the rewrite structure** of $\mathbf{\Phi}$ if for any $\phi \in \Phi$ we have $f^{ob}(\text{src}_\Phi(\phi)) = \text{src}_\Psi(f^{st}(\phi))$ and $f^{ob}(\text{tgt}_\Phi(\phi)) = \text{tgt}_\Psi(f^{st}(\phi))$. A **rewrite system morphism** is a structure $\langle f^{st}, f^{ob}\rangle : \mathbf{\Phi} \to \mathbf{\Psi}$, such that $f^{st} : \Phi \to \Psi$ and $f^{ob} : A \to B$ preserve the rewrite structure of $\mathbf{\Phi}$. The composition operator on rewrite system morphisms, $\circ$, is such that if $\langle f^{st}, f^{ob}\rangle : \mathbf{\Phi} \to \mathbf{\Psi}$ and $\langle g^{st}, g^{ob}\rangle : \mathbf{\Psi} \to \mathbf{\Upsilon}$ are rewrite system morphisms then $\langle g^{st}, g^{ob}\rangle \circ \langle f^{st}, f^{ob}\rangle : (\Phi, A, \text{src}_\Phi, \text{tgt}_\Phi) \to (\Upsilon, C, \text{src}_\Upsilon, \text{tgt}_\Upsilon) = \langle g^{st} \circ f^{st}, g^{ob} \circ f^{ob}\rangle$.

Lemma B.1.6.

- *Composition of rewrite system morphisms yields a rewrite system morphism.*
- *For every rewrite system we have an identity morphism.*
- *Composition of rewrite system morphisms is associative.*

Proof.

- Let rewriting systems $\mathbf{\Phi} = (\Phi, A, \text{src}_\Phi, \text{tgt}_\Phi)$, $\mathbf{\Psi} = (\Psi, B, \text{src}_\Psi, \text{tgt}_\Psi)$ and $\mathbf{\Upsilon} = (\Upsilon, C, \text{src}_\Upsilon, \text{tgt}_\Upsilon)$ and rewriting system morphisms $\langle f^{st}, f^{ob}\rangle : \mathbf{\Phi} \to \mathbf{\Psi}$ and $\langle g^{st}, g^{ob}\rangle : \mathbf{\Psi} \to \mathbf{\Upsilon}$ be arbitrary. By preservation of the rewrite structure, we get that for all $\phi \in \Phi$ we have $f^{ob}(\text{src}_\Phi(\phi)) = \text{src}_\Psi(f^{st}(\phi))$ and $f^{ob}(\text{tgt}_\Phi(\phi)) = \text{tgt}_\Psi(f^{st}(\phi))$ and for all $\psi \in \Psi$ we have $g^{ob}(\text{src}_\Psi(\psi)) = \text{src}_\Upsilon(g^{st}(\psi))$ and $g^{ob}(\text{tgt}_\Psi(\psi)) = \text{tgt}_\Upsilon(g^{st}(\psi))$. For any $\phi \in \Phi$ we get:

  $$g^{ob}(f^{ob}(\text{src}_\Phi(\phi))) = g^{ob}(\text{src}_\Psi(f^{st}(\phi))) = \text{src}_\Upsilon(g^{st}(f^{st}(\phi)))$$

  and

  $$g^{ob}(f^{ob}(\text{tgt}_\Phi(\phi))) = g^{ob}(\text{tgt}_\Psi(f^{st}(\phi))) = \text{tgt}_\Upsilon(g^{st}(f^{st}(\phi)))$$

  So $\langle g^{st} \circ f^{st}, g^{ob} \circ f^{ob}\rangle : \mathbf{\Phi} \to \mathbf{\Upsilon}$ is a rewrite system morphism.

- Let $\mathbf{\Phi} = (\Phi, A, \text{src}, \text{tgt})$ is a rewrite system then $\langle \text{id}_\Phi, \text{id}_A\rangle$, where id_Φ is the identity function on Φ and id_A is the identity function on A, is an identity morphism on $\mathbf{\Phi}$ because:

  - $\text{id}_A(\text{src}_\Phi(\phi)) = \text{src}_\Psi(\text{id}_\Phi(\phi))$ and $\text{id}_A(\text{tgt}_\Phi(\phi)) = \text{tgt}_\Psi(\text{id}_\Phi(\phi))$ and hence $\langle \text{id}_\Phi, \text{id}_A\rangle : \mathbf{\Phi} \to \mathbf{\Phi}$ is a rewrite system morphism.
  - Let $\langle f^{st}, f^{ob}\rangle$ be another rewrite system morphism. We have:

    $$\langle f^{st}, f^{ob}\rangle \circ \langle \text{id}_\Phi, \text{id}_A\rangle = \langle f^{st} \circ \text{id}_\Phi, f^{ob} \circ \text{id}_A\rangle = \langle f^{st}, f^{ob}\rangle$$

    and

    $$\langle \text{id}_\Phi, \text{id}_A\rangle \circ \langle f^{st}, f^{ob}\rangle = \langle \text{id}_\Phi \circ f^{st}, \text{id}_A \circ f^{ob}\rangle = \langle f^{st}, f^{ob}\rangle$$

- Composition of rewrite system morphisms is associative. Let $\langle f^{st}, f^{ob}\rangle$, $\langle g^{st}, g^{ob}\rangle$ and $\langle h^{st}, h^{ob}\rangle$ be rewrite system morphisms.

$$\begin{aligned}\langle h^{st}, h^{ob}\rangle \circ (\langle g^{st}, g^{ob}\rangle \circ \langle f^{st}, f^{ob}\rangle) &= \langle h^{st}, h^{ob}\rangle \circ \langle g^{st} \circ f^{st}, g^{ob} \circ f^{ob}\rangle \\ &= \langle h^{st} \circ (g^{st} \circ f^{st}), h^{ob} \circ (g^{ob} \circ f^{ob})\rangle \\ &= \langle (h^{st} \circ g^{st}) \circ f^{st}, (h^{ob} \circ g^{ob}) \circ f^{ob}\rangle \\ &= \langle h^{st} \circ g^{st}, h^{ob} \circ g^{ob}\rangle \circ \langle f^{st}, f^{ob}\rangle \\ &= (\langle h^{st}, h^{ob}\rangle \circ \langle g^{st}, g^{ob}\rangle) \circ \langle f^{st}, f^{ob}\rangle \quad \square\end{aligned}$$

So the class of rewrite systems together with the class of rewrite systems morphisms form a category, we'll refer to it as the category of rewrite systems.

Definition B.1.7. We denote the category of rewrite systems with their morphisms as **Rws**

Because morphisms in the category of rewrite systems preserve the rewrite structure they also preserve reductions.

Lemma B.1.8. **Rws** *is a concrete category.*

Proof. Consider the functor $F : \mathbf{Rws} \to \mathbf{Set}$ that is such that:

- $F^{ob}((\Phi, A, \mathrm{src}, \mathrm{tgt})) = \Phi \uplus A$.
- $F^{mor}(\langle f^{st}, f^{ob}\rangle) = \langle g\rangle$ where $g(x) = f^{ob}(x)$ if $x \in A$ and $g(x) = f^{st}(x)$ if $x \in \Phi$

We now prove that F is faithful and hence **Rws** is concrete. Let $\mathbf{\Phi} = (\Phi, A, \mathrm{src}_\Phi, \mathrm{tgt}_\Phi)$ and $\mathbf{\Psi} = (\Psi, B, \mathrm{src}_\Psi, \mathrm{tgt}_\Psi)$, F induces functions $F_{\mathbf{\Phi},\mathbf{\Psi}} : \mathrm{mor}_{\mathbf{Rws}}(\mathbf{\Phi}, \mathbf{\Psi}) \to \mathrm{mor}_{\mathbf{Set}}(F(\mathbf{\Phi}), F(\mathbf{\Psi}))$. We prove that $F_{\mathbf{\Phi},\mathbf{\Psi}}$ is injective.

