

Innermost rewriting is worse than rewriting for right-linear overlay term rewrite systems*

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Let $\triangleright$ be the rewrite system underlying a TRS that is RLOV, *i.e.* *right-linear* (variables occur at most once in rhss) and *overlay* (lhs may only root-overlap), and let $\blacktriangleright := \bigcup_p \triangleright_p$ be its innermost strategy, for p ranging over the positions of the contracted (innermost) redex-patterns.

Lemma. $\blacktriangleright$ copies $\triangleright$ in the sense of [2, Definition 8], *i.e.* for $s \triangleleft t \triangleright_p$, either $s \triangleleft \cdot_p \triangleleft t$ or q exists with $(t \triangleright_p t' \text{ implies } s \triangleright_q \cdot^+ \triangleleft t')$, and $s \triangleright_q s' \text{ implies } t \triangleright_p \cdot \triangleright^+ s'$.

Proof. Consider a $\triangleright$ -step ϕ for rule $\ell \rightarrow r$ from $t := C[\ell^\sigma]_q$ to $C[r^\sigma]_q =: s$, and let p be the position of an innermost redex-pattern in t . We distinguish cases on the relative positions of p and q :

($p \leq q$) by p and ϕ being innermost $p = q$ so the *either*-case of copying holds: $s \triangleright_p t$;

($q \prec p$) by RLOV, then $p = q \cdot p_\ell \cdot p'$ for positions p_ℓ, p' such that $\ell(p_\ell) = x$ for some variable x :

- if x does not occur in r , picking any $\sigma(x) \triangleright_{p'} u$ the *either*-case holds: $s = C[r^\sigma]_q = C[r^v]_q \triangleleft C[\ell^v]_q \triangleleft \cdot_p \triangleleft C[\ell^\sigma]_q = t$ for $v := \sigma[x \mapsto u]$ and the $\triangleleft$ contracting in the other occurrences of x ;

- otherwise x occurs once in r , say at position p_r , by RLOV. Then the *or*-case holds for $q \cdot p_r \cdot p'$:

(i) suppose $t := D[\sigma(x)]_{q \cdot p_\ell} \triangleright_p D[u]_{q \cdot p_\ell} =: t'$ for some D, u, t' . Then $s = C[r^\sigma] \triangleright_{p \cdot p_r \cdot p'} C[r^v] \triangleleft C[\ell^v] \triangleleft D[u] = t'$ for v and $\triangleleft$ defined as in the previous item;

(ii) suppose $s := C[r^\sigma]_q \triangleright_{q \cdot p_r \cdot p'} s'$ for some s' . By RLOV we can write s' as $C[r^v]$ for $v := \sigma[x \mapsto u]$ and $\sigma(x) \triangleright_{p'} u$ for some u . Therefore, $t' = C[\ell^\sigma] \triangleright_p \cdot \triangleright C[\ell^v] \triangleright C[r^v] = s'$ as before;

($p \parallel q$) then we proceed as in the second, *otherwise*-case of the previous ($q \prec p$)-item.¹ $\square$

By [2, Definition 6, Theorem 5] $\blacktriangleright$ is *worse* than $\triangleright$: for every maximal $\triangleright$ -reduction from a term t there is a $\blacktriangleright$ -reduction from t that is *at least as long* and that ends in the same term if finite. This *strengthens* the main results, Theorems 5 and 3.8 of [4] respectively [1], stating only that termination of $\blacktriangleright$ entails that of $\triangleright$,² and by a *simpler* proof. More generally, we expect that the *term rewriting* / λ -calculus literature is riddled with results that, similarly, are consequences / instances of the qua(l)itative techniques for comparing rewrite systems and strategies as developed in [2, 3], and think it should be worthwhile to *refactor* their complex proofs.

References

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- [4] M. Sakai. Innermost terminating right-linear overlay term rewriting systems are terminating, October 2003. URL: <https://www.trs.css.i.nagoya-u.ac.jp/~sakai/papers/sin-sn-031015.pdf>.

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¹But the analysis is simpler since steps directly commute; there is no need for the *balancing* $\triangleright$ -reductions.

²The result of [1] is a bit weaker, as its proof uses finite branchingness (contrary to what is claimed on its p.2).