# Trivial

A term rewrite step $s \to t$ is *trivial* if $s = t$. One would expect that if a term allows a trivial step, it cannot be normalising... Unless, of course, another step eliminating the trivial one can be performed. The term $a$ in the term rewrite system (TRS) $\{a \to a, a \to b\}$ allows a trivial step, but can be normalised to $b$ as well. The term $f(a)$ in the orthogonal TRS $\{a \to a, f(x) \to b\}$ allows a trivial step, but can be normalised to $a$. The elimination is caused by a critical step in the former, and by an erasing step in the latter case. These are the only problems. A term allowing a trivial head-step cannot be normalising in an (almost) orthogonal TRS by the results of [2, 3].

**Lemma** *A term $s$ allowing a trivial head-step $\phi$ by rule $\varrho : l \to r$ is not normalising in a weakly orthogonal term TRS, i.e. a left-linear TRS such that $s = t$ for every critical pair $(s, t)$.*

**Proof** We construct a prefix $C$, such that $l^\Omega \leq C \leq t$ for any $s \twoheadrightarrow t$, where $l^\Omega = l^{[\vec{x}:=\Omega]}$:

- Let $C = \bigcup_{i \geq 0} C_i$, where $C_0 = l^\Omega$, and for $i \geq 0$, $C_{i+1} = \lhd_\phi C_i$ [1]. (See below for examples.)

Remark that for any prefix $D$ of $s$, $\lhd_\phi D$ is a prefix of $s$ again [3]. Hence to show $l^\Omega \leq C \leq s$, it suffices by $C_0 = l^\Omega$ to show monotonicity: $C_i \leq C_{i+1}$, $\forall i \geq 0$, by induction on $i$. The base case $l^\Omega \leq C_1$ holds since the head-symbol of $r$ traces back to any position in $l$. In the induction step, suppose $p \in C_i$ for some $i > 0$. By definition of $C_i$, there exists some $q \in C_{i-1}$ such that $p \rhd_\phi q$. By the induction hypothesis $q \in C_i$, hence $p \in C_{i+1}$.

To show $C \leq t$, for any $s \twoheadrightarrow t$, it suffices to show that $C[s_1, \ldots, s_n] \to t$ implies $t = C[t_1, \ldots, t_n]$, for any $s_1, \ldots, s_n$. Remark that this holds for the special case of a head-step by rule $\varrho$, since any position in $C$ descends to some position in $C$ again, by construction of $C$. Consider a general step. If it takes places in one of $s_1, \ldots, s_n$, then it is clear again. For a proof by contradiction, consider a step $\psi$ at position $p$, overlapping with $C$ and modifying some symbol in $C$ at position $q$. Let $C_i$ be the first prefix containing $q$, for $i \geq 0$. Then by construction, $q$ has a unique trace through the $C_i, \ldots, C_0$ to some position in $l^\Omega$, along a reduction $\mathcal{R}$ consisting of $i$ head-$\varrho$-steps. Since $p$ is above $q$, this induces a unique trace of $p$ through $C_i, \ldots, C_0$ along $\mathcal{R}$ as well, until it overlaps $l^\Omega$. Let $q'$ be the descendant of $q$, and $\psi'$ be the residual of $\psi$, at that moment. Contracting $\psi'$ modifies position $q'$ since $\psi'$ is a residual of $\psi$, but contracting the overlapping rule $\varrho$ would not modify $q'$ as was seen in the special case. This contradicts weak orthogonality.

The result follows, since any reduct of $s$ is of shape $C[t_1, \ldots, t_n]$, hence a redex for rule $\varrho$. $\Box$

We give some examples illustrating the construction of $C$.

1. Consider the trivial head-step $f(a, a) \to f(a, a)$ in the TRS $\{f(x, y) \to f(y, x), a \to b\}$. Then $C = f(\Omega, \Omega)$. Note that projecting the infinite trivial head-reduction over the step $f(a, a) \to f(b, a)$ yields an infinite non-trivial head-reduction: $f(b, a) \to f(a, b) \to f(b, a) \to \ldots$.

2. Consider the trivial head-step $f(a, a, \ldots, a) \to f(a, a, \ldots, a)$ in the TRS $\{f(a, x_1, \ldots, x_n) \to f(x_1, \ldots, x_n, x_n)\}$. Then the $C_i$ stabilise only after $n$ steps: $C_0 = f(a, \Omega, \ldots, \Omega)$, $C_1 = f(a, a, \ldots, \Omega)$,..., $C_n = f(a, a, \ldots, a) = C$.

This proof is terribly ad hoc. A theory of descendants for non-orthogonal rewriting seems required.

## References


[1] I. Bethke, J.W. Klop, and R. de Vrijer. Descendents and origins in term rewriting. *I&C*, ?? 72 pp.

[2] A. Middeldorp. Call by need computations to root-stable form. POPL97, pp. 94–105, 1997.

[3] V. van Oostrom. Normalisation in weakly orthogonal rewriting. RTA99, LNCS 1631, pp. 60–74, 1999.