Let $\langle f_1^{st}, f_1^{ob}\rangle : \mathbf{\Phi} \to \mathbf{\Psi}$ and $\langle f_2^{st}, f_2^{ob}\rangle : \mathbf{\Phi} \to \mathbf{\Psi}$ be different rewrite system morphisms. Let $F^{mor}(\langle f_1^{st}, f_1^{ob}\rangle) = \langle g_1\rangle$ and $F^{mor}(\langle f_2^{st}, f_2^{ob}\rangle) = \langle g_2\rangle$ .We either have $f_1^{ob} \neq f_2^{ob}$ or $f_1^{st} \neq f_2^{st}$.

- $f_1^{ob} \neq f_2^{ob}$. We get an $a \in A$ such that $f_1^{ob}(a) \neq f_2^{ob}(a)$, so $g_1(a) \neq g_2(a)$ and hence $F^{mor}(\langle f_1^{st}, f_1^{ob}\rangle) = \langle g_1\rangle \neq \langle g_2\rangle = F^{mor}(\langle f_2^{st}, f_2^{ob}\rangle)$.
- $f_1^{st} \neq f_2^{st}$. We get an $\phi \in \Phi$ such that $f_1^{st}(\phi) \neq f_2^{st}(\phi)$, so $g_1(\phi) \neq g_2(\phi)$ and hence $F^{mor}(\langle f_1^{st}, f_1^{ob}\rangle) = \langle g_1\rangle \neq \langle g_2\rangle = F^{mor}(\langle f_2^{st}, f_2^{ob}\rangle)$. $\square$

B.1.3 Functors

We can define functors for the relationships of inducing and underlying.

Definition B.1.9. The functor expressing how rewrite relations induce rewrite systems is $I = \langle I^{ob}, I^{mor}\rangle : \mathbf{Rwr} \to \mathbf{Rws}$ where, if $\mathcal{A} = (A, \to_A)$ and $\mathcal{B} = (B, \to_B)$ are rewrite relations and $\langle f^{ob}\rangle : \mathcal{A} \to \mathcal{B}$ is a rewrite relation morphism, then:

- $I^{ob}(\mathcal{A}) = (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}, A, \pi_1, \pi_2)$ where $\pi_1(\langle x,y\rangle) = x$ and $\pi_2(\langle x,y\rangle) = y$.
- $I^{mor}(\langle f^{ob}\rangle) = \langle f^{st}, f^{ob}\rangle$ where $f^{st} : (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}) \rightarrow (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_B\})$ is such that $f^{st}(\langle x,y\rangle) = \langle f^{ob}(x), f^{ob}(y)\rangle$.

Lemma B.1.10. $I : \mathbf{Rwr} \rightarrow \mathbf{Rws}$ *is a functor*

Proof.

- $I^{mor} : \mathrm{mor}_{\mathbf{Rwr}} \rightarrow \mathrm{mor}_{\mathbf{Rws}}$ is well-defined.

  Let $\mathcal{A} = (A, \rightarrow_A)$ and $\mathcal{B} = (B, \rightarrow_B)$ be rewrite relations and let $\langle f^{ob}\rangle : \mathcal{A} \rightarrow \mathcal{B}$ be a rewrite relation morphism. For all $x, y \in A$ we get $x \rightarrow_A y \Rightarrow f^{ob}(x) \rightarrow_B f^{ob}(y)$ because the morphism preserves the rewrite structure. We get $I^{ob}(\mathcal{A}) = (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}, A, \pi_1, \pi_2)$ where $\pi_1(\langle x,y\rangle) = x$ and $\pi_2(\langle x,y\rangle) = y$. Let $\langle a,b\rangle \in \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}$ we get:

  $$\begin{aligned} f^{ob}(\mathrm{src}_\Phi(\langle a,b\rangle)) &= f^{ob}(a) \\ &= \mathrm{src}_\Psi(\langle f^{ob}(a), f^{ob}(b)\rangle) \\ &= \mathrm{src}_\Psi(f^{st}(\langle a,b\rangle)) \end{aligned}$$

  And:

  $$\begin{aligned} f^{ob}(\mathrm{tgt}_\Phi(\langle a,b\rangle)) &= f^{ob}(a) \\ &= \mathrm{tgt}_\Psi(\langle f^{ob}(a), f^{ob}(b)\rangle) \\ &= \mathrm{tgt}_\Psi(f^{st}(\langle a,b\rangle)) \end{aligned}$$

  So $\langle f^{st}, f^{ob}\rangle : I^{ob}(\mathcal{A}) \rightarrow I^{ob}(\mathcal{B})$ preserves the rewrite structure and hence is a rewrite system morphism.

- For all $X \in \mathrm{ob}_{\mathbf{Rwr}}$ we have $I^{mor}(\mathrm{id}_X) = \mathrm{id}_{I^{ob}(X)}$

  Let $\mathcal{A} = (A, \rightarrow)$ be a rewrite relation. Its identity morphism is $\langle \mathrm{id}_A\rangle : \mathcal{A} \rightarrow \mathcal{A}$. We get $I^{mor}(\langle \mathrm{id}_A\rangle) = \langle f^{st}, \mathrm{id}_A\rangle$ where $f^{st} : (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}) \rightarrow (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\})$ is such that $f^{st}(\langle x,y\rangle) = \langle \mathrm{id}_A(x), \mathrm{id}_A(y)\rangle$. We get $f^{st}(\langle x,y\rangle) = \langle x,y\rangle$, so $f^{st} = \mathrm{id}_{\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}}$, so $\langle f^{st}, \mathrm{id}_A\rangle$ is the identity morphism on $I^{ob}((A, \rightarrow))$.

- For all $f, g \in \mathrm{mor}_{\mathbf{Rwr}}$ we have $I^{mor}(g \circ f) = I^{mor}(g) \circ I^{mor}(f)$

  Let $\mathcal{A} = (A, \rightarrow_A)$, $\mathcal{B} = (B, \rightarrow_B)$ and $\mathcal{C} = (C, \rightarrow_C)$ be rewrite relations and let $\langle f^{ob}\rangle : \mathcal{A} \rightarrow \mathcal{B}$ and $\langle g^{ob}\rangle : \mathcal{B} \rightarrow \mathcal{C}$ be rewrite relation morphisms:

  $$\begin{aligned} I^{mor}(\langle f^{ob}\rangle \circ \langle g^{ob}\rangle) &= I^{mor}(\langle f^{ob} \circ g^{ob}\rangle) \\ &= \langle h^{st}, f^{ob} \circ g^{ob}\rangle \end{aligned}$$

Where $h^{st} : \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\} \rightarrow \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_C\}$ is such that for all $\langle a,b\rangle \in \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}$ we have $h^{st}(\langle a,b\rangle) = \langle f^{ob} \circ g^{ob}(a), f^{ob} \circ g^{ob}(b)\rangle$.

