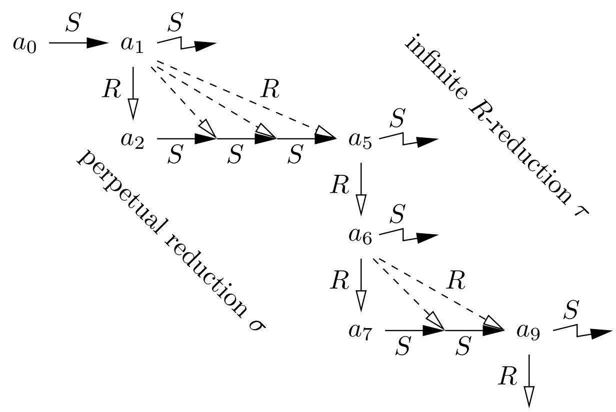
Theorem 1. Let $\to = R \cup S$. If $a\ R\,;S\ b$ entails either ($a\ S\,;\to^*\ b$) or ($a\ S\,;\to^\omega$) or ($a\ R\ b$ and not $b\ S^\omega$), for a,b such that $a \to^\omega$, then $a_0 \to^\omega$ implies $a_0\ S^\omega$ or $a_0\ S^*\,;R^\omega$.

Proof.



Assume $a_0 \to^\omega$ but not $a_0\ S^\omega$. We first construct an $\to$-reduction σ from a_0 by giving preference to S-steps. That is, a given prefix of σ ending in a_i such that $a_i \to^\omega$, is extended with any step $a_i \to a_{i+1}$ which is perpetual, i.e. such that $a_{i+1} \to^\omega$, under the condition that it be an S-step if such exists. Per construction and by $a_0 \to^\omega$, σ is an infinite reduction and by not $a_0\ S^\omega$ there exists an n such that $a_n\ R\ a_{n+1}$ is the first step in σ which is not an S-step (in the figure $n = 1$). Next, we construct an infinite R-reduction τ from a_n by skipping the S-steps in σ. That is, a given prefix of τ ending in an object a_i such that $a_i\ R\ a_{i+1}$ is a step of σ, is extended with a step $a_i\ R\ a_j$ as follows. If $a_{i+1}\ R\ a_{i+2}$ is a step of σ then we set $j = i + 1$. Otherwise $a_{i+1}\ S\ a_{i+2}$ is a step of σ, hence by assumption either $a_i\ S\,;\to^*\ a_{i+2}$ or $a_i\ S\,;\to^\omega$ or ($a_i\ R\ a_{i+2}$ and not $a_{i+2}\ S^\omega$). Since the first two would conflict with the construction of σ giving preference to perpetual S-steps for a_i, the third must be the case. Since not $a_{i+2}\ S^\omega$ and σ is infinite, the maximal sequence of S-steps from a_{i+1} in σ ends in some object which we call a_j. An easy induction shows, using this assumption, that in fact $a_i\ R\ a_k$ for all $i < k \le j$ from which we conclude. □

Corollary 2 ([1])**.** If $R\,;S \subseteq (S\,;\to^*) \cup R$, then $\to$ is terminating if R and S are.

Proof. As S is terminating, the assumption entails the assumption of the theorem. By termination of S, R neither disjunct in the conclusion of the theorem can hold, so $\to$ is terminating. □

Corollary 3 (Geser, [3] Exc. 1.3.20)**.** If $\to$ is transitive, then $\to$ is terminating iff R and S are.

Proof. By the previous corollary using that transitivity, i.e. $\to\,;\to\ \subseteq\ \to$, entails $R\,;S \subseteq S \cup R$. □

Corollary 4 ([2] Lemma 8)**.** If $R\,;S \subseteq S\,;R$, then $a_0 \to^\omega$ implies $a_0\ S^*\,;R^\omega$ if not $a_0\ S^\omega$.

Proof. By $R \subseteq\ \to$ the assumption of the theorem and hence its second disjunct hold. □

The result: if $R\,;S \subseteq S\,;\to^*$, then $R^*\,;S\,;R^*$ is terminating iff S is (Bachmair and Dershowitz, [3] Exc. 1.3.19) is not a corollary. The third disjunct ($a\ R\ b$ and not $b\ S^\omega$) in the premiss of the theorem is satisfied for $a\ R\ a\ R\ b\ S\ a$, but although $R^*\,;S\,;R^*$ is not terminating, S is. Removing that disjunct and $a_0\ S^*\,;R^\omega$ in the conclusion allows to adapt the method to that result as well.

References

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