Redeeming Newman; orthogonality in rewriting
Past, present and future in a 1-algebraic setting

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Abstract

Despite sixty percent of Newman's seminal 1942 paper being devoted to residual theory, that remains obscure due to that his instantiation of the theory there to the (non-erasing) $\lambda\beta$-calculus was fatally flawed. We redeem the approach showing: 1) any rewrite system instantiating his theory induces a so-called 1-ra, an axiomatically orthogonal rewrite system, entailing co-initial reductions have *least* upperbounds; 2) the rewrite system underlying any (non-erasing) syntactically orthogonal TRS instantiates his theory.

Rewriting The primary notion in rewriting is from [17][1]: a *rewrite* system $\to := \langle \mathcal{O}, \mathcal{S}, \mathsf{src}, \mathsf{tgt} \rangle$ comprises $\mathsf{objects}$ $\mathcal{O}$ and steps $\mathcal{S}$ with source, target maps src, tgt from the latter to the former [26, Def. 8.2.2]. Steps ϕ, ψ are *co-initial* if $\mathsf{src}(\phi) = \mathsf{src}(\psi)$, *co-final* if $\mathsf{tgt}(\phi) = \mathsf{tgt}(\psi)$ and *parallel to* each other if both. A *morphism* from $\to$ to $\to' := \langle \mathcal{O}', \mathcal{S}', \mathsf{src}', \mathsf{tgt}' \rangle$ preserves structure; it maps $\phi \in \mathcal{S}$ with source a and target b, denoted by $\phi : a \to b$, to $\phi' : a' \to b'$ in $\mathcal{S}'$. Rewrite properties [2, 26] pertain to various rewrite systems *constructed from* $\to$. *E.g.*, the *Church–Rosser* property [5, 17] expresses that for any *conversion* there exists a valley of co-final *reductions* parallel to it, with (finite) reductions and conversions rewrite systems *constructed from* $\to$. As constructions here we will use the 1-operations of *loop* 1, *composition* $\cdot$, *reverse* ${}^{-1}$ and *residuation* $/$, where by a 1-*operation* we mean an operation respecting sources and

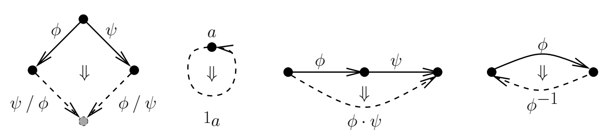

Figure 1: Step-forming operations: *residuation* $/$, *loop* 1, *composition* $\cdot$, *reverse* ${}^{-1}$

targets as depicted by their *generalised* arities [22, Ex. 5.3] in Fig. 1. *E.g.*, composition has two consecutive steps as input (the *full* arrows) and a single step as output (the *dashed* arrow), parallel to each other as depicted. That is, for each 1-operation its *input* arity is the universally quantified (full) subsystem, and its *output* arity is the existentially quantified (dashed) subsystem. The steps in the input arity of composition being *consecutive* captures that composition is only defined on consecutive steps, i.e. if $\mathsf{tgt}(\phi) = \mathsf{src}(\psi)$ for steps ϕ, ψ. Similarly, residuation $/$ requires *co-initial* steps in its input.

1-algebras To algebraically deal with 1-operations requires to enrich universal algebra. Whereas 1, $\cdot$ and ${}^{-1}$ and laws for them (see Def. 1) are known to be covered by *essentially algebraic theories* [20, Ex. 4], the generalisations needed to smoothly deal with $/$ are *in statu nascendi* [22]. Acknowledging this, we refer to algebras having a rewrite system $\to$ as carrier and 1-operations among those in Fig. 1 as 1-*algebras*. This allows to reuse, as we do, terminology from algebra.