For functions $f^{st} : \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\} \rightarrow \{\langle x,y\rangle \mid \langle a,b\rangle \in \rightarrow_B\}$ and $g^{st} : \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_C\} \rightarrow \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_C\}$ such that for all $\langle a,b\rangle \in \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}$ we have $f^{st}(\langle a,b\rangle) = \langle f^{ob}(a), f^{ob}(b)\rangle$ and for all $\langle a,b\rangle \in \{\langle a,b\rangle \mid \langle a,b\rangle \in \rightarrow_B\}$ we have $f^{st}(\langle a,b\rangle) = \langle f^{ob}(a), f^{ob}(b)\rangle$ we get:

$$\begin{aligned}\langle h^{st}, f^{ob} \circ g^{ob}\rangle &= \langle f^{st} \circ g^{st}, f^{ob} \circ g^{ob}\rangle \\ &= \langle f^{st}, f^{ob}\rangle \circ \langle g^{st}, g^{ob}\rangle \\ &= I^{mor}(\langle f^{ob}\rangle) \circ I^{mor}(\langle g^{ob}\rangle)\end{aligned}$$ □

We already know from our non-categorical reasoning that, on objects, I is injective but not surjective. Because **Rwr** and **Rws** are concrete categories, that means that I is a monofunctor. On morphisms I behaves as follows:

Lemma B.1.11. *$I : \mathbf{Rwr} \rightarrow \mathbf{Rws}$ is fully faithful.*

Proof. Let $\mathcal{A} = (A, \rightarrow_A)$ and $\mathcal{B} = (B, \rightarrow_B)$ be rewrite relations. I induces the function $I_{\mathcal{A},\mathcal{B}} : \mathrm{mor}_{\mathbf{Rwr}}(\mathcal{A},\mathcal{B}) \rightarrow \mathrm{mor}_{\mathbf{Rwr}}(I(\mathcal{A}), I(\mathcal{B}))$ from the class of rewrite relation morphisms from $\mathcal{A}$ to $\mathcal{B}$ to the class of rewrite system morphisms from $I(\mathcal{A})$ to $I(\mathcal{B})$. We have that:

- $I_{\mathcal{A},\mathcal{B}}$ is injective and hence faithful.

  Let $\langle f_1^{ob}\rangle, \langle f_2^{ob}\rangle \in \mathrm{mor}_{\mathbf{Rwr}}(\mathcal{A},\mathcal{B})$ be such that $\langle f_1^{ob}\rangle \neq \langle f_2^{ob}\rangle$ we get $f_1^{ob} \neq f_2^{ob}$. We also get $I(\langle f_1^{ob}\rangle) = \langle f_1^{st}, f_1^{ob}\rangle$ and $I(\langle f_2^{ob}\rangle) = \langle f_2^{st}, f_2^{ob}\rangle$ for some f_1^{st} and f_2^{st}, we get $\langle f_1^{st}, f_1^{ob}\rangle \neq \langle f_2^{st}, f_2^{ob}\rangle$ and hence $I(\langle f_1^{ob}\rangle) \neq I(\langle f_2^{ob}\rangle)$.

- $I_{\mathcal{A},\mathcal{B}}$ is surjective and hence full.

  Let $\langle f^{st}, f^{ob}\rangle \in \mathrm{mor}_{\mathbf{Rws}}(I(\mathcal{A}), I(\mathcal{B}))$. We get that $I(\mathcal{A}) = (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}, A, \pi_1, \pi_2)$ where $\pi_1(\langle x,y\rangle) = x$ and $\pi_2(\langle x,y\rangle) = y$ and that $I(\mathcal{B}) = (\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_B\}, B, \pi_1, \pi_2)$ where $\pi_1(\langle x,y\rangle) = x$ and $\pi_2(\langle x,y\rangle) = y$. Now:

  – $\langle f^{ob}\rangle$ is a rewrite relation morphism.
    Assume that $a \rightarrow_A b$, we get $\langle a,b\rangle \in \{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_A\}$. By $\langle f^{st}, f^{ob}\rangle$ preserving the rewrite structure we get that $f^{ob}(\pi_1(\langle a,b\rangle) = \pi_1(f^{st}(\langle a,b\rangle))$ and $f^{ob}(\pi_2(\langle a,b\rangle)) = \pi_2(f^{st}(\langle a,b\rangle))$ and hence that $f^{ob}(a) = \pi_1(f^{st}(\langle a,b\rangle))$ and $f^{ob}(b) = \pi_2(f^{st}(\langle a,b\rangle))$. So there is a step in $I(\mathcal{B})$ such that its source is $f^{ob}(a)$ and its target $f^{ob}(b)$. The set of steps of $I(\mathcal{B})$ is $\{\langle x,y\rangle \mid \langle x,y\rangle \in \rightarrow_B\}$, so we get $f^{ob}(a) \rightarrow_B f^{ob}(b)$. That means that $\langle f^{ob}\rangle$ preserves the rewrite structure of $\mathcal{B}$ and hence is a rewrite relation morphism.

– $I(\langle f^{ob} \rangle) = \langle f^{st}, f^{ob} \rangle$

By definition of our functor I we must have that $I(\langle f^{ob} \rangle) = \langle g^{st}, f^{ob} \rangle$ where $g^{st} : \{\langle x, y \rangle \mid \langle x, y \rangle \in \rightarrow_A\} \rightarrow \{\langle x, y \rangle \mid \langle x, y \rangle \in \rightarrow_B\}$. Let $\langle a, b \rangle \in \rightarrow_A$, because $\langle g^{st}, f^{ob} \rangle$ preserves the rewrite structure we get $f^{ob}(\pi_1(\langle a, b \rangle) = \pi_1(g^{st}(\langle a, b \rangle))$ and $f^{ob}(\pi_2(\langle a, b \rangle)) = \pi_2(g^{st}(\langle a, b \rangle))$. Because $\langle f^{st}, f^{ob} \rangle$ also preserves the rewrite structure we also get $f^{ob}(\pi_1(\langle a, b \rangle) = \pi_1(f^{st}(\langle a, b \rangle))$ and $f^{ob}(\pi_2(\langle a, b \rangle)) = \pi_2(f^{st}(\langle a, b \rangle))$. This means that $\pi_1(g^{st}(\langle a, b \rangle)) = \pi_1(f^{st}(\langle a, b \rangle))$ and $\pi_2(g^{st}(\langle a, b \rangle)) = \pi_2(f^{st}(\langle a, b \rangle))$ and hence $g^{st}(\langle a, b \rangle) = f^{st}(\langle a, b \rangle)$, so $f^{st} = g^{st}$ and $I(\langle f^{ob} \rangle) = \langle g^{st}, f^{ob} \rangle = \langle f^{st}, f^{ob} \rangle$. □

Definition B.1.12. The functor that expresses how rewrite relations underlie rewrite systems is $U = \langle U^{ob}, U^{mor} \rangle : \mathbf{Rws} \rightarrow \mathbf{Rwr}$ such that, if $\mathbf{\Phi} = (\Phi, A, \mathrm{src}_\Phi, \mathrm{tgt}_\Phi)$ and $\mathbf{\Psi} = (\Psi, B, \mathrm{src}_\Psi, \mathrm{tgt}_\Psi)$ are rewrite systems and $\langle f^{st}, f^{ob} \rangle : \mathbf{\Phi} \rightarrow \mathbf{\Psi}$ is a rewrite system morphism, then:

- $U^{ob}(\mathbf{\Phi}) = (A, \{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle \mid \phi \in \Phi\})$.
- $U^{mor}(\langle f^{st}, f^{ob} \rangle) = \langle f^{ob} \rangle$.

