More modular termination

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Abstract

We discuss known *modularity* results stating a finite *family* of rewrite system to be terminating iff its union is, under various additional conditions. Taking a transformational approach, relating reductions in the union to reductions for the family members, we refine some of these results.

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Background This draft note (3-7-2023) extends our earlier note [16] on *preponement* by applying its methodology of taking reductions as first-class citizens, to Podelski and Rybalchenko's extension [20] of Geser's result [8] on *disjunctive termination* from binary to arbitrary unions, and to Dawson, Dershowitz, and Goré's similar extension [5] of Doornbos and von Karger's result [7] on *jumping*. We elaborate and combine various small observations dating back to 2006 [13], 2011 [16], and 2016, taken 'out of the drawer' last month when reminded of them. Updated (4-7-2023) to use the final version of [5] instead of a preliminary version. Updated (16-7-2023) to discuss (potential) use in and of tools. Comments welcome.

Modularity of termination by transforming reductions. Our starting point is the modularity result that $\to := \bigcup_{i \in I} \to_i$ is terminating iff all $\to_i$ are, if $\to$ is transitive, a result due to Geser [8][1] for $\#I \leq 2$, as generalised by Podelski and Rybalchenko [20, Thm. 1, Cor. 1] to index sets I of arbitrary finite cardinality, and by relaxing transitivity by Doornbos and von Karger [7] for $\#I \leq 2$ and subsequently by Dawson, Dershowitz and Goré [5] to index sets I of arbitrary finite cardinality.

Throughout, our approach is based on transforming reductions, either finite or infinite [14, 22]. That is, we treat reductions as first-class citizens. Here, a reduction from a given object a in a rewrite system $\to$ is either the empty reduction from a to itself, or a step $\phi : a \to b$ followed by a reduction from b, coinductively. A reduction is a reduction from some object, called its source. It is a reduction to some object, called its target, if it has the empty reduction to that object as tail (subreduction / suffix). Then the reduction is finite; otherwise infinite (then without target). Reductions / steps having the same source (target) are called *co-initial* (*cofinal*). To indicate the length $\alpha \leq \omega$ of a $\to$-reduction we superscript the latter with the former, $\to^\alpha$. We assume familiarity with rewriting (terminology) [1, 22].

The transformational approach not only allows us to express *that* the union of a family of rewrite systems is terminating iff all its family members are, but to obtain this via transforming reductions in the former into reductions in the latter in the spirit of [14, 18]. This in turn allows us to refine termination statements *sec* by also relating the *shapes* of the reductions. To stress this, we will speak of *termination* (all reductions are finite) instead of *well-foundedness* (every non-empty subset has a minimal element).[2]

To demarcate what we are interested in here, and what not, note that taking *disjoint* unions is useful but trivial; termination is trivially preserved.[3] The other way around, without further

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[1] See Thm. (termination inheritance by transitivity) on p. 31 of [8].

[2] Though they are equivalent, assuming dependent choice. Beware that using our conventions [22] termination of $\to$ corresponds to well-foundedness of its *converse* $\leftarrow$; cf. the conclusion.

[3] Note that in term rewriting [1, 22] the study of modularity concerns taking the union of *term* rewrite systems having *disjoint* signatures. Such disjoint unions do *not* preserve termination as shown by

conditions *non-disjoint* unions trivially fail to preserve termination; consider e.g. the union of $a \blacktriangleright b$ and $b \triangleright a$. Moreover, unions of *infinite* families enable non-termination by 'going through' infinitely many family members; consider e.g. the family $(\{(n,m) \mid n < m\})_{n \in \mathbb{N}}$ having trivially terminating (at most one step) members. Accordingly, we focus on non-disjoint unions of finite families of terminating rewrite systems, with conditions [20, 6, 5].

Jumping. We first introduce some no(ta)tions to conveniently and concisely recapitulate and refine[4] the result of [7], whose condition we will refer to as *jumping* following [5].

We say a reduction δ *protracts* a reduction γ if it is co-initial to it and either δ is infinite or both are finite and also have the same target,[5] and that δ is *preferential* (for γ) if replacing any tail headed by a $\triangleright$-step, by any reduction headed by a $\blacktriangleright$-step does not protract γ; the intuition is that the reduction δ *prefers* $\blacktriangleright$-steps, as long as they preserve *reachability* of the goal, of the target of γ, or are *perpetual*, i.e. preserve non-termination.[6] Note that protracting is transitive and preserved under concatenation.[7]

▶ **Lemma 1.** *For any $\to$-reduction γ there is a preferential reduction $\hat{\gamma}$ protracting γ of shape $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\alpha}$ for $\alpha \leq \omega$, if* jumping *holds: $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup (\blacktriangleright \cdot \twoheadrightarrow)$ for $\to := \blacktriangleright \cup \triangleright$.*

Proof. We proceed by iteratively constructing, starting with the empty reduction on the source of γ, ever longer[8] finite reductions $\delta : a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright\, b$ preferential for γ, that can be extended to reductions protracting γ.

If b is the target of γ (if that exists) we conclude setting $\hat{\gamma} := \delta$, and if it is the source of an infinite $\blacktriangleright$-reduction ϵ, we conclude by setting $\hat{\gamma} := \delta \cdot \epsilon$. Otherwise, all reductions extending δ to protract γ are non-empty and b is $\blacktriangleright$-terminating. Then:

- If there exists a $\triangleright$-step ψ such that $\delta \cdot \psi$ is a preferential reduction too, that can be extended to a reduction protracting γ, we iterate on $\delta \cdot \psi$.
- Otherwise, there must be a $\blacktriangleright$-step ψ such that $\delta \cdot \psi$, again preferential, can be extended to a reduction protracting γ. If δ is empty or ends with a $\blacktriangleright$-step, we iterate on $\delta \cdot \psi$. Otherwise, the last step of δ is a $\triangleright$-step, say ϕ. Then the jumping assumption applies to the pair $\phi \cdot \psi$ of shape $\triangleright \cdot \blacktriangleright$:
  - The second disjunct cannot apply since then replacing the pair by $\blacktriangleright \cdot \twoheadrightarrow$ in (the extension of $\delta \cdot \psi$ to) the reduction protracting γ would contradict δ being preferential.
  - Hence the first disjunct applies, yielding another $\triangleright$-step ϕ' to b' having the same source (target) as ϕ (ψ). Then we iterate on δ' obtained by replacing ϕ by ϕ' in δ.

Note that eventually a longer reduction is obtained as the final case cannot occur consecutively infinitely often since that would give rise to an infinite $\blacktriangleright$-reduction through the targets $b, b', \ldots$ of the successive reductions $\delta, \delta', \ldots$, contradicting $\blacktriangleright$-termination of b. ◀

Toyama's counterexample [22, Ex. 5.9.1]; the point is that the rewrite system $\to_{\mathcal{T} \cup \mathcal{S}}$ of the disjoint union of two TRSs $\mathcal{T}$ and $\mathcal{S}$ is *not* the disjoint union $\to_{\mathcal{T}} \cup \to_{\mathcal{S}}$ of their rewrite systems $\to_{\mathcal{T}}$ and $\to_{\mathcal{S}}$, but rather a rewrite system on terms over the union of their (disjoint) signatures.

[4] Preliminary versions of Lem. 1 and its proof are in [13], [16], and [19, Lem. 3.5 and Fig. 7].

[5] In the λ-calculus, that reductions have the same sources and targets is sometimes called *Hindley-equivalence*. In graphs, such edges are called *parallel*.

