Non-height-increasing term rewrite systems without critical peaks are confluent*

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Definition. A term rewrite system is *non-height-increasing* (NHI) if for every step $t \to s$ the height of t is greater than or equal to that of s.[1] We refer to (a subterm of a) left-hand side of a rule as a (*sub-*)*pattern*, *proper* if neither the whole left-hand side nor a variable. A *compound* is a term obtained by the repeated substitution of proper sub-patterns starting from a sub-pattern. A term rewrite step $\phi : C[\ell^\sigma]_p \to C[r^\sigma]_p$ is *away* from a set of positions O in its source if $p \notin O$. A reduction $t^\sigma \twoheadrightarrow s$ is *away* from t if every step in it is away from the non-variable positions of t.

Lemma. *An NHI term rewrite system without critical peaks is confluent.*

Proof. Define a conversion (peak, valley, ..., term) to be (*strictly*) *below* n if the height of each term occurring in it is $\leq n$ ($< n$) For every natural number n that, we claim that:

(i) for every reduction $t^\sigma \twoheadrightarrow s$ below n away from compound t and step ϕ from s, then either ϕ is away from t or a root-step for a rule $\ell \to r$ with ℓ the pattern (at the root) of t;

(ii) every $\trianglerighteq \cdot \blacktriangleright^=$-peak below n is joinable either by a $\twoheadrightarrow$-valley strictly below n, or by a $\trianglerighteq \cdot \blacktriangleright^=$-valley, where $\triangleright$ denotes non-root-steps and $\blacktriangleright$ denotes root-steps.

From the claim we conclude to confluence below n by decreasing diagrams, ordering $\to$-steps below $(\trianglerighteq \cdot \blacktriangleright)$-steps. We prove both items of the claim simultaneously, by induction on n.

(i) We show that if $t^\sigma \twoheadrightarrow s$ is a reduction below n away from compound t and $\phi : s = C[\ell^\sigma]_p \to C[r^\sigma]_p =: u$ not away from t, then ϕ is a step for ℓ the pattern of t, by sub-induction on the multiset of (n minus depth) for variable postions in t, and cases on the positions of the redex-pattern of ϕ.

If all positions are in t, then by the main IH for the arguments to t in s, there is a term $s' = t^\upsilon = D[\ell^\tau]_p$ for substitutions υ, τ, such that $s \trianglerighteq s'$. Then $p = \epsilon$ with ℓ determined by t, as compounds are constructed from sub-patterns, proper if not at the root, and absence of critical peaks.

Otherwise, $s(q) = \ell'^\sigma$ for some proper sub-pattern ℓ' of ℓ and position q with $t(q) = x$ for some variable x. By the main IH for the arguments to t in s, there is a term $s' = t[x \mapsto \ell']^\upsilon$ for substitution υ such that $s \trianglerighteq s'$.[2] Since then $s' \trianglerighteq D[\ell^\tau]_p$ for some context D and substitution τ, the sub-IH (applicable as $t[x \mapsto \ell']$ extends t) yields $p = \epsilon$ for ℓ the pattern of $t[x \mapsto \ell']$ hence t.

(ii) Suppose $t \trianglerighteq s_i = \ell_i^{\sigma_i} \blacktriangleright r_i^{\sigma_i}$ for $i \in \{1, 2\}$. By (ii) of the IH for some υ, $s_i \trianglerighteq \upsilon$, away from ℓ_i by (i), hence $\upsilon \trianglerighteq \ell_i^{\upsilon_i}$ away from ℓ_1 *and* ℓ_2 for some substitution υ_i, so $\ell_1 = \ell = \ell_2$, $r_1 = r = r_2$ and $\upsilon_1 = \upsilon = \upsilon_2$ by (ii). Therefore, $r_i^{\sigma_i} = r^{\sigma_i} \twoheadrightarrow r^\upsilon$ if r is a single variable and otherwise $r^{\sigma_i} \trianglerighteq r^\upsilon$. □

Example. Each of $\{(x = x) \to \top, s(x) \to (x = s(x)), \infty \to s(\infty)\}$ (Klop), $\{(x = x) \to \top, (x = s(x)) \to \bot, \infty \to s(\infty)\}$ (Huet) and $\{(x = x) \to \top, \infty \to m(\infty, s(\infty)), m(x, s(x)) \to (x = m(x, s(\infty)))\}$ (Oyamaguchi & Gomi; *depth-preserving*) is non-confluent, has no critical peaks but fails to be NHI.

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[1] NHI can be decided by checking whether for every term rewrite rule $\ell \to r$ the height of its left-hand side ℓ is greater than or equal to that of its right-hand side r, and either there are only nullary function symbols or for every variable x and position q such that $r(q) = x$ there is a position p such that $\ell(p) = x$ and $|p| \geq |q|$.

[2] With (ii) of the IH used to guarantee the reduction is away from *both* t *and* sub-patterns of ℓ 'sticking out'.