Lemma B.1.13. $U : \mathbf{Rws} \rightarrow \mathbf{Rwr}$ *a functor.*

Proof.

- $U^{mor} : \mathrm{mor}_{\mathbf{Rws}} \rightarrow \mathrm{mor}_{\mathbf{Rwr}}$ is well-defined

  Let $\mathbf{\Phi} = (\Phi, A, \mathrm{src}_\Phi, \mathrm{tgt}_\Phi)$ and $\mathbf{\Psi} = (\Psi, B, \mathrm{src}_\Psi, \mathrm{tgt}_\Psi)$ be rewrite systems and let $\langle f^{st}, f^{ob} \rangle : \mathbf{\Phi} \rightarrow \mathbf{\Psi}$ be a rewrite system morphism. We have:

  $$U^{ob}(\mathbf{\Phi}) = (A, \{\langle \mathrm{src}_\Phi\, \phi, \mathrm{tgt}_\Phi\, \phi \rangle \mid \phi \in \Phi\})$$

  and:

  $$U^{ob}(\mathbf{\Psi}) = (B, \{\langle \mathrm{src}_\Psi\, \psi, \mathrm{tgt}_\Psi\, \psi \rangle \mid \psi \in \Psi\})$$

  Let $x, y \in A$ be such that, in $(A, \{\langle \mathrm{src}_\Phi\, \phi, \mathrm{tgt}_\Phi\, \phi \rangle \mid \phi \in \Phi\})$, we have $x \rightarrow y$. That is, there is a $\upsilon \in \Phi$ such that $\mathrm{src}_\Phi\, \upsilon = x$ and $\mathrm{tgt}_\Phi\, \upsilon = y$. Because $\langle f^{st}, f^{ob} \rangle$ preserves the rewrite structure, for all $\phi \in \Phi$ we have that $f^{ob}(\mathrm{src}_\Phi(\phi)) = \mathrm{src}_\Psi(f^{st}(\phi))$ and $f^{ob}(\mathrm{tgt}_\Phi(\phi)) = \mathrm{tgt}_\Psi(f^{st}(\phi))$, so we get $\mathrm{src}_\Psi(f^{st}(\upsilon)) = f^{ob}(x)$ and $\mathrm{tgt}_\Psi(f^{st}(\upsilon)) = f^{ob}(y)$. So $f^{st}(\upsilon)$ witnesses that, in $(B, \{\langle \mathrm{src}_\Psi\, \psi, \mathrm{tgt}_\Psi\, \psi \rangle \mid \psi \in \Psi\})$, we have $f^{ob}(x) \rightarrow f^{ob}(y)$ and hence $\langle f^{ob} \rangle$ preserves the rewrite structure of $(A, \{\langle \mathrm{src}_\Phi\, \phi, \mathrm{tgt}_\Phi\, \phi \rangle \mid \phi \in \Phi\})$ and hence is a rewrite relation morphism.

- For all $X \in \mathrm{ob}_{\mathbf{Rws}}$ we have $U^{mor}(\mathrm{id}_X) = \mathrm{id}_{U^{ob}(X)}$

  Let $\mathbf{\Phi} = (\Phi, A, \mathrm{src}, \mathrm{tgt})$ be a rewrite system. Its identity morphism is $\langle \mathrm{id}_\Phi, \mathrm{id}_A \rangle : \mathbf{\Phi} \rightarrow \mathbf{\Phi}$. We have $U^{mor}(\langle \mathrm{id}_\Phi, \mathrm{id}_A \rangle) = \langle \mathrm{id}_A \rangle : U^{ob}(\mathbf{\Phi}) \rightarrow U^{ob}(\mathbf{\Phi})$, which is indeed the identity morphism of $U^{ob}(\mathbf{\Phi})$.

- For all $f, g \in \text{mor}_{\mathbf{Rwr}}$ we have $I^{mor}(g \circ f) = I^{mor}(g) \circ I^{mor}(f)$

  Let $X, Y, Z \in \text{ob}_{\mathbf{Rws}}$ such that we have $\langle f^{st}, f^{ob}\rangle : X \to Y, \langle g^{st}, g^{ob}\rangle : Y \to Z \in \text{mor}_{\mathbf{Rws}}$:

$$\begin{aligned} U^{mor}(\langle f^{st}, f^{ob}\rangle \circ \langle g^{st}, g^{ob}\rangle) &= U^{mor}(\langle f^{st} \circ g^{st}, f^{ob} \circ g^{ob}\rangle) \\ &= \langle f^{ob} \circ g^{ob}\rangle \\ &= \langle f^{ob}\rangle \circ \langle g^{ob}\rangle \\ &= U^{mor}(\langle f^{st}, f^{ob}\rangle) \circ U^{mor}(\langle g^{st}, g^{ob}\rangle) \quad \square \end{aligned}$$

From our non-categorical reasoning we get that, on objects, U is surjective but not injective. Because **Rwr** and **Rws** are concrete categories, that means that I is an epifunctor. On morphisms U behaves as follows:

Lemma B.1.14. *U is full but not faithful.*

Proof. Let $\mathbf{\Phi} = (\Phi, A, \text{src}_\Phi, \text{tgt}_\Phi)$ and $\mathbf{\Psi} = (\Psi, B, \text{src}_\Psi, \text{tgt}_\Psi)$ be rewrite systems. U induces the functions $U_{\mathbf{\Phi},\mathbf{\Psi}} : \text{mor}_{\mathbf{Rws}}(\mathbf{\Phi}, \mathbf{\Psi}) \to \text{mor}_{\mathbf{Rwr}}(U(\mathbf{\Phi}), U(\mathbf{\Psi}))$. We have that:

- $U_{\mathbf{\Phi},\mathbf{\Psi}}$ is surjective and hence full.

  By the definition of U, we get $U(\mathbf{\Phi}) = (A, \{\langle \text{src}(\phi), \text{tgt}(\phi)\rangle \mid \phi \in \Phi\})$ and $U(\mathbf{\Psi}) = (B, \{\langle \text{src}(\psi), \text{tgt}(\psi)\rangle \mid \psi \in \Psi\})$.

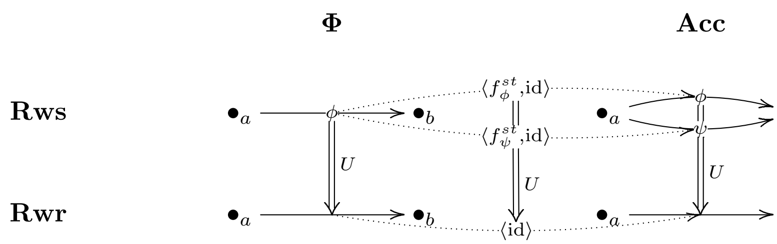
  Let $\langle f^{ob}\rangle \in \text{mor}_{\mathbf{Rws}}(U(\mathbf{\Phi}), U(\mathbf{\Psi}))$. Also let $\phi \in \Phi$, we get $\text{src}_\Phi(\phi) \to \text{tgt}_\Phi(\phi)$ in $U(\mathbf{\Phi})$. By $\langle f^{ob}\rangle$ preserving the rewrite structure of $U(\mathbf{\Phi})$ we get $f^{ob}(\text{src}_\Phi(\phi)) \to f^{ob}(\text{tgt}_\Phi(\phi))$ in $U(\mathbf{\Psi})$. So, by the structure of Ψ there must be some step $\psi \in \Psi$ such that $\text{src}_\Psi(\psi) = f^{ob}(\text{src}_\Phi(\phi))$ and $\text{tgt}_\Psi(\psi) = f^{ob}(\text{tgt}_\Phi(\phi))$. Now, if we let f^{st} be the function that maps each $\phi \in \Phi$ to its respective $\psi \in \Psi$ we get that $\text{src}_\Psi(f^{st}(\phi)) = f^{ob}(\text{src}_\Phi(\phi)$ and $\text{tgt}_\Psi(f^{st}(\phi)) = f^{ob}(\text{tgt}_\Phi(\phi)$. This means that $\langle f^{st}, f^{ob}\rangle$ preserves the rewrite structure of $(\Phi, A, \text{src}_\Phi, \text{tgt}_\Phi)$, hence is a rewrite system morphism and we have $U(\langle f^{st}, f^{ob}\rangle) = \langle f^{ob}\rangle$.