[6] Thinking of $\blacktriangleright$-steps as being *worse* than $\triangleright$-steps [14], preferential reductions are particularly bad; there are no worse ones, assuming that all infinite reductions are worse than all finite ones in that they don't even reach the goal, the target; cf. also [10] or [22, Sect. 9.5]. This could be made quantative using the framework of [18].

[7] Here we use the convention that concatenating to an infinite reduction yields the infinite reduction.

[8] With only its final $\triangleright$-steps possibly non-stable.

▶ Remark 2. To find $\hat{\gamma}$ only the source and target (if there is one) of γ are used. In particular, unlike Lem. 15 below, the objects in $\hat{\gamma}$ need not be among those of γ, as witnessed, e.g., by that the reduction $\gamma : a \triangleright b \blacktriangleright c$ is transformed into the preferential reduction $\hat{\gamma} : a \blacktriangleright a' \blacktriangleright a' \blacktriangleright \dots$ of shape $\blacktriangleright^{\omega}$, for rewrite systems $\blacktriangleright, \triangleright$ given by the steps in $\gamma, \hat{\gamma}$ combined with $a' \triangleright a$ to make the jumping criterion hold. In this example, one easily finds that in fact $\hat{\gamma}$ is the only preferential reduction protracting γ, but in general the construction is ineffective as it involves checking reachability.

We can be a bit more liberal while preserving the result, by allowing $\blacktriangleright \cdot \to^{\infty}$ as an additional, third, disjunct in the jumping condition, where $\to^{\infty}$ is the rewrite system having a *step* $a \to^{\infty} b$ for any object b and any infinite $\to$-reduction from a; cf. [16, 18]. The idea is that it is sufficient to known *that* there is an infinite reduction starting with a $\blacktriangleright$-step and that the second disjunct $\blacktriangleright \cdot \twoheadrightarrow$ of jumping is only one way in which that can be brought about (in case γ is infinite).[9] For instance, we may omit $a' \triangleright a$ in the above example.

In [5] also a contrapositive (for the special case of an infinite reduction γ) is proven, showing that $\to$ is terminating if $\blacktriangleright \cup \triangleright^{\sharp}$ is, where $\triangleright^{\sharp}$ denotes [5, Def. 15 (Constriction)] $\triangleright$ with all steps from objects also allowing a $\blacktriangleright$-step to a non-$\to$-terminating object removed.[10] Note that for *infinite* reductions γ this corresponds exactly to removing non-preferential $\triangleright$-steps, i.e. steps from objects from which a $\blacktriangleright$-step is preferred.[11]

▶ **Corollary 3** ([16]). *Let* $\to := \blacktriangleright \cup \triangleright$.

1. $\to$ *is terminating iff* $\blacktriangleright, \triangleright$ *are, if* $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup (\blacktriangleright \cdot \twoheadrightarrow)$ *(jumping) [7];*
2. $\to$ *is terminating iff* $\blacktriangleright, \triangleright$ *are, if* $\to \cdot \to \subseteq \to$ *(transitivity) [8][22, Ex. 1.3.20];*
3. $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$ *if* a *is* $\blacktriangleright$*-terminating,* $a \to^{\omega}$ *and* $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleright$ *(diamond) [9, Lem. 51];*
4. $\triangleright\!\!\triangleright \cdot \blacktriangleright \cdot \triangleright\!\!\triangleright$ *is terminating iff* $\blacktriangleright$ *is, if* $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \twoheadrightarrow$ *(quasi-commutation) [2].*

Proof.
1. The only–if-direction being trivial, it suffices to note that the if-direction follows from Lem. 1, since that entails any infinite $\to$-reduction γ would give rise to another such $\hat{\gamma}$ protracting it, tailing off in either $\blacktriangleright$ or $\triangleright$, with $\hat{\gamma}$ infinite as a reduction protracting the infinite reduction γ;
2. Immediate from item 1 since transitivity of $\to$ entails jumping of $\blacktriangleright, \triangleright$;
3. Lem. 1 applied to the reduction $\gamma : a \to^{\omega}$, yields a reduction $\hat{\gamma}$ of shape either $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$. We conclude by the first disjunct being impossible, since diamond entails $\triangleright\!\!\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleright\!\!\triangleright$ hence the infinite $\blacktriangleright$-suffix would induce an infinite $\blacktriangleright$-prefix contradicting $\blacktriangleright$-termination of a;
4. The only–if-direction being trivial, for the if-direction suppose there were an infinite $\triangleright\!\!\triangleright \cdot \blacktriangleright \cdot \triangleright\!\!\triangleright$-reduction γ from a. Then γ is an infinite $\to$-reduction and Lem. 1 applied to it would yield a reduction $\hat{\gamma}$ of shape either $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$. The first disjunct is impossible by the assumed termination of $\blacktriangleright$. The second disjunct is seen to be impossible by noting that the reduction $\hat{\gamma}$ constructed in the proof of Lem. 1 has at least as many $\blacktriangleright$-steps as γ (i.e. infinitely many here) in case of quasi-commutation (the right disjunct in the assumption of Lem. 1 always holds; its lhs (rhs) having (at least) one $\blacktriangleright$). ◀

▶ Remark 4. See e.g. [8, p. 32] for various consequences of Geser's result, i.e. of Cor. 3(2).

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[9] In fact, any reduction starting with a $\blacktriangleright$-step yielding a reduction protracting γ would do, but that condition seems not nicely captured by some operations on relations.

[10] With further variations in [5, Lem. 19 and Def. 23].

[11] For the case of an *arbitrary* reduction γ, finite or infinite, as in Lem. 1, $\triangleright^{\sharp}$ should denote $\triangleright$ with all steps from objects allowing a $\blacktriangleright$-step to an object that is *either* non-$\to$-terminating *or* from which the target of γ can be reached by a $\to$-reduction, removed.

We recapitulate and refine the extension of Cor. 3(1) from two rewrite systems $\blacktriangleright, \triangleright$ to finite families of rewrite systems of [5]. To that end, we identify the set I of indices with $\{i \mid 1 \leq i \leq n\}$ for $n := \#I$ totally ordered by $\leq$, yielding a family $(\to_i)_{1 \leq i \leq n}$ of n rewrite systems $\to_i$. We then say a reduction is ind, short for *i*ndex-*n*on-*d*ecreasing, if its sequence of indices is non-decreasing with respect to the given total order on the indices of the family, i.e. of shape $\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \cdot \ldots \cdot \twoheadrightarrow_n (\cdot \to_i^\omega)$ with the last infinite part (for some index i) optional, for the less–than–or–equal order $\leq$ on the indices $\{1,\ldots,n\}$. Note that the specification of the reductions obtained by Lem. 1 is equivalent to the special case of ind where $n = 2$, defining the family by $\to_1 := \blacktriangleright$ and $\to_2 := \triangleright$, and that for that special case the notion of jumping in Lem. 6 coincides with the earlier one in its special case, Lem. 1.

▶ Remark 5. Since reindexing is notationally cumbersome, we allow ourselves to also speak about families such as $(\to_i)_{2 \leq i \leq n+1}$, then meaning the family $(\to'_i)_{1 \leq i \leq n}$ with $\to'_i := \to_{i+1}$.