[1] In Newman's combinatorial topology-inspired words: *We are concerned with two kinds of entities, "objects" and the "moves" performed on them, and each move is associated with two objects, "initial" and "final."*

l-unit(ϱ)	:	$1 \cdot \varrho$	$\Rightarrow$	ϱ		*invol-id*	:	1^{-1}	$\Rightarrow$	1
r-unit(ϱ)	:	$\varrho \cdot 1$	$\Rightarrow$	ϱ		*anti-auto*(ϱ,ς)	:	$(\varrho \cdot \varsigma)^{-1}$	$\Rightarrow$	$\varsigma^{-1} \cdot \varrho^{-1}$
assoc$(\varrho,\varsigma,\zeta)$	:	$(\varrho \cdot \varsigma) \cdot \zeta$	$\Rightarrow$	$\varrho \cdot (\varsigma \cdot \zeta)$		*invol*(ϱ)	:	$(\varrho^{-1})^{-1}$	$\Rightarrow$	ϱ

Table 1: 1-algebra laws / proofterm rewrite rules

Definition 1. A 1-*monoid* is a 1-algebra with 1-operations 1 and $\cdot$ satisfying the (pertaining) laws in Tab. 1.[2] We use $\twoheadrightarrow$ to denote (the carrier of) the *free* 1-monoid induced by $\to$ and refer to its elements as *reductions* [26, Def. 8.2.10]. A 1-*involutive* 1-*monoid* is a 1-algebra with 1-operations 1, $\cdot$ and $^{-1}$ satisfying the laws in Tab. 1.[2]. We use $\leftrightsquigarrow$ to denote (the carrier of) the *free* 1-involutive 1-monoid, *cf.* [7], induced by $\to$ and refer to its elements as *conversions*.

Example 1. Any algebra can be viewed as a 1-algebra by viewing its carrier as a single-object rewrite system having a step on it for each element of the algebra. Accordingly, algebra examples of 1-monoids are $\langle \mathbb{Z}, 0, + \rangle$ and $\langle \mathbb{N}, 0, + \rangle$, and algebra examples of 1-involutive 1-monoids are $\langle \mathbb{Z}, 0, +, (-) \rangle$ and $\langle \mathbb{N}, 0, +, \mathsf{id} \rangle$. A 1-algebra example is (finite) walks in a graph with operations *empty*, *composition*, and *reverse*, or more generally paths in space.

Freeness of $\twoheadrightarrow$ means that any morphism from $\to$ to a 1-monoid factors into a morphism from $\to$ to $\twoheadrightarrow$ and (a 1-monoid preserving) one from $\twoheadrightarrow$ to the 1-monoid, and similarly for $\leftrightsquigarrow$.

Example 2. The *length*-morphism maps each step ϕ to 1 in the involutive monoid $\langle \mathbb{N}, 0, +, \mathsf{id} \rangle$. It factors through mapping the *step* ϕ to the *conversion* ϕ, and so does the *relation*-morphism mapping ϕ to $(\mathsf{src}(\phi), \mathsf{tgt}(\phi))$ in the *equivalence* closure, *convertibility*, of the rewrite relation of $\to$, equipped with the expected operations. Similarly, that same morphism into the *reflexive–transitive* closure, *reducibility*, factors through mapping the *step* ϕ to the *reduction* ϕ.

(1-involutive) 1-monoids are (dagger) categories, and them being essentially algebraic means that free such can be defined syntactically [20]: letting the operations 1, $\cdot$ and $^{-1}$ double as *function symbols* of arities 0, 2 and 1, one can inductively build terms from the steps of $\to$ respecting sources and targets. Such terms we refer(red) to as *proofterms* [26, Ch. 8] as they are *terms* that can be conceived of as *proofs* (of *reducibility* of their source and target in case of $\twoheadrightarrow$ and of their *convertibility* in case of $\leftrightsquigarrow$) in (sub-)equational logic(s) [18] induced by $\to$. To quotient out the (pertaining) laws from proofterms, one may use (proof)term rewriting itself: orienting the laws into rules on proofterms as in Tab. 1 yields a complete (confluent and terminating) (proof)term rewrite system $\Rightarrow$, with a $\Rightarrow$-normal form being either a single 1 or a right-branching $\cdot$-tree of (reversed) $\to$-steps, *i.e.* $\Rightarrow$-normal forms are in 1–1 correspondence with the usual notion of a reduction (conversion) as sequence of forward (and backward) $\to$-steps [5, 26]. This then allows to *define* the 1-operations 1, $\cdot$ (and $^{-1}$) on reductions (and conversions) as the proofterm-forming operation of applying the corresponding function symbol followed by $\Rightarrow$-normalisation [7, App. A].