- $U_{\mathbf{\Phi},\mathbf{\Psi}}$ is not injective and hence not faithful.

  Consider, again, the following rewrite systems:

  - $\mathbf{\Phi} = (\{\phi\}, \{a, b\}, \text{src}, \text{tgt})$ with $\text{src}(\phi) = a$ and $\text{tgt}\,\phi = b$
  - $\mathbf{Acc} = (\{\phi, \psi\}, \{a, b\}, \text{src}', \text{tgt}')$ such that $\text{src}'(\phi) = a$, $\text{tgt}'\,\phi = b$, $\text{src}'(\psi) = a$ and $\text{tgt}'\,\psi = b$ (the syntactic accident rewrite system).

  We have $\langle f^{st}_\phi, \text{id}\rangle : \Phi \to \mathbf{Acc}$ with $f^{st}_\phi(\phi) = \phi$ and $\langle f^{st}_\psi, \text{id}\rangle : \mathbf{\Phi} \to \mathbf{Acc}$ with $f^{st}_\psi(\phi) = \psi$ as different rewrite system morphisms. But $U(\langle f^{st}_\phi, \text{id}\rangle) = U(\langle f^{st}_\psi, \text{id}\rangle) = \langle \text{id}\rangle : U(\mathbf{\Phi}) \to U(\mathbf{Acc})$.



□

From our non-categorical reasoning we get that $U^{ob} \circ I^{ob} = \mathrm{ID}^{ob}_{\mathbf{Rwr}}$ but $I^{ob} \circ U^{ob} \neq \mathrm{ID}^{ob}_{\mathbf{Rws}}$. That means that, when viewed as morphism, I^{ob} is a section of U^{ob} and U^{ob} is a retract of I^{ob}.

When we compose these functors, we get:

Proposition B.1.15. $U \circ I : \mathbf{Rwr} \to \mathbf{Rwr}$ is the functor such that:

- $(U \circ I)^{ob}((A, \to)) = (A, \to)$
- $(I \circ U)^{mor}(\langle f^{ob} \rangle) = \langle f^{ob} \rangle$

That is, $U \circ I = \mathrm{ID}_{\mathbf{Rwr}}$

Proposition B.1.16. $I \circ U : \mathbf{Rws} \to \mathbf{Rws}$ is the functor such that for a rewrite systems $\mathbf{\Phi} = (\Phi, A, \mathrm{src}_\Phi, \mathrm{tgt}_\Phi)$ and $\mathbf{\Psi} = (\Psi, A, \mathrm{src}_\Psi, \mathrm{tgt}_\Psi)$:

- $(I \circ U)^{ob}(\mathbf{\Phi}) = (\{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle \mid \phi \in \Phi\}, A, \pi_1, \pi_2)$ where $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$
- $(I \circ U)^{mor}(\langle f^{st}, f^{ob} \rangle) : \mathbf{\Phi} \to \mathbf{\Psi}) = \langle f^{st'}, f^{ob} \rangle$ where $f^{st'} : \{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle \mid \phi \in \Phi\} \to \{\langle \mathrm{src}(\psi), \mathrm{tgt}(\psi) \rangle \mid \psi \in \Psi\}$ such that $f^{st'}(\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle) = \langle \mathrm{src}(f^{st}(\phi)), \mathrm{tgt}(f^{st}(\phi)) \rangle$

This is not a faithful functor. However, we do have that:

Definition B.1.17. $\eta : \mathrm{ID}_{\mathbf{Rws}} \to (I \circ U)$ is the natural transformation such that for any rewrite system $\mathbf{\Phi} = (\Phi, A, \mathrm{src}, \mathrm{tgt})$ we have $\eta_{\mathbf{\Phi}} = \langle \eta^{st}_{\mathbf{\Phi}}, \eta^{ob}_{\mathbf{\Phi}} \rangle : \mathbf{\Phi} \to I(U(\mathbf{\Phi}))$ where:

- $\eta^{st}_{\mathbf{\Phi}}(\phi \in \Phi) = \langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle$
- $\eta^{ob}_{\mathbf{\Phi}} = id_A$

Lemma B.1.18. *η is a proper natural transformation from $\mathrm{ID}_{\mathbf{Rws}}$ to $(I \circ U)$.*

Proof. Let $\mathbf{\Phi} = (\Phi, A, \mathrm{src}_\Phi, \mathrm{tgt}_\Phi)$ and $\mathbf{\Psi} = (\Psi, B, \mathrm{src}_\Psi, \mathrm{tgt}_\Psi)$ be rewrite systems and let $f \in \mathrm{mor}_{\mathbf{Rws}}(\mathbf{\Phi}, \mathbf{\Psi})$. We have:

$$\eta_{\mathbf{\Psi}} \circ \mathrm{ID}_{\mathbf{rws}}(f), (I(U(f)) \circ \eta_{\mathbf{\Phi}} \in \mathrm{mor}_{\mathbf{Rws}}(\mathbf{\Phi}, I(U(\mathbf{\Psi})))$$

We also have $\mathrm{ID}_{\mathbf{rws}}(f) = f$. Now let $f = \langle f^{ob}, f^{st} \rangle$:

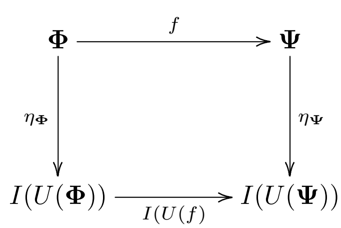
$$\begin{aligned}\eta^{ob}_{\mathbf{\Psi}} \circ f^{ob} &= \mathrm{id}_B \circ f^{ob} \\ &= f^{ob} \\ &= f^{ob} \circ \mathrm{id}_A \\ &= f^{ob} \circ \eta^{ob}_{\mathbf{\Phi}}\end{aligned}$$

Also let $\phi \in \Phi$, and $f^{st'} : \{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle \mid \phi \in \Phi\} \to \{\langle \mathrm{src}(\psi), \mathrm{tgt}(\psi) \rangle \mid \psi \in \Psi\}$ such that $f^{st'}(\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle) = \langle \mathrm{src}(f^{st}(\phi)), \mathrm{tgt}(f^{st}(\phi)) \rangle$ we have:

$$\begin{aligned}\eta^{st}_{\mathbf{\Psi}}(f^{st}(\phi)) &= \langle \mathrm{src}(f^{st}(\phi)), \mathrm{tgt}(f^{st}(\phi)) \rangle \\ &= f^{st'}(\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi) \rangle) \\ &= f^{st'}(\eta^{st}_{\mathbf{\Phi}}(\phi))\end{aligned}$$

