Residuation = Skolemised Confluence

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Abstract

We express local confluence and the diamond property by means of residuation on peaks of steps. We extend residuation to peaks of reductions by means of tiling, and to 3-peaks of faces by means of bricklaying, and investigate some ramifications of our approach.

Skolemising local confluence into residuation. Recall [18, 1] a rewrite system $\to$ is locally confluent (has the diamond property) if for every local *peak* [3] ϕ, ψ of co-initial steps there exist reductions (steps) ψ', ϕ' constituting a *confluence* $C(\phi, \psi, \psi', \phi')$, i.e. reductions such that ϕ, ψ are co-initial, ψ', ϕ' are co-final, and ϕ, ψ' and ψ, ϕ' both compose. From the statement we obtain by introducing two skolem-functions $\backslash$ and $/$ for ψ' respectively ϕ' (binary as they depend on ϕ, ψ), the (equisatisfiable) statement that $C(\phi, \psi, \phi \backslash \psi, \phi / \psi)$ for every local peak ϕ, ψ; see Fig. 1. We will refer to such skolem-functions from peaks to reductions as *residuations*.

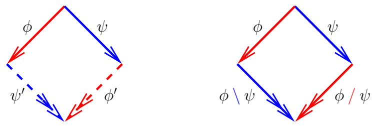

Figure 1: Local confluence (left) and its skolemisation (right)

Exploiting C is symmetric, a single skolem-function $|$ (notation of [12, Sect. 8–12]) will do:

Lemma 1. *$\to$ is locally confluent (has the diamond property) iff there is a* single *residuation $|$ to reductions (steps) such that $C(\phi, \psi, \psi \mid \phi, \phi \mid \psi)$, for all local peaks of steps ϕ, ψ.*

Stated differently, we may assume $\backslash$ is the *converse* of $/$, i.e. $\phi \backslash \psi = \psi / \phi$ for local peaks ϕ, ψ, hence that $C(\phi, \psi, \phi \backslash \psi, \phi / \psi) = C(\phi, \psi, \psi / \phi, \psi \backslash \phi)$.

Residuation by tiling. The aim of both introducing local confluence and the diamond property in [12] was to provide a way to establish *global* confluence by means of repeated *tiling* [3, 12, 10, 17] with *local* confluences. Rephrased in terms of residuation, the aim was to construct residuation for the rewrite system $\twoheadrightarrow$ of *reductions* from the residuation for *steps* $\to$. Tiling can be described by means of a rewrite system $\Rightarrow$ on *conversions* [3, 12, 18]. Tiling rules transform peaks into *valleys* [3]. Formally, to do so we associate to each given confluence $C(\phi, \psi, \chi, \omega)$ a rule $\phi^{-1} \cdot \psi \Rightarrow \chi \cdot \omega^{-1}$, where $\cdot$ and $^{-1}$ denote *composition* and *converse*. Applying such a rule $\ell \Rightarrow r$ to a conversion ς of shape $\zeta \cdot \ell \cdot \xi$ yields $\zeta \cdot r \cdot \xi$. If there is *at least one* local confluence for every local peak ϕ, ψ then $\Rightarrow$-normal forms are valleys, and if there is *at most one* then $\Rightarrow$ has *random descent* [12, 14] meaning that if there exists a $\Rightarrow$-reduction to normal form, then all (maximal) $\Rightarrow$-reductions end in that normal form and all such $\Rightarrow$-reductions have the same length. Thus, since skolemising local confluence or the diamond property yields *exactly one* local confluence $C(\phi, \psi, \phi \backslash \psi, \phi / \psi)$ for every local peak ϕ, ψ, normal forms for the corresponding rules $\phi^{-1} \cdot \psi \Rightarrow (\phi \backslash \psi) \cdot (\phi / \psi)^{-1}$ are valleys and unique (if they exist); cf. [19, Lem. 2].[1] Accordingly, we may extend $\backslash, /$ on peaks of steps to partial functions (with the same notations) on peaks ϕ, ψ of reductions by setting $\phi \backslash \psi := \chi$ and $\phi / \psi := \omega$ if $\chi \cdot \omega^{-1}$ is the $\Rrightarrow$-normal form of $\phi^{-1} \cdot \psi$, and both $\phi \backslash \psi$ and ϕ / ψ to undefined otherwise. This preserves $\backslash$ and $/$ being each other's converse since $^{-1}$ is an involution:

Proposition 2. *$\backslash$ is the converse of $/$ on $\to$ iff the same holds for their extension to $\twoheadrightarrow$.*

Thus, if local confluence or the diamond property is expressed by means of a *single* residuation $|$ on peaks of steps, then so is (partially) its extension to peaks of reductions.

