FD à la Melliès[1]

We assume the reader to be familiar with the basic syntactic rewrite theory of first-order term rewriting systems (TRSs), the lambda calculus (Λ), combinatory reduction systems (CRSs), and higher-order pattern rewriting systems (PRSs), in particular with residuals and developments [Klo80, Bar84, Nip91, Klo92, Mel93, Oos97].

Definition 1 *A binary relation on redex(occurrenc)es is called* parting *for a class of rewriting systems, if it can never be satisfied by the residuals of any redex along any rewrite sequence.*

Nesting A redex is said to *nest* another redex if the latter is a proper subterm of the former. Parting of nesting is known as the disjointness property (DP [Klo80, Def. I.4.3.1]). Nesting is trivially parting for TRSs since subterms can only be duplicated, not nested, by performing a rewrite step [Klo80, Rem. I.4.3.2]. As remarked there as well, nesting is non-parting for Λ, hence for PRSs, as witnessed by the rewrite sequence $(\lambda x.xx)\lambda y.R(y) \to_\beta (\lambda y.R(y))\lambda y.R(y) \to_\beta R(\lambda y.R(y))$, where $R(M) =^{\mathrm{def}} (\lambda x.M)N$. Note that in the second step of this sequence a redex created in the first step has been contracted. Indeed, as shown in [Hyl73] contraction of such a created redex is necessary in order to nest residuals of a redex. We present a simple proof of this fact, employing the following well-known formalisation of 'lambda calculus without creation of redexes'.

Definition 2 *The set* $\underline{\Lambda}$ *of* underlined *lambda terms is defined by:*

($\underline{\text{var}}$) $x \in \underline{\Lambda}$*, for all (countable many) variables* x*,*

($\underline{\text{app}}$) $M, N \in \underline{\Lambda} \implies MN \in \underline{\Lambda}$*,*

($\underline{\text{abs}}$) $M \in \underline{\Lambda} \implies \lambda x.M \in \underline{\Lambda}$*,*

($\underline{\text{beta}}$) $M, N \in \underline{\Lambda} \implies (\underline{\lambda} x.M)N \in \underline{\Lambda}$.

The rewrite relation on $\underline{\Lambda}$ *is generated by the rule:* $\underline{\boldsymbol{\beta}}\colon (\underline{\lambda} y.P)Q \to P^{[y\mapsto Q]}$.

A development in the ordinary lambda calculus Λ is defined as a rewrite sequence in the underlined lambda calculus $\underline{\Lambda}$ from which the underlining has been removed. We want to prove that nesting is parting for $\underline{\Lambda}$. Let P denote the subset of $\underline{\Lambda}$ for which nesting is parting.

Lemma 3 ([Hyl73]) $\underline{\Lambda} = \mathrm{P}$.

Proof One proves *for all* $\underline{\Lambda}$*-terms* M *and substitutions* σ *mapping the free variables of* M *to terms in* P*, it holds that* $M^\sigma \in \mathrm{P}$, by induction on the derivation of $M \in \underline{\Lambda}$, from which the lemma follows taking the identity for σ (variables are in P). The only interesting case is for underlined beta redexes.

($\underline{\text{beta}}$) $((\underline{\lambda} x.M)N)^\sigma = (\underline{\lambda} x.M^\sigma)N^\sigma$. M^σ and N^σ are in P by the induction hypothesis, so a non-P rewrite sequence must be of the form $(\underline{\lambda} x.M^\sigma)N^\sigma \twoheadrightarrow_{\underline{\beta}} (\underline{\lambda} x.M')N' \to_{\underline{\beta}} M'^{[x\mapsto N']} \twoheadrightarrow \ldots$. Since (by the substitution lemma) $(\underline{\lambda} x.M^\sigma)N^\sigma \to_{\underline{\beta}} M^{\sigma[x\mapsto N^\sigma]} \twoheadrightarrow_{\underline{\beta}} M'^{[x\mapsto N']}$ and the middle term is in P by the induction hypothesis, it suffices to show P is reflected by the first step. For this, the only problem might be that redexes in distinct copies of N^σ eventually get nested, but since no variable in any copy is bound by a lambda outside it the copies remain non-nested, hence the residuals in them as well. (Formally one shows that if $M^\sigma \to_{\underline{\beta}} \hat{M}$, then $\hat{M} = M'^{\sigma'}$ and either $M \to_{\underline{\beta}} M'$, $\sigma = \sigma'$ or $M = M'^\rho$, $\rho;\sigma \to_{\underline{\beta}} \sigma'$ for renaming ρ.) $\Box$

[1] Prompted by questions of one of the referees of [Oos97] inquiring after the relationship between the proof of finiteness of developments in the paper and the one due to Melliès [Mel93, Mel96]. These remarks to be included in the conference version of that paper.