So we get $\eta^{st}_{\mathbf{\Psi}} \circ f^{st} = f^{st'} \circ \eta^{st}_{\mathbf{\Phi}}$ and hence:

$$\begin{aligned}\eta_{\mathbf{\Psi}} \circ \mathrm{ID}_{\mathbf{rws}}(f) &= \eta_{\mathbf{\Psi}} \circ f \\ &= \langle \eta^{st}_{\mathbf{\Psi}}, \eta^{ob}_{\mathbf{\Psi}} \rangle \circ \langle f^{st}, f^{ob} \rangle \\ &= \langle \eta^{st}_{\mathbf{\Psi}} \circ f^{st}, \eta^{ob}_{\mathbf{\Psi}} \circ f^{ob} \rangle \\ &= \langle f^{st'} \circ \eta^{st}_{\mathbf{\Phi}}, f^{ob} \circ \eta^{ob}_{\mathbf{\Phi}} \rangle \\ &= \langle f^{st'}, f^{ob} \rangle \circ \langle \eta^{st}_{\mathbf{\Phi}}, \eta^{ob}_{\mathbf{\Phi}} \rangle \\ &= I(U(\langle f^{st}, f^{ob} \rangle)) \circ \langle \eta^{st}_{\mathbf{\Phi}}, \eta^{ob}_{\mathbf{\Phi}} \rangle \\ &= I(U(f)) \circ \eta_{\mathbf{\Phi}}\end{aligned}$$



□

We get that:

Lemma B.1.19. *U is left adjoint to I*

Proof. Let:

- $\mathbf{\Phi} = (\Phi, A, \mathrm{src}, \mathrm{tgt}) \in \mathrm{ob}_{\mathbf{Rws}}$
- $\mathcal{A} = (A, \to) \in \mathrm{ob}_{\mathbf{Rwr}}$
- $f = \langle f^{st}, f^{ob} \rangle \in \mathrm{mor}_{\mathbf{Rws}}(\mathbf{\Phi}, I(\mathcal{A}))$.

We get:

- $U(\mathbf{\Phi}) = (A, \{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi)\rangle \mid \phi \in \Phi\})$
- $I(\mathcal{A}) = (\{\langle x, y\rangle \mid \langle x, y\rangle \in \rightarrow\}, A, \pi_1, \pi_2)$ with $\pi_1(\langle x, y\rangle) = x$, $\pi_2(\langle x, y\rangle) = y$
- $I(U(\mathbf{\Phi})) = (\{\langle \mathrm{src}(\phi), \mathrm{tgt}(\phi)\rangle \mid \phi \in \Phi\}, A, \pi_1, \pi_2)$ with $\pi_1(\langle x, y\rangle) = x$, $\pi_2(\langle x, y\rangle) = y$

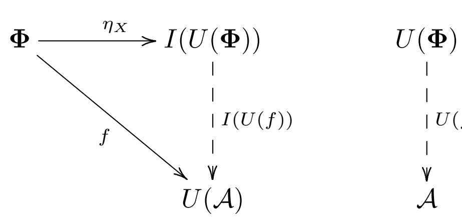
Now:

- We have a $f' \in \mathrm{mor}_{\mathbf{Rwr}}(U(\mathbf{\Phi}), \mathcal{A})$ such that $I(f') \circ \eta_{\mathbf{\Phi}} = f$, $U(f)$.

  Let $U(f) = \langle g^{ob}\rangle$. By Lemma B.1.18 we get $I(U(f)) \circ \eta_{\mathbf{\Phi}} = \eta_{I(\mathcal{A})} \circ \mathrm{ID}_{\mathbf{rws}}(f) = \eta_{I(\mathcal{A})} \circ f$. Also, for any step in $I(\mathcal{A})$ we have that it is of the form $\langle a, b\rangle$ and $\eta^{st}_{I(\mathcal{A})}(\langle a, b\rangle) = \langle \pi_1(\langle a, b\rangle), \pi_2(\langle a, b\rangle\rangle = \langle a, b\rangle$. That means that $\eta^{st}_{I(\mathcal{A})}$ is the identity function, so $\eta_{I(\mathcal{A})}$ is an identity morphism and we get $\eta_{I(\mathcal{A})} \circ f = f$. Now $I(U(f)) \circ \eta_{\mathbf{\Phi}} = \eta_{I(\mathcal{A})} \circ f = f$.

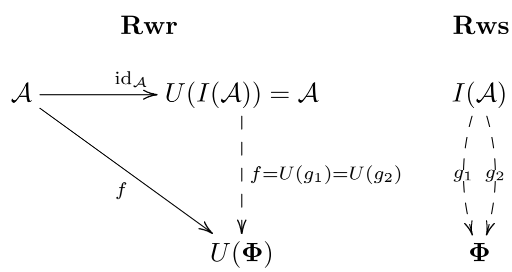
- $U(f) \in \mathrm{mor}_{\mathbf{Rwr}}(\mathcal{U}(\mathbf{\Phi}), \mathcal{A})$ such that $I(f') \circ \eta_{\mathbf{\Phi}} = f$ is unique.

  Let $g = \langle g^{st}, g^{ob}\rangle \in \mathrm{mor}_{\mathbf{Rwr}}(U(\mathbf{\Phi}), \mathcal{A})$ be such that $I(g) \circ \eta_{\mathbf{\Phi}} = f$. Let $I(g)^{st}$ be such that $I(\langle g^{st}, g^{ob}\rangle) = \langle I(g)^{st}, g^{ob}\rangle$. Because $f = I(g) \circ \eta_{\mathbf{\Phi}}$, we get that $\langle f^{st}, f^{ob}\rangle = \langle I(g)^{st}, g^{ob}\rangle \circ \langle \eta^{st}_{\mathbf{\Phi}}, \eta^{ob}_{\mathbf{\Phi}}\rangle = \langle I(g)^{st} \circ \eta^{st}_{\mathbf{\Phi}}, g^{ob} \circ \eta^{ob}_{\mathbf{\Phi}}\rangle$. That means that $f^{ob} = g^{ob} \circ \eta^{ob}_{\mathbf{\Phi}} = g^{ob} \circ \mathrm{id}_{\Phi} = g^{ob}$. Now $U(f) = U(\langle f^{st}, f^{ob}\rangle) = \langle f^{ob}\rangle = \langle g^{ob}\rangle = g$.



□

We also have that I is not left adjoint to U. This is because the only natural transformation between the identity functor and the identity functor maps any object to its identity morphism, so if, for any $\mathcal{A} \in \mathrm{ob}_{\mathbf{Rwr}}$ we chose $\mathbf{\Phi}$ as $I(\mathcal{A}))$ only with some step duplicated, then we get two different rewrite system morphisms, $g_1, g_2 : I(\mathcal{A})) \rightarrow \mathbf{\Phi}$, each mapping the duplicated step another duplicate, such that $U(g_1) = U(g_2)$ (exploiting the non-faithfulness of U, using the same type of example we used to show that). This means we get the following diagram showing non-adjointness.



Remark. The diagram has rewrite relations on the left and rewrite systems on the right, this way it is an instance of the well-known adjointness diagram (Appendix A.5). For rewriting purposes, it might also be logical to turn the diagram 90 degrees anti-clockwise to get rewrite relations at the bottom and rewrite systems at the top.

B.2 Kennaway '92

An early formalism for transfinite abstract rewriting is the one put forth by Richard Kennaway in his 1992 article "On Transfinite Abstract Reduction Systems" ([13]). It is a classic paper in the field of infinitary rewriting. I read and analyzed it and think the following observation is worth sharing.

