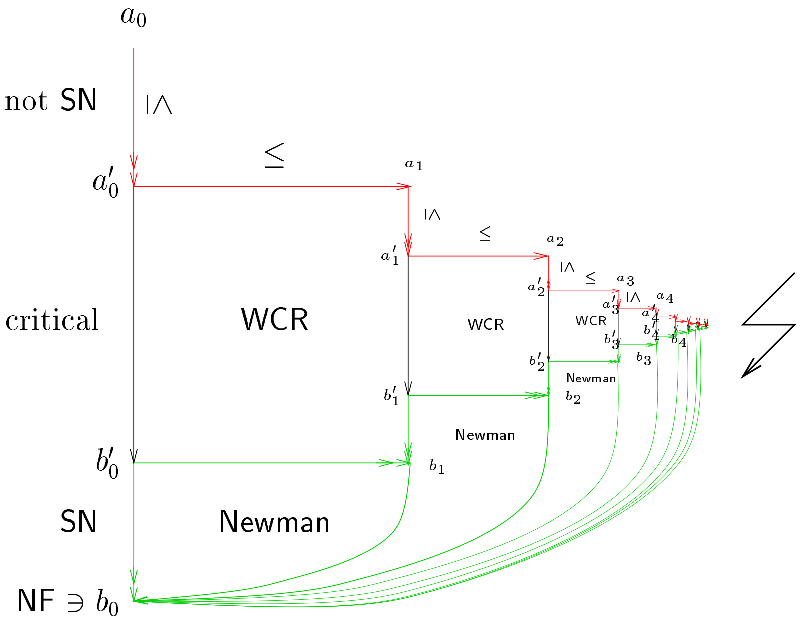
Eventually Increasing

An ARS $\langle A, \to \rangle$ is *eventually increasing* (EI) if there's a map $m{:}A \to \mathbb{N}$ such that, if $a \to b$ then $m(a) \leq m(b)$, and the m-image of any infinite rewrite sequence is eventually increasing. The last part is formalised as : $\mathsf{SN}(\to \cap =_m)$, where $=_m$ denotes m-equality. See [Klo92] for not(at)ions.

Lemma [1] EI & WCR & WN $\Rightarrow$ SN (&CR)

Proof Suppose there's a sequence $\sigma{:}a'_0 \twoheadrightarrow b_0 \in \mathsf{NF}$ which is *unsafe*, i.e. a'_0 also allows an infinite sequence $a'_0 \to a_1 \to \ldots$. Due to well-foundedness of $< \times_{\mathsf{lex}} (\to \cap =_m)^+$, we may require that σ is minimal when measured as $(m(b_0) - m(a'_0)) \times_{\mathsf{lex}} a'_0$. Remark that $\sigma{:}a'_0 \to b'_0 \twoheadrightarrow b_0$ for some b'_0. Since $b'_0 \twoheadrightarrow b_0$ is smaller than σ, it's safe, hence $\mathsf{SN}(b'_0)$. Due to $\mathsf{WCR}(a'_0)$ a_1 and b'_0 have a common reduct b_1, which rewrites to b_0 by Newman's Lemma applied to b'_0. Since $a_1 \twoheadrightarrow b_0$ is smaller than σ, it's safe, hence $\mathsf{SN}(a_1)$. $\square_1$ Instead of using the complex well-founded order, one[2] can reason that an unsafe sequence $a_0 \twoheadrightarrow b_0$ must contain a *critical step*, i.e. a step $a'_0 \to b'_0$ such that not $\mathsf{SN}(a'_0)$, but $\mathsf{SN}(b'_0)$. Using WCR repeatedly (see picture), we find an infinite sequence $a_0 \twoheadrightarrow a_1 \twoheadrightarrow \ldots$, such that $m(a_i) \leq m(b_0)$. So the sequence cannot be eventually increasing. $\square_2$



The lemma is a slight variation on [Klo80, Corollary I.5.19], which instead of EI requires *increasingness* (INC), i.e. if $a \to b$ then $m(a) < m(b)$. To apply it to some rewrite system one first shows that $\mathsf{erasing}$ steps can be transformed away or postponed (possibly introducing $\mathsf{bookkeeping}$ steps), reflecting SN. Then, by constructing a map m which is INC on non-erasing steps and non-decreasing and SN on bookkeeping steps, one reduces SN to (the hopefully simpler) WN. (1) For CRS terms with memory WN $\Rightarrow$ SN, with $\to_{\mathtt{shift}}$ as bookkeeping rule [Klo80, Section II.4]. (2) $\mathsf{SN}(\lambda^{\to})$ is obtained via the non-erasing rule β_I and the bookkeeping rule β_S [Gro93]. (3) $\mathsf{SN}(\mathsf{PN})$ via the bookkeeping rules $\mathtt{box} - \mathtt{box}$ and $\mathtt{contraction} - \mathtt{box}$ and the non-weakening rules as non-erasing rules [Raa96, Section 3.3]. The lemma can also be used to show that (head) needed reduction is a (head) hyper-normalising strategy and to show the finiteness of developments theorem.

References

[Gro93] P. de Groote. The conservation theorem revisited. In TLCA'93, LNCS 664, 1993.

[Klo80] J.W. Klop. Combinatory reduction systems. PhD RUU, 1980.

[Klo92] J.W. Klop. Term rewriting systems. In Handbook of LICS 2, CUP, 1992.

[Raa96] F. van Raamsdonk. Confluence and normalisation for higher-order rewriting. PhD VUA, 1996.

[1] Independently observed by Femke van Raamsdonk.

[2] This proof due to Marc Bezem.