Partiality of extending residuation by tiling. The proviso in the definition of residuation by tiling, turning the extensions $\backslash, /$ into *partial* functions only, is needed as tiling need not terminate; an injudicious choice of residuations for steps may lead to their extension to reductions being partial, even if the rewrite system is confluent.[2] Still, a judicious choice then always *is* available (though non-computably so) due to completeness of *decreasing diagrams* [13, Prop. 2.3.28]:

Theorem 3. *For any countable confluent rewrite system there are residuations on peaks of steps that extend by tiling to residuations on peaks of reductions.*

By inspection of the proof of [13, Prop. 2.3.28], we see the constructed residuations to be each other's converse, so that a locally confluent countable rewrite system is confluent iff there exists a *single* residuation on peaks of steps that extends to a total residuation on peaks of reductions.

Although it is undecidable whether a locally confluent rewrite system $\to$ is confluent, cf. [6], various conditions sufficient for tiling to terminate are known [15]. For instance, if $\to$ is terminating then its local confluence entails termination of tiling by Newman's Lemma [12, Thm. 3], and if $\to$ has the diamond property then tiling is terminating by [12, Thm. 1].

Residuation for reductions by recursion. The following (left, right) unit and composition laws of *residual systems* [18, Sect. 8.7][12, 2, 9] are seen to hold *by tiling*; see Fig. 2:

$$
\begin{array}{rcl}
\phi / \varepsilon & = & \phi \\
\phi \backslash \varepsilon & = & \varepsilon \\
\phi / (\psi \cdot \chi) & \simeq & (\phi / \psi) / \chi \\
\phi \backslash (\psi \cdot \chi) & \simeq & (\phi \backslash \psi) \cdot ((\phi / \psi) \backslash \chi)
\end{array}
\qquad
\begin{array}{rcl}
\varepsilon \backslash \phi & = & \phi \\
\varepsilon / \phi & = & \varepsilon \\
(\phi \cdot \psi) \backslash \chi & \simeq & \psi \backslash (\phi \backslash \chi) \\
(\phi \cdot \psi) / \chi & \simeq & (\phi / \chi) \cdot (\psi / (\phi \backslash \chi))
\end{array}
$$

where ϕ, ψ, χ range over reductions and where $\simeq$ is Kleene-equality expressing that either both sides denote and are equal, or that neither side denotes. The laws justify *defining* extend-

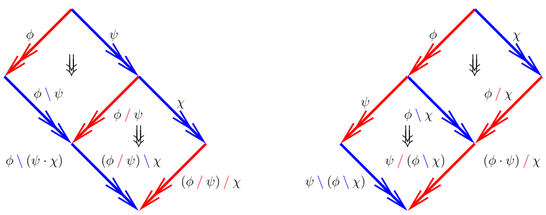

Figure 2: Composition laws for residuation

[1] Unlike [18, 4, 19] we do not assume that $\phi \backslash \phi = \varepsilon$ or $\phi / \phi = \varepsilon$ for a local peak ϕ, ϕ of identical steps.

[2] Seen e.g. by varying on Kleene's [18, Fig. 1.2] example of a locally confluent but not confluent system [12].

ing residuation from steps to reductions by *recursion*, having as base cases peaks where both reductions are steps or one of them is empty, and as recursive clauses:

$$
\begin{array}{lcl}
(\phi \cdot \psi) \,/\, (\chi \cdot \omega) & := & ((\phi \,/\, \chi) \cdot (\psi \,/\, (\phi \,\backslash\, \chi))) \,/\, \omega \\
(\phi \cdot \psi) \,\backslash\, (\chi \cdot \omega) & := & \psi \,\backslash\, ((\phi \,\backslash\, \chi) \cdot ((\phi \,/\, \chi) \,\backslash\, \omega))
\end{array}
$$

where ϕ, χ range over steps and ψ, ω over non-empty reductions, which turn into the following single recursive clause[3] in case of a single residuation $|$, i.e. if $\backslash$ is the converse of $/$:

$$
(\phi \cdot \psi) \,|\, (\chi \cdot \omega) \quad := \quad ((\phi \,|\, \chi) \cdot (\psi \,|\, (\chi \,|\, \phi))) \,|\, \omega
$$

It is justified by the above laws governing the interaction between residuation and composition: $(\phi \cdot \psi) \,|\, (\chi \cdot \omega) \simeq ((\phi \cdot \psi) \,|\, \chi) \,|\, \omega \simeq ((\phi \,|\, \chi) \cdot (\psi \,|\, (\chi \,|\, \phi))) \,|\, \omega$. Vice versa, since the laws give rise to a (this) tiling strategy, the recursive definition is the least (when representing functions as sets of pairs, ordering them by subset) extension of $|$ satisfying them; cf. [4, II Lem. 4.32].