▶ **Lemma 6.** *For any $\to$-reduction there is an ind reduction protracting it, if* jumping *holds: $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ for all $1 \leq i \leq n$, where $\to := \bigcup_{1 \leq i \leq n} \to_i$.*

Proof. By induction on n. In the base case there's only 1 rewrite system. In the step case, given a $\to$-reduction δ, Lem. 1 for $\blacktriangleright := \to_1$ and $\triangleright := \bigcup_{2 \leq i \leq n}$ yields there is a preferential reduction $\hat{\delta}$ protracting δ of shape either $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^\omega$ or $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^\alpha$ for $\alpha \leq \omega$. In either case let γ be the $\triangleright$-subreduction, i.e. comprising steps having indices ≥ 2. The induction hypothesis for γ applies to it since jumping is preserved for the subset $\{2,\ldots,n\}$ of indices, and yields an ind reduction $\hat{\gamma}$ protracting γ. That is, $\hat{\gamma}$ is of shape $\twoheadrightarrow_2 \cdot \ldots \cdot \twoheadrightarrow_n (\cdot \to_i^\omega)$ with the last infinite part, for some index i, optional. We see that the reduction obtained by replacing[12] γ in $\hat{\delta}$ by $\hat{\gamma}$ is as desired, i.e. protracting δ[13] and satisfying the ind-criterion. ◀

▶ **Corollary 7** ([5]). **1.** *$\to := \bigcup_{1 \leq i \leq n} \to_i$ is terminating iff all $\to_i$ are, if jumping holds;*
2. *$\to := \blacktriangleright \cup \triangleright \cup \gg$ is terminating iff each of $\blacktriangleright, \triangleright, \gg$ is, if $\gg \cdot \triangleright \subseteq \gg \cup (\triangleright \cdot (\triangleright \cup \gg)^*)$ and $(\triangleright \cup \gg) \cdot \blacktriangleright \subseteq (\triangleright \cup \gg) \cup (\blacktriangleright \cdot \to^*)$ [5, Thm. 8 (Jumping II)].*

Proof. 1. As for Cor. 3(1) but using Lem. 6 (instead of Lem. 1);
2. The instance of item 1 for $n := 3$ and $\to_1 := \blacktriangleright$ and $\to_2 := \triangleright$ and $\to_3 := \gg$. ◀

▶ Remark 8. Jumping can be iterated both downward and upward. We only address the former here, but note that the latter was addressed in [5, Thm. 7 and Cor. 20 (Jumping I)].

Affluence. We introduce some further no(ta)tions to conveniently and concisely state our refinement[14] of the disjunctive termination result of [8, 20].

We say $\blacktriangleright, \triangleright$ is *affluent* [19, Def. 3][15] if $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup \blacktriangleright$, and that $\blacktriangleright, \triangleright$ is *affluent for* a $(\blacktriangleright \cup \triangleright)$-reduction γ if for all a, b, c in γ, if $a \triangleright b \blacktriangleright c$, then $a \triangleright c$ or $a \blacktriangleright c$. Observe that affluence of $\blacktriangleright, \triangleright$ entails its affluence for any $(\blacktriangleright \cup \triangleright)$-reduction.

▶ Remark 9. Though affluence is a special case of jumping, both are incomparable when generalised to families; due to that affluence does not *introduce* objects[16] it affords a stronger invariant, namely that we obtain a reduction *through* objects of the original one (that typically fails for jumping as was noted above).

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[12] By our convention on concatenation, if the $\hat{\gamma}$ is infinite any part of $\hat{\delta}$ after it is dropped by replacing.
[13] By transitivity and preservation under concatenation of protracting as $\hat{\delta}$ protracts δ and $\hat{\gamma}$ protracts γ.
[14] See the appendix for our original account [12] of it.
[15] Formally, this is *one-step affluence of* $\triangleleft, \blacktriangleright$ in the nomenclature of [19].
[16] Jumping may *introduce* objects, namely, when replacing consecutive steps $\triangleright \cdot \blacktriangleright$ by a reduction of shape $\blacktriangleright \cdot \twoheadrightarrow$ all objects along the latter reduction, other than its source and target, are introduced.

We say for a reduction γ that an object is *on* γ if it is the source of a step in γ, and that γ is $\blacktriangleright$*-normal* if each source of a $\triangleright$-step in γ is $\blacktriangleright\restriction\gamma$*-normal*, i.e. on γ and in normal form w.r.t. $\blacktriangleright\restriction\gamma := \{\phi : b \blacktriangleright c \mid b \text{ is on } \gamma \,\&\, c \text{ is in } \gamma\}$.[17] We say a reduction δ is *through* a reduction γ if all objects in the former are objects in the latter. The idea is then the usual one in Ramsey theory, to *zoom-in on* (*constrict to*) a subset of the objects that has good closure properties, here: that preserves reachability of the target / having an infinite reduction.[18]

▶ **Lemma 10.** *For any $\to$-reduction γ there is a $\blacktriangleright$-normal* ind *reduction $\hat{\gamma}$ protracting and through γ, for $\to := \blacktriangleright \cup \triangleright$, if $\blacktriangleright, \triangleright$ is affluent for γ.*

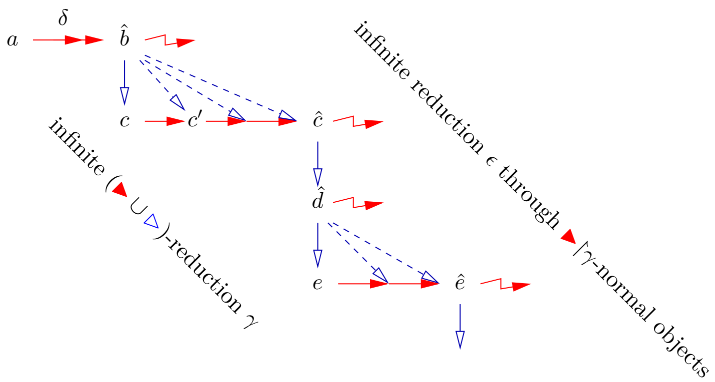

Figure 1 Transformation of γ into reduction of shape $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$ in $(*)$ in the proof of Lem. 15

▶ Remark 11. In Fig. 1 we visualised the simple but key idea of the proof of Lem. 15, the transformation in the iteration step marked $(*)$ below. The figure displays a situation where there are no infinite $\blacktriangleright$-reductions through the objects of the infinite $(\blacktriangleright \cup \triangleright)$-reduction γ, and we find an infinite reduction $\delta \cdot \epsilon$ of shape $a \blacktriangleright\!\!\blacktriangleright \hat{b} \triangleright \hat{c} \triangleright \ldots$, with its $\blacktriangleright$-tail through objects $\hat{b}, \hat{c}, \hat{d}, \hat{e}, \ldots$ in normal form w.r.t. $\blacktriangleright$ restricted to objects on γ, as visualised by lightnings.[19] Note that whereas the original reduction γ was not ind, the resulting reduction $\delta \cdot \epsilon$ is; $\blacktriangleright$s precede $\triangleright$s.

Proof of Lem. 10. Given a reduction γ from a, we construct a $\blacktriangleright$-normal ind reduction $\hat{\gamma}$ through γ that protracts γ. If a is not $\blacktriangleright\restriction\gamma$-terminating, i.e. if there is an infinite $\blacktriangleright$-reduction from a through objects on γ, we may define $\hat{\gamma}$ to be that reduction as it clearly protracts γ, is $\blacktriangleright$-normal and satisfies the ind-criterion. Otherwise, a is $\blacktriangleright\restriction\gamma$-terminating and we let δ be a maximal such reduction from a.[20] If its target is not on γ (but still *in* γ), it is the target of γ and we conclude as before.

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[17] Note that we do not require the ϕ to be steps in γ, only that they are $\blacktriangleright$-steps between objects in γ.

[18] See e.g. [10, Sect. 5 (Conclusion and related work)] or [22, Sect. 9.5] for more on the history and usage of perpetual and maximal strategies in term rewriting, by constriction or other means, including first-order TRSs and the λ-calculus. See e.g. [14, 18] for more on establishing perpetuality and maximality (for abstract rewriting) by means of *local* diagrams in the spirit of [11].