Example 3. As running example we employ Kleene's rewrite system $\to$ [26, Fig. 1.2] comprising the four steps $\phi : a \to a'$, $\phi' : a' \to a$, $\psi : a \to b$, and $\chi : a' \to c$. Then $\phi \cdot \psi$ is not a proofterm since the target a' of its 1$^{\text{st}}$ step ϕ is distinct from the source a of its 2$^{\text{nd}}$ step ψ. Among proofterms $\varrho := (\phi'^{-1} \cdot \phi^{-1})^{-1}$, $\varrho' := \phi \cdot \phi'$, $\varsigma := (\phi' \cdot \phi) \cdot \phi^{-1}$ and $\varsigma' := \phi' \cdot (\phi \cdot \phi^{-1})$, both ϱ' and ς' are conversions, but only the former is a reduction. Because both ϱ and ς are $\Rightarrow$-reducible neither is a conversion; their $\Rightarrow$-normal forms are, ϱ' and ς'.

[2] For the moment reading the proofterm rewrite rules as laws (symmetrically) for the 1-operations. We have left the assumptions [20, Ex. 4] on sources and targets of $\varrho, \varsigma, \zeta$ and 1 implicit, to stress similarity with algebra.

Orthogonality We recast the diamond and cube properties, *cf.* [26, Sec. 8.7], 1-algebraically.

Definition 2. $\to$ has the *diamond* property (DP), if *for all* co-initial ϕ,ψ, *there exist* co-final ψ',ϕ' such that ${}^{\phi}_{\psi'}\diamond^{\psi}_{\phi'}$, where ${}^{\phi}_{\psi'}\diamond^{\psi}_{\phi'}$ denotes that ϕ,ψ,ψ',ϕ' constitute a *diamond*: $\mathsf{src}(\phi) = \mathsf{src}(\psi)\,\&\,\mathsf{tgt}(\phi) = \mathsf{src}(\psi')\,\&\,\mathsf{tgt}(\psi) = \mathsf{src}(\phi')\,\&\,\mathsf{tgt}(\psi') = \mathsf{tgt}(\phi')$. $\to$ is *confluent* if $\twoheadrightarrow$ has the DP.

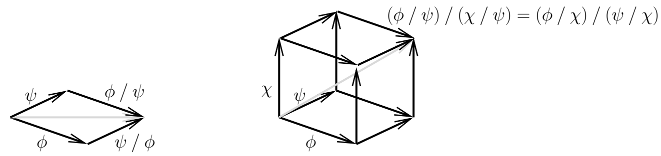

Figure 2: The diamond property ($\diamond$; left) *vs.* the cube property (⧉; right)

Using that a *peak* (*valley*) [5] is a conversion of shape $\leftarrow\cdot\to$ ($\to\cdot\leftarrow$), DP expresses that for every peak there exists a valley parallel to it. Using a skolem-function / (*residuation* in Fig. 1) to witness ψ' by $\psi\,/\,\phi$ and ϕ' by $\phi\,/\,\psi$,[3] DP can be expressed 1-algebraically as: *for all* co-initial ϕ,ψ, ${}^{\phi}_{\psi/\phi}\diamond^{\psi}_{\phi/\psi}$. Thus, $\to$ has the DP iff $\langle\to,/\rangle$ is a 1-algebra for *some* residuation /.

Example 4. In Ex. 3, for the $\twoheadrightarrow$-peak $\psi^{-1}\cdot\phi$ the $\twoheadrightarrow$-valley $\psi^{-1}\cdot\phi'^{-1}$ is parallel to it, but $\to$ is not confluent since for the peak $\psi^{-1}\cdot(\phi\cdot\chi)$ there is no valley (b and c are $\to$-normal forms).