Remark that substituting SN (strong normalisation) for P yields the proof of the finiteness of developments theorem (FD) for Λ as presented in [Oos97].[2] An alternative way to derive FD, using the lemma as a black box, is due to Micali (see [Klo80, Lem. I.4.3.3]). We give an account of that proof replacing his multiset argument by an appeal to recursive path orders (RPO). $\underline{\Lambda}$-terms are interpreted as first-order terms via the mapping T defined by

$$\begin{aligned} T(x) &=^{\mathrm{def}} c \\ T(MN) &=^{\mathrm{def}} @(T(M), T(N)) \\ T(\lambda x.M) &=^{\mathrm{def}} T(M) \\ T((\underline{\lambda} x.M)N) &=^{\mathrm{def}} i(T(M), N) \end{aligned}$$

where the *weight* i of an underlined beta redex $(\underline{\lambda} x.M)N$ is the number of distinct residuals (i.e. residuals of initially distinct redexes) it nests. By choosing the precedence $j > i > @, c$, for all $j > i$, we have $M \rightarrow_\beta N \implies T(M) >_{RPO} T(N)$, since by P the weight of a redex is always larger than the weight of all redexes it nests, from which FD follows.

In the case of PRSs nesting is non-parting even for developments as remarked in (cf. [Klo80, Rem. II.2.13]) and witnessed by the PRS (which is second-order hence a CRS):

$$\begin{aligned} \mu x.Z(x) &\rightarrow Z(\mu x.Z(x)) \\ a(x) &\rightarrow b(x) \end{aligned}$$

After the step $\underline{\mu} x.\underline{a}(x) \rightarrow \underline{a}(\mu x.\underline{a}(x))$ the residual of the a-redex on the left nests the one on the right.

Holding The problem we are faced with is to find a relation which is parting for developments in higher-order rewriting. We will show that the notion of gripping as introduced by Melliès, in his abstract approach to FD [Mel93], meets these requirements for second-order rewriting (CRSs). A redex is said to *grip* another redex if the latter nests a variable bound by the (pattern of the) former. *Holding* is the transitive closure of gripping. (In the initial term of previous example, the μ-redex holds the a-redex.)

Holding is trivially parting for TRSs, but is non-parting for Λ, hence for PRSs, as witnessed by $(\lambda x.xx)\lambda y.R(y) \rightarrow_\beta (\lambda y.R(y))\lambda y.R(y) \rightarrow_\beta R(\lambda y.R(y)) \rightarrow_\beta (\lambda x.R(x))N$, where $R(M) =^{\mathrm{def}} (\lambda x.Mx)N$. We prove that holding is parting for developments in CRSs. To that end, the definition of underlined system is generalised to PRSs (cf. [Oos97]).

Definition 4 *The underlined version $\underline{\mathcal{P}}$ of a PRS $\mathcal{P}$ is defined by*

1. *The alphabet of $\underline{\mathcal{P}}$ is the alphabet of $\mathcal{P}$ extended with a function symbol $\underline{l \rightarrow r}$ of type θ for every rule $l \rightarrow r$ of type θ in $\mathcal{P}$.*
2. *For every rule $l \rightarrow r$ in $\mathcal{P}$, there's a rule $\underline{l \rightarrow r} \rightarrow r$ in $\underline{\mathcal{P}}$.*

Let P denote the subset of $\underline{\mathcal{P}}$ for which holding is parting, and $\mathrm{P}^{\rightarrow}$ its subset consisting of terms of the form $\vec{X}.\mathsf{a}(\vec{M})$ such that $\mathsf{a}(\vec{M})^{\sigma} \in \mathrm{P}$ for every substitution $\sigma \in \mathrm{P}^{\rightarrow}$ (on a subset of $\vec{X}$). This is well-defined since the types of the terms in σ are smaller than the type of $\vec{X}.\mathsf{a}(\vec{M})$. The set P^n is obtained by bounding substitutions by order n in the definition of $\mathrm{P}^{\rightarrow}$.

Lemma 5 *Let $\mathcal{P}$ be a CRS, then every $\underline{\mathcal{P}}$ term is in* P.