[19] The figure is similar to [19, Fig. 7] (indeed we used the same source file; there used to illustrate the proof of a result corresponding to Lem. 1 here), but note that the lightnings mean different things: In Fig. 1 a lightning means that from that object there is no $\blacktriangleright$-step to an object in the reduction γ itself. In [19, Fig. 7] a lightning means that no $\blacktriangleright$-step from that object could be extended to a reduction protracting the original reduction γ; since there the original reduction was assumed infinite, this boils down to there not being an infinite reduction from the target of any $\blacktriangleright$-step.

[20] Recall [22] a reduction is *maximal* if it cannot be extended, either is infinite or ends in a normal form. *Computations* in [20] are maximal reductions.

Otherwise, the target of δ is $\blacktriangleright\!\upharpoonright\!\gamma$-normal. We claim that any $\blacktriangleright\!\upharpoonright\!\gamma$-normal object $\hat{b}$ is the source of a step $\phi : \hat{b} \triangleright c$ with $\hat{c}$ in γ, such that $\hat{c}$ is either $\blacktriangleright\!\upharpoonright\!\gamma$-normal too or not on γ or not $\blacktriangleright\!\upharpoonright\!\gamma$-terminating. From the claim we conclude $(*)$ by defining ϵ to be a reduction from the target of δ maximally concatenating such steps through $\blacktriangleright$-normal objects. Then ϵ is $\blacktriangleright$-normal per construction. We distinguish cases on whether or not ϵ is finite. If ϵ is finite, we define $\hat{\gamma}$ to be the concatenation of δ, ϵ, followed by an infinite $\blacktriangleright$-reduction if the target of ϵ is not $\blacktriangleright\!\upharpoonright\!\gamma$-terminating. Per construction these do compose since if ϵ is finite then (using the claim) its target is in but not on γ, i.e. it is the target of γ, and $\hat{\gamma}$ is $\blacktriangleright$-normal and ind, by δ comprising $\blacktriangleright$-steps, ϵ comprising $\triangleright$-steps from $\blacktriangleright\!\upharpoonright\!\gamma$-normal objects, and any trailing infinite reduction comprising only $\blacktriangleright$-steps. If ϵ is infinite, we define $\hat{\gamma}$ to be the concatenation of δ and ϵ. Then that $\hat{\gamma}$ protracts γ is trivial, and $\hat{\gamma}$ is $\blacktriangleright$-normal and ind as before.

To prove the claim, assume $\hat{b}$ is $\blacktriangleright\!\upharpoonright\!\gamma$-normal so the source of a step of shape $\hat{b} \triangleright c$ by $\hat{b}$ being $\blacktriangleright$-normal, with c in γ. ▬ If c is $\blacktriangleright\!\upharpoonright\!\gamma$-normal too or is not on γ, we conclude by setting $\hat{c} := c$. ▬ Otherwise, there is a step $c \blacktriangleright\!\upharpoonright\!\gamma\; c'$ with c' in γ. By the assumed affluence for γ and $\blacktriangleright\!\upharpoonright\!\gamma$-normality of $\hat{b}$ there exists $\hat{b} \triangleright c'$. Repeating the second case, we either eventually end up in the first case, or find an infinite reduction $c \blacktriangleright\!\upharpoonright\!\gamma\; c' \blacktriangleright\!\upharpoonright\!\gamma \dots$ so may set $\hat{c}$ to c. ◀

▶ **Corollary 12.** *$\to\; := \;\blacktriangleright \cup \triangleright$ is terminating iff $\blacktriangleright, \triangleright$ are, if $\blacktriangleright, \triangleright$ is affluent.*

This allows to factor Geser's result [8], i.e. Cor. 3(2), through Lem. 10, noting that transitivity of $\to$ entails affluence of $\blacktriangleright, \triangleright$.

▶ **Example 13.** Though it is trivial to observe that affluence is symmetric in its two constituting rewrite systems, it may be instructive to spell out some consequences:

If $\blacktriangleright, \triangleright$ are terminating, then any $\to$-reduction γ from a not only is finite by Cor. 12, but if it has, say, target b we have *both* $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright b$ and $a \triangleright\!\!\triangleright \cdot \blacktriangleright\!\!\blacktriangleright b$ (by switching rôles).

If there is an infinite $\to$-reduction from a, then there are infinite reductions from a of shape $(\blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega})$ *and* of shape $(\triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\triangleright\!\!\triangleright \cdot \blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega})$.[21]

These considerations extend to families of rewrite systems if affluence holds.

▶ Remark 14. In general, the transformation from γ into $\hat{\gamma}$ in the proof of Lem. 10 is not effective since it requires deciding whether or not a given object is $\blacktriangleright\!\upharpoonright\!\gamma$-terminating. However, if we assume $\blacktriangleright$ is terminating, as is the case in Cor. 16 below, then the decision is always 'yes'. Assuming also the other actions (choosing a step in γ given its source object in γ, and finding a step witnessing transitivity) are effective, the transformation itself is.

To obtain the generalisation of Cor. 12 for two rewrite systems, to finite families of rewrite systems [20], we accordingly extend Lem. 10 from two rewrite systems $\blacktriangleright, \triangleright$ to finite families of rewrite systems. The extension and its proof structure are analogous to how Lem. 6 extends Lem. 1, differing only in the invariant employed (using the $\triangleright$-reduction is preferential there vs. $\blacktriangleright$-normal and through objects of the original reduction here). The notions and result coincide with those of Lem. 10 for the case $n = 2$ and $\to_1 := \blacktriangleright$ and $\to_2 := \triangleright$.

▶ **Lemma 15.** *Let $\to$ be the union of the family $(\to_i)_{1 \leq i \leq n}$. For any $\to$-reduction ϵ there is an ind reduction protracting and through it, if* affluence *of the family holds for ϵ: for all a, b, c in ϵ and $1 \leq i < k \leq n$, if $a \to_k b \to_i c$ then $a \to c$.*

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[21] In general, it's not true that the conjunction of any pair, one from each disjunct, holds. E.g. we cannot find infinite reductions from 0 of both shapes $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$ *and* $\triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$, for $\blacktriangleright$ the predecessor relation on even natural numbers and $\triangleright$ the difference of the less-than order and $\blacktriangleright$; so $\blacktriangleright \cup \triangleright \;=\; <$ is trivially transitive.

