Increasingness by Random Descent[1]

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Abstract

We show classical results by Nederpelt / Klop on inductive / increasing rewrite systems factor through random descent, trivialising the former and illustrating the power of the latter.

The main result CR & SN $\iff$ OWCR & WN on *random descent* in [3, Corollary 2] is powerful in that it allows to reduce confluence (CR) and termination (SN) of *any* rewrite system to it being ordered weakly Church–Rosser (OWCR) and normalising (WN). With the other notions standard [4], we only recall from [3, Definition 3] that a rewrite system [4, Definition 8.2.2] $\to$ on A is *ordered* weakly Church–Rosser if for all local peaks $b \leftarrow_\mu a \to_\nu c$ there is a valley $b \to^*_{\nu'} d \; {}_{\mu'}^{*}\!\!\leftarrow c$ with $\nu' + \mu \geq \mu' + \nu$, or there is an infinite rewrite sequence from b, where steps are *measured* $(\mu, \nu, \ldots)$ by non-unit elements of a derivation monoid and rewrite sequences by the sum of their steps from tail to head. Here a *derivation* monoid comprises a monoid with unit $\perp$ and operation $+$, and a well-founded partial order $\leq$ on the carrier, having $\perp$ as least element, and with $\leq$ monotonic in both arguments of $+$, strictly so in the second [3, Definition 2]. We say rewrite sequences that are parallel, i.e. have the same source and target, are *commensurate* (COμ) if they have the same measure, and that $\to$ is COμ if all rewrite sequences that are parallel are. Recall:

Definition (Nederpelt / Klop; cf. Def. 1.1.15 of [4])**.** *(i)* $\to$ *is* inductive *(*Ind*) if for all* $a_0 \to a_1 \to \ldots$, $\exists a \in A$ *such that* $a_i \to^* a$ *for all* $i \in \mathbb{N}$.

(ii) $\to$ *is* increasing *(*Inc*) if there is a map* $|\ | : A \to \mathbb{N}$ *such that* $\forall a, b \in A (a \to b \Rightarrow |a| < |b|)$.

Lemma. *(nulla)* Inc $\Rightarrow$ COμ

(i) COμ & Ind $\Rightarrow$ SN

(ii) COμ & WCR $\Rightarrow$ OWCR

Proof. (nulla) Define the measure of a step $a \to b$ to be $|b| \dot{-} |a|$. By Inc, this yields a measure with respect to zero 0 and addition $+$ for $\mathbb{N}$, well-foundedly partially-ordered by less-than-or-equal $\leq$, and COμ holds since the measure of *any* reduction $a \to^* b$ is $|b| \dot{-} |a|$.[2]

(i) Suppose to have a rewrite sequence $a_0 \to_{\mu_0} a_1 \to_{\mu_1} \ldots$. By Ind, we have an object a and rewrite sequences $a_i \to^*_{\nu_i} a$ for all $i \in \mathbb{N}$. By COμ we have $\nu_{i+1} + \mu_i = \nu_i$. Then $\nu_{i+1} < \nu_i$, since by the assumptions on measures $+$ is strictly increasing in its second argument and $\perp < \mu_i$. By well-foundedness of $\leq$ we conclude the rewrite sequence must be finite.

(ii) The legs of the local confluence diagram obtained by WCR are commensurate by COμ. $\square$

Nederpelt's Inc & Ind $\Rightarrow$ SN and Klop's Inc & WCR & WN $\Rightarrow$ SN (cf. [4, Thm. 1.2.3]), both follow trivially; the former from (nulla) and (i), and the latter from (nulla), (ii) and the main result on random descent above. This goes to show that *measures* can be fruitfully transferred from objects (the map $|\ |$) to steps (via (nulla)) for *random descent*, in much the same way as *labels* can be transferred from objects to steps for *decreasing diagrams* (cf. [2, Example 12]).

References

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[3] V. van Oostrom. Uniform completeness. In *IWC*, pages 19–24, 2022. URL: http://cl-informatik.uibk.ac.at/iwc/2022/proceedings.pdf.

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[2] Cf. [1]. This would fail when measuring in (standard) ordinals by non-commutativity of $+$; cf. $0 \to_1 1 \to_\omega \omega$.