From vertical to horizontal tiling. We show any diagram tiled top–down by confluences can be obtained by tiling left–right, now associating to a confluence $C(\phi, \psi, \chi, \omega)$ both a *vertical* and *horizontal* tiling rule, $\phi^{-1} \cdot \psi \Rightarrow_v \chi \cdot \omega^{-1}$ respectively $\phi \cdot \chi \Rightarrow_h \psi \cdot \omega$. The idea is then to

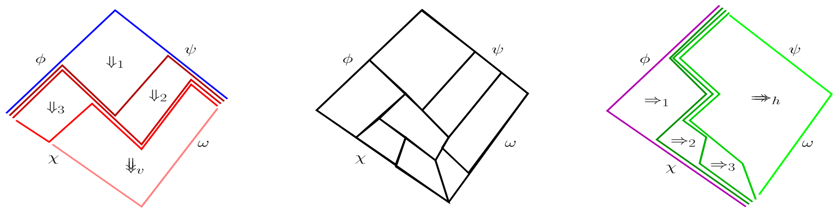

Figure 3: From vertical tiling (left) via tiled diagram (middle) to horizontal tiling (right)

reuse the same tiles, but from left to right instead of from top to bottom; Fig. 3.

Theorem 4. *If $\phi^{-1} \cdot \psi \Rrightarrow_v \chi \cdot \omega^{-1}$ then $\phi \cdot \chi \Rrightarrow_h \psi \cdot \omega$, for reductions ϕ, ψ, χ, ω.*

As Fig. 3 suggests, the proof can easily be adapted to show the numbers of vertical and horizontal tiles indeed to be the same, and to the case of *commutations* $C(\phi, \psi, \chi, \omega)$ where ϕ, ω are reductions of one rewrite system $\rightarrow$ and ψ, χ of another $\rightsquigarrow$; that conversely $\phi \cdot \chi \Rrightarrow_h \psi \cdot \omega$ entails $\phi^{-1} \cdot \psi \Rrightarrow_v \chi \cdot \omega^{-1}$ is then obtained by *symmetry* [16], for $\leftarrow$ and $\rightsquigarrow$. The theorem is simpler to state and prove, and holds for *global* confluences and commutations, compared to just for *local* confluences as in [4, II Sect. 4.2, Lem. 4.24, Prop. 4.34].

3-confluence. Call a rewrite system $\rightarrow$ (*locally*) 3-*confluent* if every 3-*peak* ϕ, ψ, χ of co-initial reductions (steps) can be completed by 9 reductions into a (*local*) *brick* as in Fig. 4 left. (If local 3-confluence holds using *steps*, $\rightarrow$ has the *cube* property.) If local confluence is given by a residuation $|$ that extends to a total residuation on reductions, local 3-confluence may fail due to failure of the *cube* law [11] $(\varsigma \,|\, \zeta) \,|\, (\xi \,|\, \zeta) = (\varsigma \,|\, \xi) \,|\, (\zeta \,|\, \xi)$. This failure is well-known (since at least the 90s) for systems such as positive braids, TRSs and the $\lambda\beta$-calculus; cf. [16]. (Even if the diamond property holds, the cube property may fail; cf. gadget qp2 in Fig. 5.) We

[3] In Haskell: `resred (i:u) (j:v) = resred ((resstp i j)++(resred u (resstp j i))) v`, where `resstp` is the given residuation on steps and `resred` its extension to reductions, represented as lists of steps.

give ways to extend local 3-confluence to 3-confluence by *bricklaying*. *Decreasing diagrams* [13] (DD) is one way, when defining a brick to be 3-*decreasing* if its 6 *faces* are decreasing. This is shown by induction, measuring a 3-peak by the multiset sum of the *lexicographic maximum* measures [13] of the 3 reductions in it, with the decrease of measure from the 3-peak $\phi, \psi \cdot \bar{\psi}, \chi$ to $\phi', \bar{\psi}, \chi'$ visualised on the right in Fig. 4, and the induction step(s) on the left.