Proof. By induction on n. In the base case there's only 1 rewrite system. In the step case, given a $\to$-reduction δ, Lem. 10 applies to $\blacktriangleright := \to_1$ and $\triangleright := \bigcup_{2\leq i\leq n} \to_i$ since affluence of the original family for δ entails affluence of $\blacktriangleright, \triangleright$ for δ (seen as a $(\blacktriangleright \cup \triangleright)$-reduction), yielding there is a $\blacktriangleright$-normal reduction $\hat{\delta}$ protracting δ and through it of shape either $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\alpha}$ for $\alpha \leq \omega$. In either case let γ be the $\triangleright$-subreduction, i.e. having indices ≥ 2. Then γ is a $\triangleright\restriction\gamma$-reduction (any reduction is a reduction for the rewrite system restricted to the steps in the reduction) with $\triangleright\restriction\gamma = \bigcup_{2\leq i\leq n} \to_i\restriction\gamma$ by distributivity of intersection over union. Observe that affluence of the latter family holds for γ: if $a \to_k\restriction\gamma\ b \to_i\restriction\gamma\ c$ for $i < k$, then $a \to_j c$ for some $1 \leq j \leq n$ by affluence of the original family for δ, and $j \geq 2$ since a is $\blacktriangleright$-normal per construction of $\hat{\delta}$ (and selection of γ from $\hat{\delta}$), so $a \to_j\restriction\gamma\ c$ as $a \triangleright c$ and a is on γ by $a \to_k\restriction\gamma\ b$, and c in γ by $b \to_k\restriction\gamma\ c$. Hence the induction hypothesis applies to γ yielding an ind reduction $\hat{\gamma}$ protracting and through γ. Thus, $\hat{\gamma}$ is of shape $\twoheadrightarrow_2 \cdot \ldots \cdot \twoheadrightarrow_n (\cdot \to_i^{\omega})$ with the last infinite part, for some index i, optional. We see that the reduction obtained by replacing[12] γ in $\hat{\delta}$ by $\hat{\gamma}$ is as desired, i.e. protracting and through δ and satisfying the ind-criterion.[13] ◀

Observe that *affluence* of the family: $\to_k \cdot \to_i \subseteq \to$ for all $1 \leq i < k \leq n$, entails its affluence for any $\to$-reduction.

▶ **Corollary 16.** **1.** *$\to := \bigcup_{1\leq i\leq n} \to_i$ is terminating iff all $\to_i$ are, if $\to$ is transitive [20, Thm. 1,Cor. 1];*

2. *$\to := \blacktriangleright \cup \triangleright \cup \gg$ is terminating iff $\blacktriangleright, \triangleright, \gg$ are, if $(\triangleright \cdot \blacktriangleright) \cup (\gg \cdot \blacktriangleright) \cup (\gg \cdot \triangleright) \subseteq \to$ [5, Thm. 2].*

Proof. **1.** By Lem. 15 using that transitivity of $\to$ implies affluence of the family;
2. The instance of item 1 for $n := 3$ and $\to_1 := \blacktriangleright$ and $\to_2 := \triangleright$ and $\to_3 := \gg$. ◀

▶ Remark 17. We are puzzled by the following remark in [21] (our boldface): *"As observed by Geser in [13, pag 31], the fact that given any two well-founded binary relations if their union is transitive then it is well-founded has been remarked before Podelski and Rybalchenko. However the Termination Theorem is a non-trivial generalization of this result. In fact it **cannot be directly proved** from it by induction over the number of the relations, since we cannot keep the transitivity through the inductive steps."* True though that may be, it doesn't rule out the possibility that termination of the transitive family $\bigcup_{1\leq i\leq n} \to_i$ follows from termination of the transitive family of *restrictions* $\bigcup_{2\leq i\leq n} \to_i\restriction\gamma$, which as we showed **does suffice** for a direct proof by induction from the case $n = 2$; cf. also the appendix.

Given that the strands of work of [5] and [20] are similar in spirit, both extending [8] from two rewrite systems to arbitrary finite families of such, and the absence of [20] from the references of [5], it seems that the authors of the latter were not aware of the former. Note that though the results and techniques of both are similar, as we show here, they are incomporable; cf. the text below Rem. 14.

Partite. Inspired by [5, Thm. 4] we present the notion of a family being (n-)partite, a variation on affluence, that is on the one hand more strict than affluence in that the index of the steps in its conclusion must have (weakly) *increased*, but on the other hand more liberal in that steps for the *transitive closure* are allowed. It is a generalisation of (the basis for; see below) the notion of *tripartite* [5, Thm. 4].[22]

[22] Below, it will be used as a building block to regain [5, Thm. 22 (Preferential Commutation)].

We call a family $\to := \bigcup_{1\leq i\leq n} \to_i$ *(n-)partite* if: $\to_{>i}\cdot\to_i^+ \subseteq \to_{>i}\cup\to_i^+$ for $1\leq i\leq n$. and say it is *(n-)partite for* a $\to$-reduction γ, if for all a,b,c in γ and $1\leq i<n$, if $a\to_{>i} b\to_i^+ c$, then $a\to_{>i} c$ or $a\to_i^+ c$. Note that a family being (n-)partite entails the same for any $\to$-reduction.

▶ **Lemma 18.** *For any $\to$-reduction ϵ there is an* ind *$\ggg$-reduction $\hat{\epsilon}$ protracting and through ϵ, for $\to := \bigcup_{1\leq i\leq n}\to_i$ partite for ϵ, and $\ggg$ the union of $(\to_i^+)_{1\leq i<n}$ and $\to_n$.*

Proof. By induction on n, with the base case $n=1$ being trivial. For the step case suppose γ is a $\to$-reduction for a family of size $n+1$. Then γ can be seen as a $(\blacktriangleright\cup\triangleright)$-reduction for $\blacktriangleright := \to_1^+$ and $\triangleright := \to_{>1}$, for which Lem. 10 yields an ind $(\blacktriangleright\cup\triangleright)$-reduction $\hat{\gamma}$ protracting and through γ, since affluence of $\blacktriangleright,\triangleright$ for γ follows from the original family for γ being partite for $i=1$.

Let δ be the $\triangleright$-subreduction of $\hat{\gamma}$. It is a $\to$-reduction for the family $(\to_i)_{2\leq i\leq n+1}$ with the family being partite for δ inherited (via $\hat{\gamma}$) from that of the original family for γ. The induction hypothesis for δ then yields an ind $((\bigcup_{2\leq i\leq n}(\to_i)^+)\cup\to_{n+1})$-reduction (hence also an ind $\ggg$-reduction) $\hat{\delta}$ protracting and through δ. Finally, we obtain an ind $\ggg$-reduction protracting[13] and through γ, by replacing[12] δ by $\hat{\delta}$ in $\hat{\gamma}$. ◀

Jumping, affluence and partite being special cases of commutation / factorisaton makes that known methods [15, 17] for *localising* [11] the latter are at our disposal for the former. In particular, the transitive closure in the assumption of a family being partite may be elided.

▶ Remark 19. Being *local* bipartite $\triangleright\cdot\blacktriangleright \subseteq \triangleright\cup\blacktriangleright^+$ entails $\triangleright\cdot\blacktriangleright^+ \subseteq \triangleright\cup\blacktriangleright^+$, i.e. $\blacktriangleright,\triangleright$ being bipartite.[23] This follows from $\triangleright\cdot\blacktriangleright^n \subseteq \triangleright\cup\blacktriangleright^+$ for all n, which can be proven by induction on n; cf. the proof of Lem. 25. This is the analogon of Hindley's Lemma [15, Ex. 15] and of that *semi-confluence* entails *confluence* [1]. Combining that if $\triangleright\cdot\blacktriangleright\subseteq\triangleright\cup\blacktriangleright$ then $\triangleright^+\cdot\blacktriangleright^+\subseteq\triangleright^+\cup\blacktriangleright^+$ [19, Lem. 2.4] with the above, yields that if $\triangleright\cdot\blacktriangleright\subseteq\triangleright\cup\blacktriangleright^+$ then $\triangleright^+\cdot\blacktriangleright^+\subseteq\triangleright^+\cup\blacktriangleright^+$, by $+$ being a closure operation.

▶ Remark 20. The $\ggg$-reduction obtained in Lem. 18 can be trivially transformed into a $\to$-reduction, by unfolding $\to_i^+$-steps into (non-empty) $\to_i$-reductions, preserving ind and protracting the original reduction again, but not necessarily *through* its objects as the objects introduced by unfolding need not satisfy that constraint; cf. footnote 16. (Of course, if we drop the transitive closures in being partite, the unfolding is trivial, does not introduce objects, and the resulting reduction is through the original one per Lem. 15.)

