Reduce to the max

A cofinal strategy for weakly orthogonal higher-order pattern rewrite systems (WOPRSs).

Notions and results needed for WOPRSs can be found in [Oos95, Oos99].

Definition 1 *A simultaneous set U of redex(-occurrenc)es in a term s is* maximal*, if any redex v in s overlaps the head of some redex in U.*

By the tree structure of terms maximal sets can be constructed inside-out, but need not be unique.

Lemma 2 *If $s \mathrel{\circ\!\!\!\!\twoheadrightarrow}_V t$ then $t \mathrel{\circ\!\!\!\!\twoheadrightarrow} s^*$, where s^* is obtained from s by contracting a maximal set U.*

Proof By maximality of U and simultaneity of V, we can define an injection ι mapping every redex $v \in V$ to a redex $\iota(v) \in U$ such that v overlaps the head of $\iota(v)$. By weak orthogonality $s \mathrel{\circ\!\!\!\!\twoheadrightarrow}_{\iota(V)} t$, so $t \mathrel{\circ\!\!\!\!\twoheadrightarrow}_{U/\iota(V)} s^*$ [Oos95, Theorem 5, Prism]. $\Box$

Note that distinct maximal sets may exist, but these must be equipollent hence lead to the same result, justifying our notation s^*. The lemma is a generalisation of [BBKV76, Lemma 3.2.2], [Tak95, Section 1, property (5)], [Nip96, Section 5.2], and [Raa96, Lemma 5.3.3]. Only slightly relaxing weak-orthogonality invalidates the theorem, as witnessed by the term $f(a)$ in the parallel-closed TRS $a \to a$, $f(a) \to f(b)$.

As a standard corollary, we have that the maximal strategy is (hyper-)(head-)normalising and cofinal for WOPRSs, where a strategy is maximal if it contracts maximal sets. This generalises (results for) the Gross-Knuth strategy for λ-calculus and the full-substitution strategy for orthogonal TRSs in the papers cited. Note that it also applies to $\lambda\beta\eta$-calculus, i.e. full-extendedness is not required, so the result does not follow from [Oos99, Theorem 1].

The proof of the lemma avoids notions such as chain of λ's [BBKV76], chain [Vri87] and cluster [BKV98] of redexes, which were introduced to set up a satisfactory residual theory for $\lambda\beta\eta$-calculus, having the same 'nice' properties as that of $\lambda\beta$-calculus. Instead the proof is based on the more general notion of weakly orthogonal projection [Oos99], which applies to all WOPRSs.

References

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