Proof We prove that *every $\vec{X}.\mathsf{a}(\vec{M})$ such that $\vec{X}$ is a vector of* metavariables *and* $\mathsf{a}(\vec{M})$ *a* metaterm *(in CRS-jargon) is in* P^2, *i.e. that for any substitution (*substitute *in CRS jargon)* $\sigma \in \mathrm{P}^1$, $\mathsf{a}(\vec{M})^{\sigma} \in$ P by induction, first on the number of underlined symbols and then on the number of head symbols in the term, from which the lemma follows by taking the identity for σ (variables are trivially in P^1, hence we may also assume that $\vec{X}$ is the domain of σ)[3]. By induction hypothesis we know

[2]The proof of FD is actually simpler than the proof of DP since the reflection argument is not needed.

[3]In fact, this shows P for underlined CRS metaterms implying P for underlined CRS terms.

that $\vec{X}.M_i \in \mathrm{P}^2$ for every M_i among $\vec{M}$. The proof is by cases on a, the only interesting case being the rule case.

(rule) If a is an underlined symbol $\underline{l \to r}$ for some rule $l \to r$ of $\mathcal{P}$, then since its arguments are in P^2 by the induction hypothesis, a rewrite sequence witnessing non-P must look like $\underline{l \to r}(\vec{M}^\sigma) \twoheadrightarrow \underline{l \to r}(\vec{M}') \to r(\vec{M}') \to \ldots$. Since (by the substitution lemma) $\underline{l \to r}(\vec{M}^\sigma) \to r(\vec{M}^\sigma) \twoheadrightarrow r(\vec{M}')$ and the middle term is in P by the induction hypothesis, it suffices to show P is reflected by the first step. For this, the only problem might be that redexes in distinct copies of some ${M_i}^\sigma$ eventually hold one another, but since no second-order variable in ${M_i}^\sigma$ is bound by a λ outside it,[4] the copies remain non-holding, hence the residuals in them as well. (Formally one shows that if $M^\sigma \to \hat{M}$ for non-underlined metaterm M, substitute σ, and (underlined) term $\hat{M}$, then $\hat{M} = M'^{\sigma'}$ with $M = M'^{\rho}$, and $\rho; \sigma \to \sigma'$ for renaming ρ.) $\Box$

Again, substituting SN for P (and omitting the reflection argument) yields a proof of FD for CRSs.

An alternative way to derive FD for CRSs, using the lemma as a ■,[5] is due to Melliès [Mel93]. We present a slight modification of that proof replacing his multiset argument by an appeal to RPO. $\underline{\mathcal{P}}$-terms are interpreted as first-order terms via the mapping T.

$$
\begin{aligned}
T(\vec{X}.Y(\vec{M})) &=^{\mathrm{def}} c(T(\vec{M})) \\
T(\vec{X}.f(\vec{M})) &=^{\mathrm{def}} f(T(\vec{M})) \\
T(\vec{X}.\underline{l \to r}(\vec{M})) &=^{\mathrm{def}} i(T(\vec{M}))
\end{aligned}
$$

where the *weight* i is the number of distinct redexes the redex $\underline{l \to r}(\vec{M})$ holds[6]. Taking the precedence $j > i > f, c$, for all $j > i$, we show that $M \to N \implies T(M) >_{RPO} T(N)$. The correspondence between this proof and the proof in [Mel96, Ch. 3], based on a number of axioms (A,B,C,fd-1,fd-2,fd-3,fd-4,Z-1, and Z-2), is illustrated by decorating the proofsteps with the corresponding axioms.[7]

Proof One roughly shows: $M = C[l^\sigma] \to C[r^\sigma] = N \implies C_T[l_T^{\sigma_T}] >_{RPO} C'_T[r_T^{\sigma'_T}]$, *for some* $C_T \geq_{RPO} C'_T$, $l_T >_{RPO} r_T$, *and* $\sigma_T \geq_{RPO} \sigma'_T$. This decomposition is achieved as follows:

(Ctx) We can take for C_T and C'_T just the parts of the first-order terms ($T(M)$ and $T(N)$) corresponding to the context C, in which weights can only decrease by rewriting due to P (Axiom fd-3), hence $C_T \geq_{RPO} C'_T$.