Combining. As known and shown in [5, Sect. 3] simply taking the 'union' of the conditions of modular termination results typically fails. For instance, one could surmise that $\to := \bigcup_{1\leq i\leq n}\to_i$ is terminating iff all $\to_i$ are, if $\to_{>i}\cdot\to_i \subseteq \to\cup(\to_i\cdot\twoheadrightarrow_{\geq i})$ (†) holds, a condition 'unifying' the jumping and affluence conditions (of Lem. 6 and 15). But this fails already for $n=3$, as can be seen by reusing [5, Ex. 9(a)].[24]

▶ **Example 21.** $b\blacktriangleright d$, $c\triangleright d\triangleright a\triangleright b$, and $a\ggg d$, $b\ggg c$ are terminating but their union is not, e.g. $a\ggg d\triangleright a$, despite satisfying condition (†) for $\to_1 := \blacktriangleright$, $\to_2 := \triangleright$ and $\to_3 := \ggg$.

Still, sometimes one can 'stack' the results 'on top of' each other. This technique was already employed to good effect in [5, Sect. 4 and 7]. Here we give further examples of such a modular

[23] By taking the converse being an anti-automorphic involution, this is equivalent to that if $\blacktriangleright\cdot\triangleright\subseteq\blacktriangleright^+\cup\triangleright$ then $\blacktriangleright^+\cdot\triangleright\subseteq\blacktriangleright^+\cup\triangleright$.

[24] Such finite counterexamples are also easily found automatically by the tool Carpa [24].

approach using the above three results (jumping, affluence, partite) as basic building blocks, and also show that these can be used to refactor some known results.

We first stack affluence on top of jumping for $n = 2$, i.e. on top of Lem. 1. The idea is that given a reduction γ jumping yields a reduction $\hat{\gamma}$ through *preferential* objects, and affluence can be stacked on top of it due to that it zooms-in on a *subset* of the objects of $\hat{\gamma}$, still preferential.

Formally, call an object *preferential* (for γ) if any $\blacktriangleright$-step from it yields a $\rightarrow$-terminating object from which the target (if any) of γ cannot be reached (by $\rightarrow$-steps), for $\rightarrow = \blacktriangleright \cup \triangleright$.

▶ **Lemma 22.** *For any $\rightarrow$-reduction there is an* ind *reduction protracting it, for $\rightarrow := \bigcup_{0 \leq i \leq n} \rightarrow_i$, if* jumping affluence *holds: $\rightarrow_{>i} \cdot \rightarrow_i \subseteq \rightarrow_{>0} \cup (\rightarrow_0 \cdot \twoheadrightarrow)$ for $0 \leq i \leq n$.*

Proof. Suppose to have a $\rightarrow$-reduction δ. Since jumping affluence entails jumping for $\blacktriangleright := \rightarrow_0$ and $\triangleright := \bigcup_{1 \leq i \leq n} \rightarrow_i$ and $\rightarrow = \blacktriangleright \cup \triangleright$, by Lem. 1 there is a preferential $(\blacktriangleright \cup \triangleright)$-reduction $\hat{\delta}$ protracting δ of shape either $\blacktriangleright\!\!\!\blacktriangleright \cdot \triangleright\!\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\blacktriangleright\!\!\!\blacktriangleright \cdot \triangleright^{\alpha}$ for $\alpha \leq \omega$. Let γ be the $\triangleright$-part of $\hat{\delta}$, in either case. By $\hat{\delta}$ being preferential for δ, each object on γ is preferential for δ.

By definition, γ is a $\triangleright\restriction\gamma = \bigcup_{1 \leq i \leq n} (\rightarrow_i\restriction\gamma)$-reduction. We claim affluence of this family holds, yielding an ind $(\triangleright\restriction\gamma)$-reduction $\hat{\gamma}$ protracting γ by Lem. 15. By replacing γ by $\hat{\gamma}$ in $\hat{\delta}$, we then obtain an ind $\rightarrow$-reduction protracting $\hat{\delta}$, hence δ.

To prove the claim, suppose $a\ (\rightarrow_k\restriction\gamma) \cdot (\rightarrow_i\restriction\gamma)\ b$ for some $1 \leq i < k \leq n$. Then $a \rightarrow_k \cdot \rightarrow_i b$ and a is on γ and b is in γ, so by jumping affluence either $a \rightarrow_j b$ for some $j > 0$ or $a \rightarrow_0 \cdot \twoheadrightarrow b$. In the former case, we conclude to $a \rightarrow_j\restriction\gamma\ b$ as desired, whereas the latter case cannot hold as that would contradict the object a on $\hat{\delta}$ being preferential, as it then would allow a reduction headed by a $\blacktriangleright$-step and protracting it, via b. ◀

▶ **Corollary 23.** *$\rightarrow := \bigcup_{0 \leq i \leq n} \rightarrow_i$ is terminating iff all $\rightarrow_i$ are, if jumping affluence holds.*

▶ Remark 24. Note that for the rewrite systems in Ex. 21, jumping affluence fails: $a \gg \cdot \triangleright a$ but neither $a\ (\triangleright \cup \gg)\ a$ nor $a \blacktriangleright \cdot \twoheadrightarrow a$.

Similarly, being partite may be stacked on top of jumping for $n = 2$, i.e. on top of Lem. 1.

▶ **Lemma 25.** *For any $\rightarrow$-reduction there is an* ind *reduction protracting it, for $\rightarrow := \bigcup_{0 \leq i \leq n} \rightarrow_i$ if* jumping partite *holds: $\rightarrow_{>i} \cdot \rightarrow_i \subseteq \rightarrow_{>i} \cup \rightarrow_i^{+} \cup (\rightarrow_0 \cdot \twoheadrightarrow)$ for $0 \leq i < n$.*

Proof. Suppose to have a $\rightarrow$-reduction δ. Since jumping partite entails jumping for $\blacktriangleright := \rightarrow_0$ and $\triangleright := \bigcup_{1 \leq i \leq n} \rightarrow_i$ and $\rightarrow = \blacktriangleright \cup \triangleright$, by Lem. 1 there is a preferential $(\blacktriangleright \cup \triangleright)$-reduction $\hat{\delta}$ protracting δ of shape either $\blacktriangleright\!\!\!\blacktriangleright \cdot \triangleright\!\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\blacktriangleright\!\!\!\blacktriangleright \cdot \triangleright^{\alpha}$ for $\alpha \leq \omega$. Let γ be the $\triangleright$-part of $\hat{\delta}$, in either case. By $\hat{\delta}$ being preferential for δ, each object on γ is preferential for δ.

Jumping partite entails $\rightarrow_{>i} \cdot \rightarrow_i^{+} \subseteq \rightarrow_{>i} \cup \rightarrow_i^{+} \cup (\rightarrow_0 \cdot \twoheadrightarrow)$ for $1 \leq i < n$, as shown by an easy induction; cf. Rem. 19. We claim that from this it follows that the family $(\rightarrow_i)_{1 \leq i \leq n}$ is partite for γ, yielding an ind $\triangleright$-reduction $\hat{\gamma}$ protracting γ by Lem. 18 and using Rem. 20. By replacing γ by $\hat{\gamma}$ in $\hat{\delta}$, we then obtain an ind $\rightarrow$-reduction protracting $\hat{\delta}$, hence δ.

