Newman's Proof of Newman's Lemma

Textbooks on rewriting all give one or more (short[1]) proofs of Newman's Lemma [2, Theorem 2], but none of them gives an account of Newman's original proof. It might be worthwhile to do so.

Lemma. *If $\to$ is terminating and has the weak Church–Rosser property, then it has the Church–Rosser property.*

Proof. It suffices to prove that $\to$ is confluent. This we prove by showing that any peak $\pi_1 : b \twoheadleftarrow a \twoheadrightarrow c$ can be stepwise transformed into a valley $b \twoheadrightarrow \cdot \twoheadleftarrow c$. A transformation step consists in replacing an occurrence of a local peak $b' \leftarrow a' \to c'$ in a path π_i between b and c, by a valley $b' \twoheadrightarrow \cdot \twoheadleftarrow c'$ using the assumption that $\to$ has the weak Church–Rosser property, yielding another such path π_{i+1}. Local peaks are selected during the transformation process in breadth-first fashion as illustrated in Figure 1, where the numbers indicate the successive selections, and

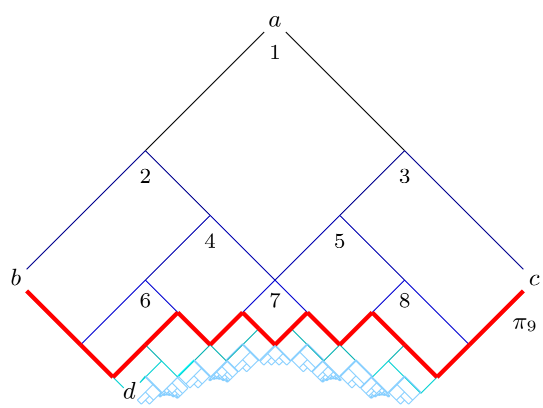

Figure 1: Attempt to transform a peak stepwise into valley

the red conversion is the path π_9 reached at stage 9 of the process. Intuitively, *if* the process terminates the final path must be a valley since it does not contain peaks, and moreover the process *must* terminate since peaks are being pushed further and further down.

To formalise the intuition that peaks are being pushed ever further down, define the *depth at stage i* of an object occurring in π_i to be the maximal length of a reduction from a to that object *only through steps up to that stage*, i.e. in the subsystem $\to_{\restriction i} = \bigcup_{j \leq i} \pi_j$. For instance, the object d in the figure has depth 8 (at stage 14). The notion of depth is well-defined, since each stage contains only finitely many objects all reachable from a, and because reduction cycles are impossible by the assumption that $\to$ is terminating. Depths increase along a *breadth-first* transformation, formalised by always selecting a peak having a source of minimal depth at that stage. For instance, the source of the peaks selected at stages 6, 7 and 8 all have depth 3, and applying transformation steps in breadth-first fashion, consecutively replaces each by a valley, yielding the conversion π_9 only having peaks of depth 4. Formally, it is first shown by induction on i that π_i contains at most i peaks. This is trivial for π_1 and the property is preserved by any transformation step since a peak is being replaced by at most two new ones. Next, one shows by induction on n that the depth at stage 2^n of any peak is at least n. This is trivial for $n = 0$ and the propery is preserved since assuming that all peaks at stage 2^n are of depth at least n, we have by the previous property that there are at most 2^n such, and these are all removed after performing 2^n further transformation steps, i.e. at stage 2^{n+1}.

Having formalised the intuition that peaks are being pushed ever further down, it remains to show that this implies that the transformation process terminates. If not we may choose, for each

[1]The first short proof of Newman's Lemma we know of is the proof of Lemma 11.1 in [1].

n, a peak of maximal depth at stage 2^n, and a witnessing reduction path σ_n from a to that peak (the red reduction paths in Figure 2). By the previous paragraph, the length of such a path is

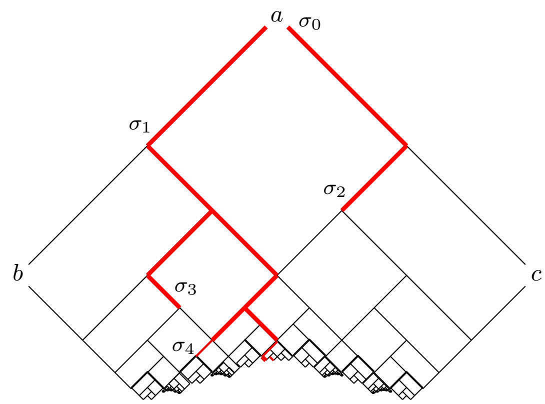

Figure 2: Reduction paths σ_n to peaks of maximal depth at stage 2^n

at least n and thus the collection of all these paths (corresponding to the red tree in the figure) constitutes a dag with root a which is infinite. We claim that this rooted dag is finitely branching,[2] from which the result follows since then it contains an infinite reduction path by König's Lemma, contradicting the assumption that $\to$ is terminating.

The claim is shown by proving that for all $i \leq n$, the ith step in the path σ_n belongs to the finite subsystem $\to_{\restriction 2^i}$. For a proof by contradiction, consider the least i for which the statement would not hold. Then for the least stage j such that the step does belong to $\to_{\restriction 2^j}$ it holds $i < j \leq n$. Thus the step was adjoined by a transformation step at some stage equal to or greater than 2^i and less than 2^j. By the above, all peaks at these stages are at depth at least i and thus the new step would be at depth at least $i + 1$, contradicting the assumed maximality of the path σ_n. □

Of course, we have but presented our own understanding of Newman's proof, and we advise the reader to read herself the seminal paper [2], in which Newman layed the foundations for both the abstract treatment of rewriting and the axiomatic treatment of residuals.[3] We have neither attempted to shorten Newman's proof nor to make it constructive.

References

[1] H. Barendregt, J. Bergstra, J.W. Klop, and H. Volken. Some notes on lambda reduction, 1976. Chapter II in "Degrees, Reductions and Representability in the Lambda Calculus".

[2] M.H.A. Newman. On theories with a combinatorial definition of "equivalence". *Annals of Mathematics*, 43(2):223–243, 1942.

[3] V. van Oostrom and R. de Vrijer. Four equivalent equivalences of reductions. *Electronic Notes in Theoretical Computer Science*, 70(6):41 pp., 2002.

[2] Looking at the figure, the claim may seem trivial as the out-degree of any 'vertex' in it is at most two. However note that distinct such 'vertices' in the figure may correspond to vertices in the graph which are identical.

[3] Despite the fact that Newman devoted more than half of his paper (p. 231–242) to axiomatic residual theory, it is hardly ever cited because of it, the reason (probably) being that the main application presented in the paper of his axiomatic results, the Church–Rosser theorem for β-reduction in the λ-calculus, was later shown to be incorrect by Schroer, a student of Rosser. See [3, Remark 6.14(ii)] for more on this.