(Rule) For l_T, we take that part of the first-order term which corresponds to the redexes which are held by l, and for r_T the descendant of that part. Firstly, we show that the so-constructed parts are convex. For l_T this is clear since redexes inbetween the contracted redex and a redex held by it, are held as well (Axiom fd-4). For r_T this holds since for a residual u' of u above a residual v' of a redex v in l_T, we must have that either u itself is held by the contracted redex and belongs to l_T so u' belongs to r_T (Axiom fd-2), or u belongs to C_T and u' belongs to C'_T. By P (Axiom Z-1), the weights of all residuals in r_T are strictly dominated by the weight of the left-hand side l_T, hence $l_T >_{RPO} r_T$.

(Sub) For σ_T and σ'_T, the remaining parts of the first-order terms are collected. Since none of the subterms in σ_T is held by the contracted redex, its descendants can only have been duplicated (Axiom FD-1), and the weights of the redexes in them has not changed, hence $\sigma_T \geq_{RPO} \sigma'_T$. $\Box$

[4] This is exactly what fails to be true in third-order PRSs.

[5] Actually, the FD proof for CRSs in [Mel96, Sec. 7.3.3] is based on the (weaker) property that gripping is acyclic.

[6] Formally c and the is have to be indexed by the number of arguments they take.

[7] The correspondence is not exact (the axioms are much more abstract, e.g. they only concern redexes and their residuals, but the RPO proof employs terms containing non-redex function symbols as well), hence some proofsteps here are not essential, and are not backed up by an axiom in the abstract approach.

Surprisingly, holding need not be parting for developments in orthogonal PRSs in general. Consider the underlined version of the following (orthogonal) PRS:

$$\begin{array}{rcl} g(M.N.X(x.M(x),N)) & \to_\gamma & X(y.X(x.y,I),I) \\ (\lambda x.M(x))N & \to_\beta & M(N) \end{array}$$

(where λ and g are unary function symbols and the binary application operator @ is dropped from the left-hand side of the β-rule as in combinatory logic.) After the step $\underline{g}(M.N.(\underline{\lambda}x.M(x)N)) \to_\gamma (\underline{\lambda}x.(\underline{\lambda}y.x)I)I$ the residual of the β-redex on the left holds the one on the right. This example shows at the same time that the axioms for FD of [Mel96, Ch. 3], are not verified in general by (non-second-order) PRSs.[8]

Since the direct inductive proof of FD above generalises without effort to PRSs (cf. [Oos97]), one could expect to lift the typical (combinatorial) second-order property of gripping to a (logical) property which holds for all orders, and show that that property is parting for developments in any orthogonal PRS. We leave this to further research.

Conclusion We've indicated some relationships between different approaches to the finiteness of developments theorem. The axiomatic approach due to Melliès which is known to apply to CRSs [Mel96], was shown not to apply (without further ado) to PRSs. This is somewhat surprising considering the close correspondence between CRSs and HRSs as established in [OR93], but as remarked there as well, although CRSs and PRSs have the same 'matching power', the latter have more 'rewriting power'. From a technical point of view, properties need to be defined (and checked) for CRSs only up to order 2 (their substitution calculus is second-order), whereas for PRSs they need to be defined for any order (their substitution calculus is simply typed lambda calculus).

References

[Bar84] H.P. Barendregt. *The Lambda Calculus, Its Syntax and Semantics.* North-Holland, 1984.

[Hyl73] J.M.E. Hyland. A simple proof of the Church-Rosser theorem, 1973. Typescript.

[Klo80] J.W. Klop. *Combinatory Reduction Systems.* Proefschrift, Rijksuniversiteit Utrecht, 1980.

[Klo92] J.W. Klop. Term rewriting systems. pp. 1–116, Handbook of Logic in Computer Science 2, 1992.

[Mel93] P.-A. Melliès. An abstract finite developments theorem. Available on: `ftp://theory.doc.ic.ac.uk/theory/CONFER/papers/Mellies/developments.ps.gz`, 1993.

[Mel96] P.-A. Melliès. *Description Abstraite des Systèmes de Réécriture.* Thèse de doctorat, Université Paris VII, 1996.

[Nip91] T. Nipkow. Higher-order critical pairs. pp. 342–349, LICS 6, 1991.

[Oos97] V. van Oostrom. Finite family developments. To be published in RTA 8, 1997.

[OR93] V. van Oostrom and F. van Raamsdonk. Comparing combinatory reduction systems and higher-order rewrite systems. pp. 276–304, HOA'93, 1993.

[8] To be precise, the axioms fd-3 and either acyclicity of gripping or axiom Z-1 cannot hold at the same time.