To prove the claim, note that if $a \rightarrow_{>i} b \rightarrow_i c$ for a, b, c in γ and $1 \leq i < n$, then $a \rightarrow_0 \cdot \twoheadrightarrow c$ cannot hold, as that would contradict (via c) a being preferential for δ. ◀

▶ **Corollary 26** ([5])**. 1.** *$\rightarrow := \bigcup_{0 \leq i \leq n} \rightarrow_i$ is terminating iff all $\rightarrow_i$ are, if jumping partite holds [5, Thm. 22 (Preferential Commutation)];*[25]

[25] *Jumping partite* is our systematic naming arising from (re)factoring into jumping and partite. It is called *preferential commutation* in [5].

2. $\blacktriangleright \cup \triangleright \cup \gg$ *is terminating iff each of* $\blacktriangleright, \triangleright, \gg$ *is, if* $(\triangleright \cup \gg) \cdot \blacktriangleright \subseteq \triangleright \cup \gg \cup (\blacktriangleright \cdot (\blacktriangleright \cup \triangleright \cup \gg)^*)$ *and* $\gg \cdot \triangleright \subseteq \gg \cup \triangleright^+ \cup (\blacktriangleright \cdot (\blacktriangleright \cup \triangleright \cup \gg)^*)$ *[5, Thm. 4 (Tripartite)].*

Proof. **1.** As for Cor. 3(1) but using Lem. 25 (instead of Lem. 1);
2. The instance of item 1 for $n := 2$ and $\to_0 := \blacktriangleright$ and $\to_1 := \triangleright$ and $\to_2 := \gg$. ◀

One can recombine the results in many ways. Here we present but two examples illustrating that: we stack jumping on top of jumping affluence and jumping partite, respectively.

▶ **Corollary 27** (Family packs). $\to := \bigcup_{0 \leq i \leq n}$ *is terminating iff all* $\to_i$ *are,*

1. *if for some* $0 \leq k \leq n$, $\to_{>i} \cdot \to_i \subseteq \to_{>0} \cup (\to_0 \cdot \twoheadrightarrow)$ *for* $0 \leq i < k$*, and* $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ *for* $k \leq i < n$*; or*
2. *if for some* $0 \leq k \leq n$, $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup \to_i^+ \cup (\to_0 \cdot \twoheadrightarrow)$ *for* $0 \leq i < k$*, and* $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ *for* $k \leq i < n$ *[5, Thm. 28 (Preferential Jumping)]; cf. p. 80 of the accompanying presentation slides).*

Proof. **1.** Suppose there were an infinite $\to$-reduction γ. This induces 'the same' infinite reduction for the family $(\to'_i)_{0 \leq i \leq k}$ where $\to'_i := \to_i$ for $0 \leq i < k$ and $\to'_k := \bigcup_{k \leq i \leq n} \to_i$ (combining all $\to_i$ for $k \leq i \leq n$). Then $\to' = \to$ (as relations) and jumping affluence holds, $\to'_{>i} \cdot \to'_i \subseteq \to'_{>0} \cup (\to'_0 \cdot \twoheadrightarrow')$ for $0 \leq i < k$, by assumption for the original family, as $\to'_{>i} = \to_{>i}$, $\to'_i = \to_i$ and $\to'_0 = \to_0$ for such i. Hence Lem. 22 applies yielding an infinite ind-reduction tailing off in an infinite $\to'_i$-reduction δ for some $0 \leq i \leq k$. Since $\to'_i = \to_i$ is assumed terminating for $0 \leq i < k$, we must in fact have that δ is an infinite $\to'_k$-reduction, inducing 'the same' infinite reduction for the family $(\to_i)_{k \leq i \leq n}$. for which jumping holds by assumption. Hence Lem. 6 applies yielding an infinite ind-reduction tailing off in an infinite $\to_i$-reduction for some $k \leq i \leq n$; contradiction.
2. Suppose there were an infinite $\to$-reduction γ. This induces 'the same' infinite reduction for the family $(\to'_i)_{0 \leq i \leq k}$ where $\to'_i := \to_i$ for $0 \leq i < k$ and $\to'_k := \bigcup_{k \leq i \leq n} \to_i$ (combining all $\to_i$ for $k \leq i \leq n$). Then $\to' = \to$ (as relations) and jumping partite holds, $\to'_{>i} \cdot \to'_i \subseteq \to'_{>i} \cup (\to'_i)^+ \cup (\to'_0 \cdot \twoheadrightarrow')$ for $0 \leq i < k$, by assumption for the original family, as $\to'_{>i} = \to_{>i}$, $\to'_i = \to_i$ and $\to'_0 = \to_0$ for such i. Hence Lem. 25 applies yielding an infinite ind-reduction tailing off in an infinite $\to'_i$-reduction δ for some $0 \leq i \leq k$. We then proceed as in the previous item. ◀

▶ Remark 28. As before the proof shows more (than preservation of termination); ind reductions are obtained in both cases.

Conclusion and future work. We have presented more modular termination results[26] much in the spirit of [6, 5] but without the (direct) aim of applying the results to path orders; the aims here were mainly methodological in nature. Though we do expect our refined results to have applications to path orders and also in the study of transition invariants [20]. We leave that to further research, but illustrate the idea by the following simple example [20].

▶ **Example 29.** Collapsing non-positive integers, the program in [20, Fig. 2 (**CHOICE**)] is faithfully modelled (qua termination) by the transition relation R relating pairs of natural numbers (x, y) and (x', y') if the latter is either $(x \dot{-} 1, x)$ or $(y \dot{-} 2, x + 1)$, assuming $x, y > 0$ (which we leave implicit below). Consider moreover $\blacktriangleright := \neg P(x, y) \wedge (Q \vee P(x', y'))$ and $\triangleright := P(x, y) \wedge Q \wedge P(x', y')$ for $Q := x + y > x' + y'$ and $P(n, m) := m \dot{-} 2 \leq n \leq m \dot{-} 1$.

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[26] Both *(more modular) (termination results)* and *more (modular termination results)*.

Then $R \subseteq \to$ for $\to := \blacktriangleright \cup \triangleright$ since P is created by the first and preserved by the second R-transition, and Q holds for the second, and both $\blacktriangleright$ and $\triangleright$ are terminating since Q is and since P and $\neg P$ do not compose for $\blacktriangleright$. For the same reason $\triangleright \cdot \blacktriangleright = \emptyset$ yielding affluence of $\blacktriangleright, \triangleright$ hence termination of $\to$ by Cor. 12, so R is terminating.

The modular termination technique of [20] relies on checking that the *transitive closure* R^+ of the transition relation is included in the so-called transition invariant[27] and it is reliance on this that *"makes the method more difficult in practice"* [4, sidebar on p. 90]. The above exemplifies that checking affluence is in general easier than checking transitivity, gives rise to fewer constraints, suggesting it might be profitable to use instead.

▶ Remark 30.