Kennaway defines the MARS formalism for transfinite abstract rewriting, which is based on a metric on the set of objects and a notion of height for reduction steps. In these senses it is similar to the MRSs of Section 5.3, however, it is based on the ARSs of [18, chapter 1], which are related to the rewrite relations of Section 3.1 (as opposed to MRSs which are based on the rewrite systems of Section 3.2. Kennaway defines weakly continuous reduction sequences as transfinite reduction sequences of which sequence of objects in continuous which respect to the metric. Strongly continuous reductions are defined as weakly continuous sequence of which the sequence of heights converges to 0 up to every limit ordinal smaller than the length of the reduction sequence. Kennaway proposes the following:

> PROPOSITION. A strongly continuous sequence converges to a limit; a weakly convergent reduction sequence need not.
>
> PROOF. This is a consequence of the property of ultrametric spaces, that if the distances between successive members of a sequence tend to zero, the sequence converges. (We thank Jan Vis of the University of Nijmegen for pointing out that this proposition depends on the space being ultrametric, and not merely metric.) □

It appears to me that even though the property holds, the proof is, at the least, not really clear. It is not the case that, for any sequence, if distances between successive members tend to zero, the sequence converges. Not even for sequences in complete ultra-metric spaces. First of all, that this doesn't hold for merely metric spaces is obviously true.

Theorem B.2.1. *In a complete space that is merely metric we don't necessarily have that if the distances between successive members in the sequence tend to zero the sequence converges.*

Proof. Consider the metric space of reals $(\mathbb{R}, d_{\mathbb{R}})$. This space is complete, metric, but not ultra-metric. In this space we have the following counterexample: let $\langle \ln(n) \rangle_{n<\omega}$. For successive sequence members we have that the distance between them is $\ln(x+1)-\ln(x) = \ln(x*(1+1/x))-\ln(x) = \ln(1+1/x)$. As x approaches ω, $1/x$ tends to 0, hence $1+(1/x)$ tends to 1 and $\ln(1+(1/x))$ tends to 0. The sequence doesn't converge though, it is unbounded. For any $\epsilon \in \mathbb{R}^+$ we have that the $\ln(e^\epsilon) = \epsilon$, so for any $n < \omega$ such that $e^\epsilon < n$ we have that $\epsilon < \ln(n)$. □

In the case of an ultrametric space such a theorem does holds for sequences of length ω. We can prove this.

Theorem B.2.2. *In a complete, ultrametric space we have that if, in a sequence of length $\omega \in On$, the distances between successive members tend to zero the sequence converges.*

Proof. Let $\langle s_n \rangle_{n<\omega}$ be such that the distances between successive members tend to zero. To prove that $\langle s_n \rangle_{n<\omega}$ is Cauchy, let $\epsilon \in \mathbb{R}^+$ be arbitrary. By assumption we get a $n < \omega$ such that for all m with $n \leq m < \omega$ we have $d(s_m, s_{m'}) < \epsilon$. Let k, l be such that $n \leq k < l < \omega$. We can prove $d(s_k, s_{k+(l-k)}) = d(s_k, s_l) < \epsilon$ by induction on $l-k$. Since $l-k$ is a successor ordinal ($k, l < \omega$), we get as induction hypothesis $d(s_k, s_{(k+(l-k))-1}) = d(s_k, s_{l-1}) < \epsilon$. We get $d(s_{l-1}, s_l) < \epsilon$ by assumption, so by STRONG TRIANGLE INEQUALITY (d is an ultrametric) we get $d(s_k, s_l) < \epsilon$. This means that $\langle s_n \rangle_{n<\omega}$ is Cauchy, and because our space is complete, $\langle s_n \rangle_{n<\omega}$ converges to some limit. □

Such a theorem does not hold for sequences of arbitrary ordinal length, though. A proof such as the one above breaks down because, in our induction, we cannot deal with limit ordinal indexes. We can prove this by counterexample.

Theorem B.2.3. *There are complete, ultrametric spaces in which there are sequences of length $> \omega$ in which the distances between successive members tend to zero, but that do not converge.*

Proof. Our counterexample is constructed as follows: let $X = \{x_{i,j}\}_{i,j \in \mathbb{N}}$ (that is, a family of families of elements indexed by the natural numbers twice) be our space. Let d be a distance measure on X such that:

- If $i_1 = i_2$ and $j_1 = j_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 0$.
- If $i_1 = i_2$ and $j_1 \neq j_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 1/(i_1 + 1)$.
- If $i_1 \neq i_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 2$.

We can prove that d is an ultrametric on X.

- REFLEXIVITY, IDENTITY OF INDISCERNIBLES and SYMMETRY

- STRONG TRIANGLE INEQUALITY

  Let $i_1, i, 2, i_3, j_1, j_2, j_3 \in \mathbb{N}$ be arbitrary to define three arbitrary points in the space, x_{i_1,j_1}, x_{i_2,j_2} and x_{i_3,j_3}.

  – If $i_1 = i_2 = i_3$ then we have that if any two of j_1,j_2 and j_3 are equal then so are two of x_{i_1,j_1}, x_{i_2,j_2} and x_{i_3,j_3} and we trivially get

  $$d(x_{i_1,j_1}, x_{i_3,j_3}) \leq \max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3}))$$

  If j_1,j_2 and j_3 are all different we get $d(x_{i_1,j_1}, x_{i_3,j_3}) = 1/(i_1 + 1)$, $d(x_{i_1,j_1}, x_{i_2,j_2}) = 1/(i_1+1)$ and $d(x_{i_2,j_2}, x_{i_3,j_3}) = 1/(i_2+1) = 1/(i_1+1)$ and hence

  $$d(x_{i_1,j_1}, x_{i_3,j_3}) = \max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3}))$$

  – If $i_1 \neq i_2$, $i_1 \neq i_3$ and $i_2 \neq i_3$ then we have $d(x_{i_1,j_1}, x_{i_3,j_3}) = 2$, $d(x_{i_2,j_2}, x_{i_3,j_3}) = 2$ and $d(x_{i_1,j_1}, x_{i_2,j_2}) = 2$ and hence

  $$d(x_{i_1,j_1}, x_{i_3,j_3}) = \max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3}))$$

  – If $i_1 = i_3 \neq i_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 2$, $d(x_{i_2,j_2}, x_{i_3,j_3}) = 2$ and $d(x_{i_1,j_1}, x_{i_3,j_3})$ is either 0 or $1/(i_1+1)$ depending on whether $j_1 = j_3$, but $\max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3})) = 2$ and both 0 and $1/(i_1 + 1)$ are smaller than 2 so

  $$d(x_{i_1,j_1}, x_{i_3,j_3}) < \max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3}))$$

  – If $i_1 = i_2 \neq i_3$ then $d(x_{i_1,j_1}, x_{i_3,j_3}) = 2$, $d(x_{i_2,j_2}, x_{i_3,j_3}) = 2$ and $d(x_{i_1,j_1}, x_{i_2,j_2})$ is either 0 or $1/(i_1+1)$ depending on whether $j_1 = j_3$, but $\max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3})) = 2$ anyway and hence

  $$d(x_{i_1,j_1}, x_{i_3,j_3}) = \max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3}))$$

