

Critically steady TRSs are confluent*

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Definition. A variable x is *steady d -depth*¹ in a term t if $t(p) = x$ entails $|p| = d$. We use *steady* as shorthand for steady 1-depth. A term / rule / TRS is *steady* if its variables / sides / rules are. A TRS is *critically steady* if it is steady and its critical peaks can be completed steadily:

- (i) for any *overlay* critical peak $s \blacktriangleleft t \blacktriangleright u$ there is a valley $s \blacktriangleright^= v \blacktriangleleft^= u$ for a term v , where $\blacktriangleright$ denotes a root-step contracting a redex-pattern at depth 0; and
- (ii) for any *outside-in* critical peak $s \blacktriangleleft t \blacktriangleright_d u$ there is a valley $s \blacktriangleright_{\leq d} v \blacktriangleleft^= u$ for a term v , where $\blacktriangleright_d$ denotes a non-root-step, the contraction a redex-pattern at depth $d > 0$, and $\blacktriangleright_{\leq d}$ a non-root-reduction comprising steps that contract redex-patterns at depth > 0 and $\leq d$.

The key observation is that a peak $s \leftarrow t \rightarrow u$ can be written as $s \leq_d \leftarrow t \rightarrow_{\leq d} u$ for some depth d , where $\rightarrow_{\leq d}$ denotes a reduction comprising only steps that contract redex-patterns at depth $\leq d$.

Lemma. *Critically steady TRSs are confluent.*

Proof. By the observation it suffices to prove $\rightarrow_{\leq d}$ is confluent for all d , by induction on d . Since $\rightarrow_{\leq d} = (\blacktriangleright_{\leq d} \cdot \blacktriangleright^=)^*$ it suffices to show $\blacktriangleright_{\leq d} \cdot \blacktriangleright^=$ has the diamond property. A peak has shape $s \blacktriangleleft^= f(\vec{s}) \leq_d \blacktriangleright f(\vec{t}) \blacktriangleright_{\leq d} f(\vec{u}) \blacktriangleright^= u$ for terms s and u , an n -ary function symbol f , and vectors of terms $\vec{s}, \vec{t}, \vec{u}$ of length n , composed from peaks $s_i \leftarrow_{< d} t_i \rightarrow_{< d} u_i$ for $1 \leq i \leq n$.

Then $\vec{s} \rightarrow_{< d} \vec{v} \leftarrow_{< d} \vec{u}$ for some vector of terms $\vec{v}$, that we may choose by the IH to be such that $(\dagger) v_i = v_j$ if v_i and v_j are $\rightarrow_{< d}$ -convertible, in particular, if $\ell(i) = x = \ell(j)$ for $\ell \rightarrow r$ applied in $s \blacktriangleleft^= f(\vec{s})$ or $f(\vec{u}) \blacktriangleright^= u$; cf. [1, Lemma 2]. Then $s \blacktriangleleft^= f(\vec{s}) \blacktriangleright_{\leq d} f(\vec{v}) \leq_d \blacktriangleright f(\vec{u}) \blacktriangleright^= u$ and we claim $s \blacktriangleright_{\leq d} v_1 \blacktriangleleft^= f(\vec{v}) \blacktriangleright^= v_2 \leq_d \blacktriangleright u$ for terms v_1 and v_2 . From the claim we conclude by (i).

To prove the claim, it suffices by symmetry that $s \blacktriangleleft^= f(\vec{s}) \blacktriangleright_{\leq d} f(\vec{v})$ entails $s \blacktriangleright_{\leq d} v_1 \blacktriangleleft^= f(\vec{v})$, which we prove by induction on the $\blacktriangleright_{\leq d}$ -reduction. If $s = f(\vec{s})$, we may set $v_1 := f(\vec{v})$. Otherwise:

- if the i^{th} argument of the redex-pattern of $s \blacktriangleleft^= f(\vec{s})$ overlaps the first step of $s_i \rightarrow_{< d} v_i$, for some i , then by rules being steady the argument is *closed* so *encompasses* the lhs of the latter. We conclude by (ii) for that peak and the IH for the $\blacktriangleleft$ -step of the valley and the tail of the $\blacktriangleright_{\leq d}$ -reduction, noting that per construction $(\dagger)$ holds for a rule $\ell \rightarrow r$ applied in that $\blacktriangleleft$ -step; and
- otherwise, $\ell(i)$ is a variable for exactly the non-empty $s_i \rightarrow_{< d} v_i$, with $\ell \rightarrow r$ the rule applied in the $\blacktriangleleft$ -step. Setting $\sigma(\ell(i)) := s_i$ and $v(\ell(i)) := v_i$ then is well-defined with $\sigma \rightarrow_{< d} v$ by $(\dagger)$ yielding $\ell^\sigma = f(\vec{s}) \blacktriangleright_{\leq d} f(\vec{v}) = \ell^v$. By rules being steady we conclude by $s = r^\sigma \blacktriangleright_{\leq d} r^v =: v_1$. $\square$

Since a TRS is *non-collapsing and shallow* iff it is steady, and critically steady if it also is *weakly non-overlapping* as then $s = u$ in both (i) and (ii), the lemma *entails* [2, Theorem 5]. In fact, it *strengthens* it and by a proof that is *simpler*.² We expect the result and its key technique of *non-depth-increasingness* can be strengthened and also exploited to *refactor* the theory of modularity.

References

- [1] V. van Oostrom. Modularity of confluence; constructed. In *IJCAR*, volume 5195 of *LNCS*, pages 348–363. Springer, 2008. doi:10.1007/978-3-540-71070-7_31.
- [2] M. Sakai and M. Ogawa. Weakly-non-overlapping non-collapsing shallow term rewriting systems are confluent. *IPL*, 110(18):810–814, 2010. doi:10.1016/j.ipl.2010.06.015.
- [3] Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.

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¹Named after the nautical notion of *steady depth*, thinking of redex-patterns and their residuals as submarines.

²Only the second - item above needs to be retained; note that well-definedness of the substitution in it fails for the non-ambiguous, but non-steady, non-confluent TRSs by Klop and Huet [3, Exercise 2.7.20 and Example 4.1.4(i)].