- We arrived at the properties P and Q by hand: first seeing the decrease of the sum in the second R-transition (modelled by Q), and then seeing that though the first R-transition may increase that sum in general, we then end up in a state (modelled by P) from which on it will not. This gives rise to the question how to automate this.
- The analysis of the **CHOICE** example in [20] is based on their notion of *transition invariant* [20, Def. 1]: a superset of the transitive closure of the transition relation R of a program restricted to its accessible states. For **CHOICE**, first the transition invariant $T = T_1 \cup T_2 \cup T_3$ for $T_1 := x' < x$ and $T_2 := x' + y' < x + y$ and $T_3 := y' < y$ is proposed in [20, Sect. 3], and next a so-called *inductive* transition invariant $I = I_1 \cup I_2 \cup I_3 \cup I_4$ for $I_1 := x' < x \wedge y' \leq x$ and $I_2 := x' < y - 1 \wedge y' \leq x + 1$ and $I_3 := x' < y - 1 \wedge y' < y$ and $I_4 := x' < x \wedge y' < y$ is proposed in [20, Sect. 5] (we again omitted positivity conditions). *How* T, I were arrived at was not given in [20],[28] but methods to find such automatically has since been the subject of a flurry of follow-up research (tools); see e.g. [4]. We think it should be interesting to try to rebase those developments, instead of on the termination theorem of [20], on the methods presented here, in particular on affluence (introduced), jumping [7], partite [5], and on the combinations thereof.
- Termination of **CHOICE** itself is (and was in 2004) automatic: the transition relation R can be faithfully modelled[29] by the reduction relation of the TRS with rules:

$$
\begin{array}{rcl}
p(s(x), s(y)) & \to & p(x, s(x)) \\
p(s(x), s(s(y))) & \to & p(y, s(s(x)))
\end{array}
$$

where natural numbers are represented in unary, and termination of the TRS is easily shown by termination tools for TRSs such as Aprove and $\mathsf{T_TT_2}$, e.g. by the polynomial interpretation $p(n, m) := 9n + m + 15$ and $s(n) := 4n + 1$, which entails termination of R.

We have opted for presenting the results in a 2D way: as transformations on transformations (reductions), extending our earlier basic approach in [16] to also cover [20, 5]. We go beyond the latter in two ways: (1) by dealing not only with infinite reductions but also with finite reductions, opening up the possibility of comparing / transforming reduction lengths; (2) by precisely characterising the shapes of the transformed reductions (as reductions whose indices are sorted in non-decreasing-order with the possible exception of an infinite tail for one of the indices). We leave reaping potential benefits from this to future work.

Due to its relationship to Ramsey Theory, (some of the) problems and results considered here have attracted attention in proof theory and constructive mathematics, studying them

[27] Cf. the specialisation of the proof rule of [20, Fig. 5] to termination, as proposed there; see Remark 30.
[28] Also *that* T, I are transition invariants is not shown but is suggested to follow by noting that I is *inductive*, i.e. $R \cup (I \cdot R) \subseteq I$, and that I entails T.
[29] Terminating *before* not *after* a negative components would be obtained though.

using various tools, e.g. *inductive* termination (which holds for an object if it does for each of its one-step reducts, inductively), *almost fulness* (which holds for a relation if its complement does not allow a homogeneous sequence [22, App. A.5]), *well-quasi orders* (WQOs; well-founded orders without infinite anti-chains), *better-quasi orders*, *open* induction, *bar* recursion, *calculational* proofs, ...; see e.g. [3, 23] and other literature cited (or not) above for more. We leave adapting those analyses for later / to others, and have focussed exclusively on (structuring) the results (and their proofs) and on the transformational 2D perspective.

That being said, we foremost hope the results and their proofs are correct, that they can be useful for (formalised) termination proofs, and that our structuring constitutes a good basis for further extensions and tools.

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▶ Remark 31. All URLs in the above bibliography active at 3-7-2023.

Appendix. We recapitulate our original[30] proof [12] of the disjunctive termination theorem of [20], for the (extension of the) predicate P on objects defined by:

$$P := \{a \mid a \text{ is } \to\text{-perpetual and all single-step } \to_{n+1}\text{-reducts of } a \text{ are } \to\text{-terminating}\}$$

where we (re)call an object (is) *perpetual* if there is an infinite reduction from it.[31]

Proof. Assume $\to := \bigcup_{i \leq n+1} \to_i$ is transitive.

To show transitivity of $\blacktriangleright := \to' \restriction P := \{\phi : a \to' b \mid a, b \in P\}$ for $\to' := \bigcup_{i \leq n} \to_i$, suppose $a \blacktriangleright b \blacktriangleright c$. For $a \blacktriangleright c$ to hold, $a \to' b$ and $a, b \in P$ must hold. That $a \in P$ holds follows from $a \blacktriangleright b$, and that $c \in P$ holds from $b \blacktriangleright c$. To see that $a \to' c$, it suffices that $a \blacktriangleright b \blacktriangleright c$ entails $a \to b \to c$ by $\blacktriangleright \subseteq \to' \subseteq \to$, hence $a \to c$ by the assumed transitivity of $\to$, from which we conclude to $a \to' c$ by observing that if all single-step $\to_{n+1}$-reducts of an object d are $\to$-terminating and $d \to e$ with e $\to$-perpetual then $d \to' e$, (as otherwise e would be $\to$-terminating as single-step $\to_{n+1}$-reduct of d), with the conditions of the observation met for $a \to c$ by $a, c \in P$.

Combining transitivity of $\to' \restriction P$ with $\to \restriction P = \to' \restriction P = \bigcup_{i \leq n} (\to_i \restriction P)$, which hold respectively by the same observation and distributivity of intersection over union, the induction hypothesis is seen to apply to yield termination of $\to \restriction P$, since each $\to_i \restriction P$ is terminating as *restriction* of $\to_i$, with the latter terminating by assumption.

We claim that for any $\to$-perpetual object b there is a $c \in P$ such that the following two properties hold: (I) $b \twoheadrightarrow_{n+1} c$; and (II) for any a whose single-step $\to_{n+1}$-reducts are $\to$-terminating, if $a \to b$ then $a \to' c$. From the claim it follows that all $a \in P$ are $\to \restriction P$-perpetual, since by $a \in P$ we have $a \to b$ for some $\to$-perpetual b, hence by (II) there is a $c \in P$ such that $a \to' c$, using that the single-step $\to_{n+1}$-reducts are $\to$-terminating by $a \in P$.

Combining the two previous paragraphs we have on the one hand that objects in P are $\to \restriction P$-perpetual, but on the other hand that $\to \restriction P$ is terminating, so that P must be empty. But then by (I), there are no $\to$-perpetual objects, i.e. $\to$ is terminating, as desired.

[30] Translated from Dutch into English.

[31] Perpetual objects are named in line with the *perpetual* strategy [22, Def. 4.9.16 and Sect. 9.5], selecting such an object if one is available. Perpetual objects are called *immortal* in [5].

It remains to prove (the two items of) the claim. We proceed by well-founded induction on b ordered by ${}_{n+1}{\leftarrow}$, distinguishing cases on whether or not $b \in P$:

- if $b \in P$, then we trivially conclude by the observation setting $c := b$; and
- if $b \notin P$, then since b *is* $\rightarrow$-perpetual there exists a $\rightarrow$-perpetual b' such that $b \rightarrow_{n+1} b'$. By the induction hypothesis for b', there is an object a $c' \in P$ such that (i) $b' \twoheadrightarrow_{n+1} c'$; and (ii) for any a whose single-step $\rightarrow_{n+1}$-reducts are $\rightarrow$-terminating, if $a \rightarrow b'$ then $a \rightarrow' c'$. Setting $c := c'$ we conclude to (I) by $b \rightarrow_{n+1} b' \twoheadrightarrow_{n+1} c' = c$ using (i); and to (II) since for any a whose single-step $\rightarrow_{n+1}$-reducts are $\rightarrow$-terminating, if $a \rightarrow b$ then $a \rightarrow b'$ by $b \rightarrow_{n+1} b'$ and the assumed *transitivity* of $\rightarrow$, from which we conclude to $a \rightarrow' c' = c$ using (ii). ◀

Note that as for the proof of Lem. 15 a restricted form of transitivity suffices for the proof to go through, but in this case it suffices to have $\rightarrow_i \cdot \rightarrow_k \subseteq \rightarrow$ only for $i < k$. However, since the order on the indices of the relations $\rightarrow_i$ was chosen arbitrarily, this is equivalent.