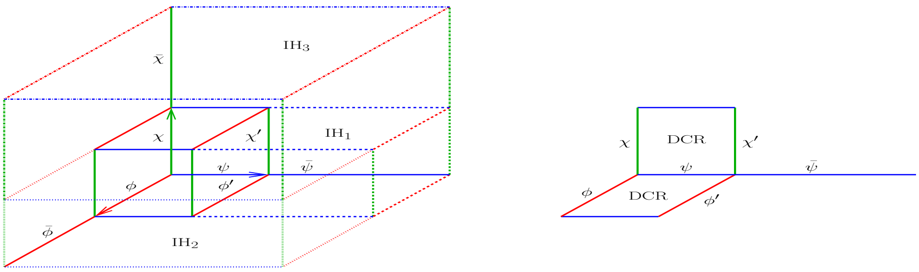

Figure 4: Induction (left) and decrease in measure (right) of 3-decreasingness by bricklaying

Theorem 5 (3-DD)**.** *A locally* 3*-decreasing rewrite system is* 3*-decreasing; cf. [13, Thm. 2.3.20].*

As corollaries we obtain that systems that are terminating and locally 3-confluent, or that have the cube property, are 3-confluent. Using that DD is complete for countable confluent systems, cf. [13, Prop. 2.3.28], we even have that *any* such system is 3-confluent, if we are free to *choose* residuation to 'go to' the *least common reduct* on a chosen spanning forest [5, 8].

Lemma 6. *A countable confluent rewrite system is locally* 3*-decreasing for some residuation.*

Bricklaying. Analogously to how tiling a peak of reductions with local confluences turns it into a valley, *bricklaying* with local bricks is a way to turn a 3-*peak* into a 3-*valley*. Also analogously: an injudicious choice of local bricks may lead to non-termination of bricklaying (even if the system is 3-confluent). We define bricklaying in an attempt to make sense of that. Whereas (2D) tiling has (1D) *conversions* as intermediate stages, we introduce (2D) *beds* as intermediate stages of (3D) bricklaying. As a first approximation to beds we use graphs having (red, blue, green) *coloured* edges to model steps in 3 dimensions, which we then embed. (Hypermaps as in [7] could be a suitable alternative; we leave this to future research.)

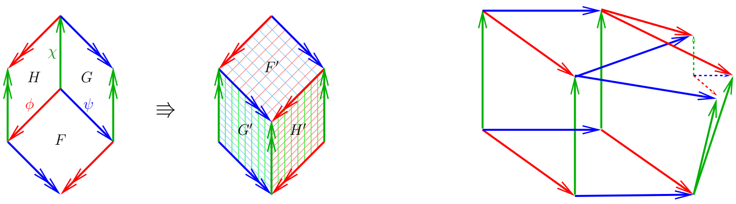

Figure 5: Bricklaying (left) and gadget qp2 (right; dashed arrows indicate recursive nature)

Definition 7. A *bed*-dag for a rewrite system → is a finite connected dag having nodes labelled by objects of → and edges having a unique colour and labelled by steps of → such that the label of the source (target) of an edge is the source (target) of its label, satisfying: (i) the dag is the union of its tiles, where a *tile* is a red–blue (blue–green, green–red) *tetragonal* cycle

$\leftarrow^+ \cdot \leftarrow \cdot \rightarrow \cdot \rightarrow^+$ such that interior nodes of its red (blue) valley path don't have blue (red) in-edges, (ii) a node has at most 1 in-edge and at most 1 out-edge of a given colour; (iii) there are no paths having edges of all 3 colours; (iv) every $\rightarrow \cdot \rightarrow$ (consecutive edges of distinct colours) belongs to some tile; and (v) the source of the edges of a local peak $\leftarrow \cdot \rightarrow$ (edges of distinct colours) not belonging to a tile, has an in-edge of the 3rd colour (here green).

A bed-dag is a *bed* if nodes are elements of $\mathbb{R}^3$ and a red (blue, green) edge is a pair of nodes such that only its 1st (2nd, 3rd) coordinates differ, with the target greater than the source.

The *boundary* of a dag is its subgraph of edges belonging to exactly one tile. A *bricklaying* step $\Rrightarrow$ replaces a bed-dag L by another R, fresh except for having the same *hexagonal* cycle $\leftarrow^+ \cdot \rightarrow^+ \cdot \leftarrow^+ \cdot \rightarrow^+ \cdot \leftarrow^+ \cdot \rightarrow^+$ as boundary, where L is the union of 3 tiles pairwise sharing only one peak edge, a *redex*, and R is without nodes of out-degree 3, a 3-*valley*; see Fig. 5 left.

We designed bed-dags such that they are preserved by $\Rrightarrow$, and beds such that for *any* node of out-degree 3, its 3 peaks lie in orthogonal planes, giving a 3-peak having the valleys as boundary.