In [17, Sec. 1], Newman explained confluence of a rewrite system $\to$ in terms of its reducibility quasi-order $\twoheadrightarrow$ having *upperbounds*. He left the determination of conditions required for having *least* upperbounds for later. We put forward such conditions in [26, Sec. 8.7] in the form of the laws on residuation in Tab. 2, proposing to call any rewrite system satisfying (1)–(4) *orthogonal*, *cf.* [15, 9] (Fig. 2 depicts law (4) going back to [17, Thm. 5(Δ_4)] dubbed *cube* in [13, Lem. 2.2.1]). Here we recast that account 1-algebraically to then instantiate it in the next sections.

$$\phi\,/\,1 = \phi \tag{1}$$
$$\phi\,/\,\phi = 1 \tag{2}$$
$$1\,/\,\phi = 1 \tag{3}$$
$$(\phi\,/\,\psi)\,/\,(\chi\,/\,\psi) = (\phi\,/\,\chi)\,/\,(\psi\,/\,\chi) \tag{4}$$
$$\chi\,/\,(\phi\cdot\psi) = (\chi\,/\,\phi)\,/\,\psi \tag{5}$$
$$(\phi\cdot\psi)\,/\,\chi = (\phi\,/\,\chi)\cdot(\psi\,/\,(\chi\,/\,\phi)) \tag{6}$$
$$1\cdot 1 = 1 \tag{7}$$

Table 2: Laws of a 1-ra (left) and of a 1-rac (also right)

Definition 3. A 1-*residual algebra* (1-*ra*) is a 1-algebra $\langle\to,1,/\rangle$ such that (1)–(4) in Tab. 2 hold. A 1-*rac* (1-ra with *composition*) is a 1-algebra $\langle\to,1,/,\cdot\rangle$ such that (1)–(7) hold.

Example 5. $\langle\mathbb{N},0,\dot{-},+\rangle$ is a 1-rac, so $\langle\mathbb{N},0,\dot{-}\rangle$ is a 1-ra, for $\dot{-}$ *monus* (cut-off subtraction).

In a 1-ra(c) there is a *natural* order on co-initial steps given by $\phi\preceq\psi := (\phi\,/\,\psi = 1)$. Quotienting out $\preceq\cap\succeq$ yields a 1-ra(c) again whose natural order is a partial order [26, Lem. 8.7.25(iii) and 8.7.41(ii)]. Key to referring to a rewrite system $\to$ constituting a 1-ra as being *orthogonal*, is that any such induces a 1-rac on $\twoheadrightarrow$ [26, Lem. 8.7.47], which then has *least* upperbounds [26, Exc. 8.7.40(ii)]. This is characterised, using categorical language to be (ex/comp)act, by:

Theorem 1 (*cf.* [25, 15, 21])**.** *$\langle\to,1,/,\cdot\rangle$ is a 1-rac whose natural order is a partial order, where $\phi\,/\,\psi := \phi'$ for every peak ϕ,ψ and its pushout valley ψ',ϕ' (in the categorical sense) iff $\langle\to,1,\cdot\rangle$ is a 1-monoid that is* left-cancellative *(each χ is* epi*: for all ϕ,ψ, if $\chi\cdot\phi = \chi\cdot\psi$ then $\phi=\psi$),* gaunt *(isomorphisms are 1) and has* pushouts*;* i.e. *lubs of peaks exist.*

[3] We *a priori* get 2 skolem-functions, $f(\phi,\psi)$ for ψ' and $g(\phi,\psi)$ for ϕ', but may assume $f(\phi,\psi) = g(\psi,\phi)$.

Redeeming Newman In 1942 in [17], Newman refactored the proof of the Church–Rosser property for the λI-calculus in [5], by abstracting from the λ-*term-structure* of objects, factoring the proof through an *axiomatisation* of a function $|$ entailing confluence [17, Sec. 8–12], and showing the axioms to be satisfied for the λI-calculus [17, Sec. 13,14]. The latter was later found to be erroneous [23] due to confusing variables when working with λ-terms modulo α-equivalence.[4] We redeem his approach, showing in this section his main result [17, Thm. 5] factors through orthogonality, and in the next that it applies to (non-erasing) OTRS. To present his result,[5] let $|$ yield for co-initial ϕ, ψ a (finite) *set* $\phi|\psi$ of $\rightarrow$-steps from $\mathsf{tgt}(\psi)$, the ψ-*derivates* of ϕ; it lifts to (finite) *sets* Φ of steps by $\Phi|\psi{:=}\bigcup_{\phi\in\Phi}\phi|\psi$ and to reductions by $\Phi|(\varrho{\cdot}\psi){:=}(\Phi|\varrho)|\psi$ and $\Phi\,|\,1 := \Phi$. A *development* of Φ is a $\rightarrow$-reduction in which only derivates of steps in Φ occur and no remain [5, 3, 26]. Using Newman's notions and our notations, his result reads:

Theorem 2 ([17])**.** *For co-initial reductions ϱ, ς and set of steps Φ, there are reductions ς', ϱ' such that ${}^{\varrho}_{\varsigma'}\!\diamond^{\varsigma}_{\varrho'}$ and $\Phi\,|\,(\varrho\cdot\varsigma') = \Phi\,|\,(\varsigma\cdot\varrho')$ if axioms Δ_1–Δ_4 hold and J_1,J_2 for a predicate J:*

- *(Δ_1) $\phi\,|\,\psi = \emptyset$ iff $\phi = \psi$;*
- *(Δ_2) if $\phi \neq \psi$, then $(\phi\,|\,\chi) \cap (\psi\,|\,\chi) = \emptyset$;*
- *(Δ_3) if $\phi \neq \psi$, then there exist co-final developments ϱ of $\psi\,|\,\phi$, and ς of $\phi\,|\,\psi$;*
- *(Δ_4) for ϱ and ς in (Δ_3), $\chi\,|\,(\phi\cdot\varrho) = \chi\,|\,(\psi\cdot\varsigma)$;*
- *(J_1) If $\phi\ J\ \psi$, then $\phi\,|\,\psi$ has precisely one member;*
- *(J_2) If $\psi_1 \in \phi_1\,|\,\chi$ and $\psi_2 \in \phi_2\,|\,\chi$, and if $\phi_1\ J\ \phi_2$ or $\phi_1 = \phi_2$, then $\psi_1\ J\ \psi_2$ or $\psi_1 = \psi_2$.*

Let the *parallel* rewrite system $\twoheadrightarrow\!\!\!\!\!\shortmid\ \ $ have as objects the objects of $\rightarrow$, and a step Φ_a if Φ is a set of steps *at* a that is a J-*set* [17, p. 232]: (distinct) steps in Φ are from a and pairwise J-related. Then the source of Φ_a is a and its target is the target of a development of Φ.

Lemma 1. *Under the assumptions of Thm. 2, $\langle \nrightarrow\!\!\!\rightarrow, \emptyset, | \rangle$ is a 1-ra, so $\nrightarrow\!\!\!\rightarrow$ is orthogonal.*

Proof. That $\nrightarrow\!\!\!\rightarrow$ is well-defined, *i.e.* that Φ_a has a unique target holds by [17, Lem. 2]. [17, Lem. 5.1] shows both that $|$ is a residuation (Fig. 1) for $\nrightarrow\!\!\!\rightarrow$, so $\nrightarrow\!\!\!\rightarrow$ has the DP, and that it has the cube property, *i.e.* law (4) holds. Laws (1)–(3) are seen to hold by easy inductions. ☐

Proof of Thm. 2. As $\rightarrow$-reductions *are* (singleton) $\nrightarrow\!\!\!\rightarrow$-reductions and the 1-ra on $\nrightarrow\!\!\!\rightarrow$ induces a 1-rac on $\nrightarrow\!\!\!\twoheadrightarrow$ by the previous section,[6] one concludes by setting $\varsigma' := \varsigma\,|\,\varrho$ and $\varrho' := \varrho\,|\,\varsigma$. ☐