  – If $i_1 \neq i_2 = i_3$ then $d(x_{i_1,j_1}, x_{i_3,j_3}) = 2$, $d(x_{i_1,j_1}, x_{i_2,j_2}) = 2$ and $d(x_{i_2,j_2}, x_{i_3,j_3})$ is either 0 or $1/(i_2+1)$ depending on whether $j_1 = j_3$, but $\max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3})) = 2$ anyway and hence

  $$d(x_{i_1,j_1}, x_{i_3,j_3}) = \max(d(x_{i_1,j_1}, x_{i_2,j_2}), d(x_{i_2,j_2}, x_{i_3,j_3}))$$

We also need to prove that (X, d) is complete. Let $\langle c_\beta \rangle_{\beta<\alpha}$ an arbitrary Cauchy sequence in (X, d). We show that $\langle c_\beta \rangle_{\beta<\alpha}$ is eventually constant. Because $\langle c_\beta \rangle_{\beta<\alpha}$ is cauchy we get some $\delta < \alpha$ such that for all γ with $\delta \leq \gamma < \alpha$ we have $d(c_\delta, c_\gamma) < 2$. So, if $c_\delta = x_{i,j_1}$ we must have $c_\gamma = x_{i,j_2}$ for all γ with $\delta \leq \gamma$, otherwise we would have had $d(c_\delta, c_\gamma) = 2$. Now, by Cauchyness of the sequence we must have some $\eta < \alpha$ with $\delta \leq \eta$ such that for all γ_1, γ_2 with $\eta \leq \gamma_1 < \gamma_2 < \alpha$ we have $d(c_{\gamma_1}, c_{\gamma_2}) < 1/(i+1)$. We can't have $c_{\gamma_1} = x_{i_1,j_1}$ and $c_{\gamma_2} = x_{i_2,j_2}$ with $i_1 \neq_2$, since $\delta \leq \eta \leq \gamma_1, \gamma_2$. We also can't have $c_{\gamma_1} = x_{i,j_1}$ and $c_{\gamma_2} = x_{i,j_2}$ with $j_1 \neq j_2$ because then we would have $d(c_{\gamma_1}, c_{\gamma_2}) = 1/(i+1)$. So we must have $c_{\gamma_1} = c_{\gamma_2} = x_{i,j}$, hence the sequence is constant from η and on and converges to c_η.

Now, for our actual counterexample, let $\langle s_\beta \rangle_{\beta<\omega^2}$ such that $s_{(\omega*m)+n} = x_{m,n}$. This is well-defined because for any $\alpha < \omega^2$ we have that it is of the form $(\omega * m) + n$ for $m, n \in N$.

We have that the distances between successive members of our sequence tend to 0, because, let $\epsilon \in \mathbb{R}^+$ be arbitrary, we have a $k \in \mathbb{N}$ such that $1/(k+1) < \epsilon$. Now for any α with $\omega * k \leq \alpha < \omega^2$ we have that α is of the form $(\omega * m) + n$ with $k \leq m$. And because $d(s_{(\omega*m)+n}, s_{((\omega*m)+n)\prime}) = 1/(m+1)$, we get $d(s_{(\omega*m)+n}, s_{((\omega*m)+n)\prime}) < \epsilon$, so $d(s_\alpha, s_{\alpha\prime}) < \epsilon$.

But our sequence does not converge. For any $\alpha < \omega^2$ of the form $(\omega*m)+n$, we have that $\alpha \leq \omega * (m+1) < \omega * (m+2)$ and $d(s_{\omega*(m+1)}, s_{\omega*(m+2)}) = 2$. So at any point in the sequence, there are two later points such that the distance between them is 2. □

We can say something even stronger here.

Proposition B.2.4. There are complete, ultrametric spaces in which there are sequences of length $> \omega$ in which the distances between successive members tend to zero *at each limit ordinal*, but that do not converge.

To show this, we can slightly modify the our space and distance measure. Our space is $Y = \{y_{i,j}\}_{i\in\mathbb{N}, j\in\mathbb{N}\cup\{\omega\}}$. The distance measure on it is:

- If $i_1 = i_2$ and $j_1 = j_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 0$.
- If $i_1 = i_2$ and $j_1 < j_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 1/(i_1 + j_1 + 1)$.
- If $i_1 \neq i_2$ then $d(x_{i_1,j_1}, x_{i_2,j_2}) = 2$.

We reuse the sequence $\langle s_\beta \rangle_{\beta<\omega^2}$ such that $s_{(\omega*m)+n} = x_{m,n}$ and in this space distances between successive sequence members tend to 0 at each limit ordinal, while the sequence still does not converge.

Kennaway's proposition does hold though, but that this isn't because "if distances between successive members in the sequence tend to zero the sequence converges" (which, as proven above, isn't true). We need to involve weak continuity. The counterexample above clearly isn't weakly continuous because of the jumps (of my, rather arbitrarily chosen, distance 2) at the limit ordinals smaller than the length of the sequence. To prove that the proposition holds, using the definitions as laid out in [13, p. 2-3]:

Theorem B.2.5. *A strongly continuous reduction sequence converges to a limit.*

Proof. Let s be a strongly continuous reduction sequence of length α, let a be its associated sequence of objects.

- If α is a successor ordinal then the length of a is $\alpha\prime$. $\alpha\prime$ is also a successor ordinal and a converges by Lemma A.2.31.

- If α is a limit ordinal and the reduction sequence is closed then the length of a is $\alpha\prime$ again and hence it converges.
- If α is a limit ordinal and the reduction sequence is open however, then the length of a is α, a limit ordinal and hence we need to prove convergence differently.

  Because s is strongly continuous we have that the associated sequence of heights of steps tends to 0. So the distances between successive members in a also tends to 0. We also have that a is a continuous sequence because s is strongly continuous and hence weakly continuous.

  Let $\epsilon \in \mathbb{R}^+$ be arbitrary, we have a $\beta < \alpha$ such that distances between successive members in a after a_β are smaller than ϵ. Let γ_1, γ_2 be such that $\beta \leq \gamma_1 < \gamma_2 < \alpha$. We show that $d(a_{\gamma_1}, a_{\gamma_2}) < \epsilon$ by induction γ_2. We get as induction hypothesis that for all $\gamma_3 < \gamma_2$ we have $d(a_{\gamma_1}, a_{\gamma_3}) < \epsilon$. We get the following cases:

  - If γ_2 is a successor ordinal, we get that there is a $\delta < \gamma_2$ such that $\delta\prime = \gamma_2$. We get that $d(a_{\gamma_1}, a_\delta) < \epsilon$ by induction hypothesis and $d(a_\delta, a_{\gamma_2}) < \epsilon$ because distances between successive members in a after a_β are smaller than ϵ. We apply the strong triangle inequality and get $d(a_{\gamma_1}, a_{\gamma_2}) < \epsilon$.
  - If γ_2 is a limit ordinal, we get that there is a $\delta_1 < \gamma_2$ such that for all δ_2 such that $\delta_1 \leq \delta_2 < \gamma_2$ we have $d(a_{\delta_2}, a_{\gamma_2}) < \epsilon$ because a is a continuous sequence (because s is strongly and hence also weakly continuous). If $\delta_1 < \gamma_1$ then we have $d(a_{\delta_2}, a_{\gamma_2}) < \epsilon$ directly, if not we have $d(a_{\delta_1}, a_{\gamma_2}) < \epsilon$, and by induction hypothesis $d(a_{\gamma_1}, a_{\delta_1})$. And when we apply strong triangle inequality, we get $d(a_{\gamma_1}, a_{\gamma_2}) < \epsilon$ again.

  So we have $d(a_{\gamma_1}, a_{\gamma_2}) < \epsilon$, this means our sequence a is Cauchy and because our space is complete, converges to a limit. □