Definition 8 (bricklaying)**.** Assuming local 3-confluence for a (total) residuation $|$ on steps, proceed in III phases: (I) We reify[4] the empty reduction as the *empty* step ε, update residuation by setting $\phi \mid \psi$ to the *step* ε if it was the empty *reduction*, and set $\varepsilon \mid \phi := \varepsilon$ and $\phi \mid \varepsilon := \phi$; (II) We represent a 3-peak as a *flat*bed, a bed that need not satisfy condition (v), in the way illustrated in Fig. 6, and associate to each tiling step $\phi^{-1} \cdot \psi \Rightarrow (\psi \mid \phi) \cdot (\phi \mid \psi)^{-1}$ a *flat* bricklaying step replacing a subgraph with boundary $\phi \cdot \varepsilon \cdot \phi^{-1} \cdot \psi \cdot \varepsilon^{-1} \cdot \psi^{-1}$ by a tile with boundary $(\psi \mid \phi)^{-1} \cdot \varepsilon \cdot \phi^{-1} \cdot \psi \cdot \varepsilon^{-1} \cdot (\phi \mid \psi)$, *scaling* it to make it fit in the bed. For reductions ϕ, ψ then $\phi^{-1} \cdot \psi \Rrightarrow (\psi \mid \phi) \cdot (\phi \mid \psi)$ iff flat bricklaying terminates for the 3-peak ϕ, ψ, ε (where ε is the empty reduction, not the empty step) in a bed, with inside it a 3-*valley* with tetragonal boundary $(\psi \mid \phi)^{-1} \cdot \phi^{-1} \cdot \psi \cdot (\phi \mid \psi)$. This allows to produce 3-valleys for each of the 6 faces F, G, H, F', G', H' of a bricklaying step $\Rrightarrow$ for 3-peak of steps ϕ, ψ, χ as in Fig. 5; (III) A lhs L for local 3-peak ϕ, ψ, χ is the bed obtained by pasting F, G, H along theses 3 shared steps.[5] The rhs R is the 3-valley obtained by pasting the red–blue F', blue–green G' and green–red H' 3-valleys along their 'shared' reductions; *caveat*: in general these are *not* shared due to having reified empty steps, as illustrated for the reduction 'shared' between F' and H' on the right in Fig. 6. We overcome this by a process we dub *buttering*, *refining* both 3-valleys by inserting εs in tiles, and *rescaling* the opposite reductions (with a knock-on effect through the bed).[6]

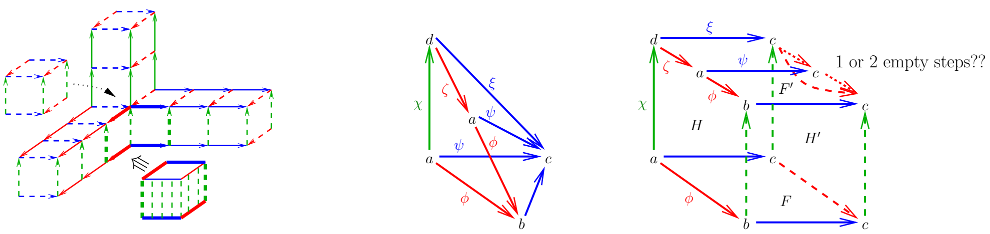

Figure 6: Embedding a conversion as a flatbed using tiles with empty (dashed) steps (left), and reifying empty reductions in a local 3-confluence may break it (right)

Just like tiling, bricklaying has random descent if $\Rrightarrow$ is deterministic (if the three faces of a 3-peak determine the rhs), hence the bricklaying *strategy* implicit in the proof of Thm. 5 (Fig. 4

[4] An idea due to Klop to keep confluence diagrams *rectangular* in drawings. It was given a *description but not a definition* in [10]. Our *buttering* keeps 3-confluence diagrams rectangular in drawings; cf. Fig. 6 right.

[5] *Caveat*: we get different Ls for different local confluences *with empty steps* for the 3 local peaks of steps.

[6] This *is* possible; in the example we insert 1 ε in H'; as the opposite side is 1 (ε-)step no rescaling is needed.

left) being terminating in the decreasing case, then entails termination of bricklaying, yielding a residuation satisfying the cube law *per construction*; residuals can be read off from the bed.

Conclusion We have identified residuation as skolemised confluence, and studied computing residuation via tiling of local peaks in 2D yielding valleys, and via bricklaying of local bricks in 3D yielding 3-valleys, giving confluence respectively 3-confluence (such that Lévy's cube law holds). We introduced 3-decreasingness as a way to show 3-confluence by means of local bricks.

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