Steps-as-terms To apply Thm. 2 to a *term* rewrite system [2, 26] $\mathcal{T} := \langle \Sigma, P \rangle$ given by a *signature* Σ and a set P of *rules* $\rho{:}\ell{\rightarrow}r$ for ℓ, r terms over function symbols in Σ (and variables), we let its *multistep* rewrite system $\multimap_{\mathcal{T}}$ have as objects terms over Σ and *steps-as-terms*, *i.e.* as steps terms over $\Sigma \sqcup P$, with src (tgt) the homomorphic extension of the function mapping rule symbols to their left-(right-)hand side. The *parallel* rewrite system $\nrightarrow\!\!\!\rightarrow_{\mathcal{T}}$ and (single step) rewrite system $\rightarrow_{\mathcal{T}}$ arise from $\multimap_{\mathcal{T}}$ by restricting rule-symbols to occur *at parallel positions* respectively *once* in steps [26, Prop. 8.2.22]. We employ that $\nrightarrow\!\!\!\rightarrow_{\mathcal{T}}$ is orthogonal (in the above sense) if $\mathcal{T}$ is an OTRS, a TRS that has $\mathsf{left\text{-}linear}$ and $\mathsf{non\text{-}overlapping}$ rules [2, 26], *i.e.* that the residuation / given in [26, Def. 8.7.4] induces a 1-ra on $\nrightarrow\!\!\!\rightarrow_{\mathcal{T}}$ [26, Prop. 8.7.7(ii)] for OTRS.

Lemma 2. *For an OTRS $\mathcal{T}$, the assumptions of Thm. 2 hold for $\rightarrow_{\mathcal{T}}$ when:*

- *defining $\phi\ J\ \psi$ if ϕ, ψ are steps (whose rule symbols are) at parallel positions; and*

[4] It being the first of its kind, the error could be called the α-α-error (lucerne-error?).

[5] See [17] for more. We present Newman's axioms and result as is, as our only goal is to *instantiate* them.

[6] Alternatively, this can be concluded from that the assumptions of Thm. 2 entail properties 1–4 of [15].

- *letting* $\phi \mid \psi$ *be the* set *of steps occurring in (the parallel step)* $\phi \,/\, \psi$.

Proof. That the axioms hold follows from well-known facts for parallel steps in OTRS [2, 26], *e.g.*, J_2 corresponds to the *Disjointness Property* in [26, Ex. 8.6.30] and Δ_4 to *cube* [13, Lem. 2.2.1], **permutation** [15]. (Using steps-as-terms, the axioms can also be verified by *inductions* on steps.) Note that non-erasingness is (only) needed for Δ_1, *cf.* [26, Ex. 8.7.24]. □

Conclusions We ruminate about the past, present and future of orthogonality.

Past (the curious case of orthogonality in rewriting) It is curious that Newman starts out [17] with stating to leave the study of *least* upperbounds (lubs) for later, to then devote most of the paper to introducing conditions that guarantee the very existence of lubs, as shown above (Lem. 1). That reductions constitute a lattice (have lubs) was shown only much later (in the 70s) for concrete rewrite systems such as recursive programs and the $\lambda\beta$-calculus, see [13], subsequently axiomatised and couched in categorical language (existence of pushouts) in [25, 15, 21]. Here we cast our account of that [26, Sec. 8.7] 1-algebraically (Def. 3 and Thm. 1).

$$\beta(x,y,z) : B\,x\,y\,z \;\to\; x\,(y\,z) \qquad \gamma(x,y,z) : C\,x\,y\,z \;\to\; x\,z\,y \qquad \iota(x) : I\,x \;\to\; x$$

Table 3: Combinatory Logic: term rewrite rules P of $\mathcal{BCI}$ in applicative notation

Example 6 (illustrating that lubs are subtle to define)**.** $\mathcal{BCI}$ is the TRS with signature $\Sigma := \{B, C, I, @\}$ and rules $P := \{\beta, \gamma, \iota\}$ (Tab. 3). Since the rules of the Combinatory Logic $\mathcal{BCI}$ in Tab. 3 are left-linear and non-overlapping, $\twoheadrightarrow\!\!\!\!\!+_{\mathcal{BCI}}$ is orthogonal by the above. Consider the peak $I\,x \;{}_{\phi}\!\leftarrow I\,(I\,x) \to_{\psi} I\,x$ for the steps $\phi := \iota(I\,x)$ and $\psi := I\,\iota(x)$. (Note that ϕ, ψ are extensionally the same but not intensionally so.) The lub (pushout) of the peak is formed *not* by the empty valley from $I\,x$, but by the valley comprising twice the step $\iota(x) : I\,x \to_{\mathcal{BCI}} x$ since $\psi \,/\, \phi = \iota(x) = \phi \,/\, \psi$. That is, that ϕ, ψ are intensionally different, perform different work, constitute a *syntactic accident* [13, p. 34], is reflected in their pushout (lub) not being empty.

