Puzzle 1 *Can $\mathbb{Q}^2$ be partitioned into Q_0, Q_1 such that if $p \in Q_i$, $q \in Q_j$ and $d(p,q) = 1$, then $i \neq j$?*

Write i' for $i+1$ where $i < n$. A *path* is a sequence $q_0, \ldots, q_n$ of points in $\mathbb{Q}^2$ such that $\forall 0 \leq i < n\, d(q_i, q_{i'}) = 1$. If $q_0 = q_n$ the path is a *cycle of length n*.

Claim 1 *There are no cycles of odd length.*

PROOF Consider for a contradiction a cycle $q_0, \ldots, q_n$ of odd length n. Wlog we may assume $q_0 = (0,0)$. Since all q_i are rational, we can write $q_i = (x_i/d, y_i/d)$ such that $x_i, y_i \in \mathbb{N}$ and $d \in \mathbb{N}$ is a common denominator and $d(q_i, q_{i'})^2 = (x_{i'}/d - x_i/d)^2 + (y_{i'}/d - y_i/d)^2 = ((x_{i'} - x_i)^2 + (y_{i'} - y_i)^2)/d^2 = 1$. Wlog we may assume that the set $D = \{x_i, y_i, d | 0 \leq i < n\}$ is relatively prime.

- Suppose d hence d^2 is odd. Then for every i exactly one of $(x_{i'} - x_i)$ and $(y_{i'} - y_i)$ must be odd, i.e. $(x_{i'} - x_i) + (y_{i'} - y_i) = 1 \bmod 2$. Hence if $x_i + y_i = k \bmod 2$, then $x_{i'} + y_{i'} = x_i + y_i + (x_{i'} - x_i) + (y_{i'} - y_i) = k + 1 \bmod 2$. Hence if n is odd, then $0 = 1 \bmod 2$. Contradiction.
- Suppose d hence d^2 is even. Then for every i either none or both of $(x_{i'} - x_i)$ and $(y_{i'} - y_i)$ must be odd, i.e. $(x_{i'} - x_i) + (y_{i'} - y_i) = 0 \bmod 2$. Suppose both are odd, i.e. $x_{i'} - x_i = 2x + 1$ and $y_{i'} - y_i = 2y + 1$ for some $x, y \in \mathbb{N}$. Then $(x_{i'} - x_i)^2 + (y_{i'} - y_i)^2 = (2x+1)^2 + (2y+1)^2 = 2 \bmod 4$. This contradicts $d^2 = 0 \bmod 4$. So all differences must be even and since x_0, y_0 are even, x_i, y_i must be even contradicting relative primality of D. $\square$

Let $e_1, e_2, \ldots$ enumerate $\mathbb{Q}^2$. Let $\vec{P_0} = \epsilon$ and repeat the following. Let $\mathbb{Q}^2$ up to e_k be partitioned as $\vec{P_k} = P_k^1, \ldots, P_k^{n_k}$ such that there's a path from $p \in P_k^i$ to $q \in P_k^j$ iff $i = j$. Let $\vec{P} = P_k^{i_1}, \ldots, P_k^{i_{m_k}} \subseteq \vec{P_k}$ consist of sets with a path to e_k.

- If $\vec{P} = \epsilon$, assign (arbitrarily) 0 to e_k and let $\vec{P_{k+1}} = \vec{P_k}, \{e_k\}$.
- Otherwise $\vec{P_{k+1}} = \vec{P_k} \backslash \vec{P}, \{e_k\} \cup \bigcup \vec{P}$, assigning 0 to the first $e_i \in \vec{P}$ and all points on even distance (thru the set), and 1 to the others.

By the claim, this can be done consistently and an ever larger number of e_i's gets a stable assignment. Put e_i's with stable assignment k in Q_k. So: yes.