Despite the prominence of Combinatory Logic and the $\lambda\beta$-calculus since the 30s, it took until the 80s to clepe them *orthogonal* term rewrite systems, making them in retrospect the 1st and 2nd (1st and 2nd-order) such. But that definition of orthogonality is *syntactic*, asks rule(s) to be left-linear and non-overlapping [2, 26], pertains to *terms* only. That led to the second curiosity that on the one hand many structured rewrite systems having lubs, *e.g.*, *interaction nets* [12], *braids* [16], *self-distributivity* [24], …were *not* covered by that syntactic definition, and that on the other hand that syntactic definition was found to be *lacking*, to not guarantee confluence let alone existence of lubs, already for minor generalisations of term rewriting:

Example 7. The rules $a \to b$ and $f(x) \to c \Leftarrow x = a$ are left-linear and non-overlapping but not even confluent as witnessed by both $f(b)$ and c being normal forms in the peak $f(b) \leftarrow f(a) \to c$.

Whence we propose(d [26, Sec. 8.9]) to factor the syntactic definition of orthogonality through its associated rewrite system being orthogonal, to constitute a 1-ra (Def. 3), making it interesting to see whether for systems in the literature the former entails the latter. For orthogonal TRS and PRS this is known to be the case [26, Sec. 8.7][4] by associating $\multimap\!\!\!\rightarrow_{\mathcal{T}}$ to $\mathcal{T}$ with the proof unifying [5] with *axiomatic* [17] and *inductive* [3, Sec. 3.2] (TML) proofs. By *steps-as-terms* being conceptually parsimonious the residuation-based proof is superior to TML, which is based on *ad hoc prooftrees*, as we illustrate for $\mathcal{BCI}$ in Ex. 8. Ex. 9 exemplifies that for certain *context-sensitive* and *conditional* TRS, orthogonality indeed induces orthogonality. We leave it to future research to check other syntactically orthogonal systems in the literature.

Example 8 (orthogonality of $\mathrel{-\!\!\circ\!\!\!\rightarrow}_{\mathcal{BCI}}$ for OTRS $\mathcal{BCI}$)**.** The (*proof*)*terms* $\phi{:=}B\,I\,\gamma(C,x,y)\,\iota(z)$ and $\psi{:=}\beta(I,C\,C\,x\,y,I\,z)$ yield a peak $B\,I\,(C\,y\,x)\,z \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!-}_\phi B\,I\,(C\,C\,x\,y)\,(I\,z) \to_\psi I\,(C\,C\,x\,y\,(I\,z))$ for which residuation (per [26, Def. 8.7.4]) yields the valley $B\,I\,(C\,y\,x)\,z \to_{\psi'} I\,(C\,y\,x\,z) \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!-}_{\phi'} I\,(C\,C\,x\,y\,(I\,z))$, the lub (pushout) of ϕ,ψ, formed by (*proof*)*terms* $\psi' := \psi \,/\, \phi := \beta(I,C\,y\,x,z)$ and $\phi' := \phi \,/\, \psi := I\,(\gamma(C,x,y)\,\iota(z))$, as one may check in Haskell or ProTeM.

Example 9. $\langle \mathrel{-\!\!\circ\!\!\!\rightarrow}, 1, / \rangle$ is a 1-ra for $/$ as in Ex. 8 [26, Def. 8.7.4], if steps ϕ are restricted by:[7]

- (i) for CSR as in [14, Thm. 8.12]: all frozen arguments are Σ-terms, or
- (ii) for orthogonal normal CTRS [26, Sec. 4.11.2]: if $\phi = \rho(\vec{\phi})$ for rule $\rho(\vec{x}) : \ell \to r \Longleftarrow \overrightarrow{\ell \twoheadrightarrow r}$, then for all j, $\ell_j^\sigma \twoheadrightarrow r_j^\sigma$ (observe $r_j^\sigma = r_j$ by normality), where $\sigma(x_i) := \mathsf{src}(\phi_i)$ for all i.

Present (inappropriate appropriation) Rewrite systems being basic small wonder they occur elsewhere nowadays, *e.g.*, as *multidigraphs*, *quivers* in representation theory [8], *pre-categories* in Garside theory [6] or 1-*polygraphs* in higher-dimensional group presentations [1]. As much as we would like to base ourselves on [6, 1], we cannot as both accounts are inadequate for our key notions *conversion* and *residuation*. More generally, *sub*systems [18] ($\leftrightarrow$ and $\to^+$, the free 1-algebras with $^{-1}$ respectively $\cdot$ satisfying the laws in Tab. 1) and *super*systems [26] (infinit(e/ary) reductions) are absent from them, and so is (must be) classical rewrite theory.

$l\text{-}inv(\varrho)$	:	$\varrho^{-1}\cdot\varrho$	$\Rightarrow$	1	$l\text{-}inv\text{-}x(\varrho,\varsigma)$	:	$\varrho^{-1}\cdot(\varrho\cdot\varsigma)$	$\Rightarrow$	ς
$r\text{-}inv(\varrho)$	:	$\varrho\cdot\varrho^{-1}$	$\Rightarrow$	1	$r\text{-}inv\text{-}x(\varrho,\varsigma)$	:	$\varrho\cdot(\varrho^{-1}\cdot\varsigma)$	$\Rightarrow$	ς

Table 4: 1-algebra laws / proofterm rewrite rules for 1-groups, extending Tab. 1

Remark. Modelling conversions as $\leftrightarrow$-reductions is *too weak*, so 1-polygraphs are [11, Sec. 2.4], as *categories* (1-monoids) miss out on involution [7],[8] and assuming cancellation is *too strong* [17, Sec. 1], as *groupoids* (1-*groups*; see Tab. 4[9]) lose embedding [19]. Indeed, [6, 1] have algebraic accounts of neither conversion nor residuation, so cannot account for orthogonality, 1-ra's (having composition without residuation is analogous to having addition without monus; we do not know of other accounts where both are treated on a par, algebraically, as we think they should).

Future (explorations) (I) We restricted attention to the axioms in [17] for the *non-erasing* $\lambda\beta$-calculus; (Δ_1). These entail additional properties, *e.g.*, the natural order on $\mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}$-steps is the *subset* order, all developments have the same length [19], and $\to$ is *normalising* (WN) iff it is *terminating* (SN) [17, Thm. 8][26, Thm. 4.8.5]. Which are retained for the axioms in [17] for the (*erasing*) $\lambda\beta$-calculus? orthogonality? (II) Does meta-theory such as that the *full* $\mathrel{-\!\!\circ\!\!\!\rightarrow}$-strategy is (hyper-)normalising, generalise? (III) Does viewing a *proof order* (see [26, Thm. 7.5.12]) as a morphism from conversions into a (well-founded) 1-involutive 1-monoid have advantages beyond those in [7]? (IV) Does the approach generalise to *infinit*(*e*/*ary*) reductions? (V) Can the *Grothendieck group construction* be based on orthogonality? (VI) How to *formalise* this algebraic approach, in particular *residuation*, *cf.* [10]? (VII) How to generalise proofs via steps-as-terms to other structures, *e.g.*, to *steps-as-port-graphs* for interaction nets [12].

Acknowledgments We thank Nao Hirokawa, Vítek Jelínek, Philip Saville and Fer-Jan de Vries for discussions and feedback on a previous version, and the IWC reviewers for reviews.

[7] As before[5], we refer the reader to the cited literature for more, for reasons of room.

[8] Involution is *essential* and *useful*; it saved half the work in formalising [7] (B. Felgenhauer; pers. comm.).

[9] The proofterm rewrite rules are obtained by 1-*completion*, see Ch. 7 of either [2, 26]: though *l-inv-x* and *r-inv-x* are *derivable* they need to be adjoined to turn $\Rightarrow$ into a *complete* proofterm rewrite system.

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