

Increasingness by Random Descent¹

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Abstract

We show classical results by Nederpelt / Klop on inductive / increasing rewrite systems factor through random descent, trivialising the former and illustrating the power of the latter.

The main result $\text{CR} \ \& \ \text{SN} \iff \text{OWCR} \ \& \ \text{WN}$ on *random descent* in [3, Corollary 2] is powerful in that it allows to reduce confluence (CR) and termination (SN) of *any* rewrite system to it being ordered weakly Church–Rosser (OWCR) and normalising (WN). With the other notions standard [5], we only recall from [3, Definition 3] that a rewrite system $\rightarrow$ on A is *ordered* weakly Church–Rosser if for all local peaks $b \leftarrow_{\mu} a \rightarrow_{\nu} c$ there is a valley $b \rightarrow_{\nu'}^* d \xleftarrow{\mu'}^* c$ with $\nu' + \mu \geq \mu' + \nu$, or there is an infinite rewrite sequence from b , where steps are *measured* $(\mu, \nu, \dots)$ by non-unit elements of a derivation monoid and rewrite sequences by the sum of their steps from tail to head. Here a *derivation* monoid comprises a monoid with unit $\perp$ and operation $+$, and a well-founded partial order $\leq$ on the carrier, having $\perp$ as least element, and with $\leq$ monotonic in both arguments of $+$, strictly so in the second [3, Definition 2]. We say rewrite sequences that are parallel, i.e. have the same source and target, are *commensurate* ($\text{CO}\mu$) if they have the same measure, and that $\rightarrow$ is $\text{CO}\mu$ if all rewrite sequences that are parallel are. Recall:

Definition (Nederpelt / Klop; cf. Def. 1.1.15 of [5]). (i) $\rightarrow$ is inductive (Ind) if for all $a_0 \rightarrow a_1 \rightarrow \dots$, $\exists a \in A$ such that $a_i \rightarrow^* a$ for all $i \in \mathbb{N}$.

(ii) $\rightarrow$ is increasing (Inc) if there is a map $|\cdot| : A \rightarrow \mathbb{N}$ such that $\forall a, b \in A (a \rightarrow b \Rightarrow |a| < |b|)$.

Lemma. (nulla) $\text{Inc} \Rightarrow \text{CO}\mu$

(i) $\text{CO}\mu \ \& \ \text{Ind} \Rightarrow \text{SN}$

(ii) $\text{CO}\mu \ \& \ \text{WCR} \Rightarrow \text{OWCR}$

Proof. (nulla) Define the measure of a step $a \rightarrow b$ to be $|b| - |a|$. By Inc, this yields a measure with respect to zero 0 and addition $+$ for $\mathbb{N}$, well-foundedly partially-ordered by less-than-or-equal $\leq$, and $\text{CO}\mu$ holds since the measure of *any* reduction $a \rightarrow^* b$ is $|b| - |a|$.²

(i) Suppose to have a rewrite sequence $a_0 \rightarrow_{\mu_0} a_1 \rightarrow_{\mu_1} \dots$. By Ind, we have an object a and rewrite sequences $a_i \rightarrow_{\nu_i}^* a$ for all $i \in \mathbb{N}$. By $\text{CO}\mu$ we have $\nu_{i+1} + \mu_i = \nu_i$. Then $\nu_{i+1} < \nu_i$, since by the assumptions on measures $+$ is strictly increasing in its second argument and $\perp < \mu_i$. By well-foundedness of $\leq$ we conclude the rewrite sequence must be finite.

(ii) The legs of the local confluence diagram obtained by WCR are commensurate by $\text{CO}\mu$. $\square$

Nederpelt’s Inc & Ind $\Rightarrow$ SN and Klop’s Inc & WCR & WN $\Rightarrow$ SN (cf. [5, Thm. 1.2.3]), both follow trivially; the former from (nulla) and (i), and the latter from (nulla), (ii) and the main result on random descent above. This goes to show that *measures* can be fruitfully transferred from objects (the map $|\cdot|$) to steps (via (nulla)) for *random descent*, in much the same way as *labels* can be transferred from objects to steps for *decreasing diagrams* (cf. [2, Example 12]).

References

- [1] V. van Oostrom. Eventually increasing, 1998. URL: <http://www.javakade.nl/research/pdf/ei.pdf>.
- [2] V. van Oostrom. Confluence by decreasing diagrams; converted. In *RTA 2008*, volume 5117 of *LNCS*, pages 306–320. Springer, 2008. doi:10.1007/978-3-540-70590-1_21.

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²Cf. [1]. This would fail when measuring in (standard) ordinals by non-commutativity of $+$; cf. $0 \rightarrow_1 1 \rightarrow_{\omega} \omega$.

- [3] V. van Oostrom. Uniform completeness. In *IWC*, pages 19–24, 2022. URL: <http://www.javakade.nl/research/pdf/uc-iwc-2022.pdf>.
- [4] V. van Oostrom and Y. Toyama. Normalisation by Random Descent. In *FSCD*, volume 52 of *LIPICs*, pages 32:1–32:18. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPICs.FSCD.2016.32.
- [5] Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.

Appendix

Let $| \cdot | : A \rightarrow \mathbb{N}$ be a map on the objects A of a rewrite system $\rightarrow$.

Definition. We say $\rightarrow$ is locally Dyck, LD, if for every local peak $b \leftarrow a \rightarrow c$ there is a conversion $b \leftrightarrow^* c$ such that $|a| < |d|$ for all objects d on the conversion.

Observe $\text{LD} \implies \text{Inc}$ (consider peaks $b \leftarrow a \rightarrow b$) and $\text{Inc} \ \& \ \text{WCR} \implies \text{LD}$ (valleys are conversions).

Remark. The name LD for the property is from [4]. There it was defined through a measure on reductions instead of on objects as here, with the two related by that the measure of a reduction $a \rightarrow^* b$ is the difference $|b| - |a|$ of the measures of its target and source objects (as used above already).

The naming derives from that the language of matching parentheses, also known as the Dyck language, can be equivalently characterised as that for every prefix of a string in the language the number of opening parentheses in it minus the number of closing parentheses in it, is non-negative (and zero for the string).

Proposition. If LD and $\text{WN}(a)$, then $\text{SN}(a)$.

Proof. We show that if a reduces to normal form z then a is z -complete, i.e. $\text{SN}(a)$ and any maximal reduction from a ends in z , by induction on $|z| - |a|$. If a is a normal form, then it is trivial. Otherwise, we show that if $a \rightarrow c$ then c is z -complete, by induction on the conversion $b \leftrightarrow^* c$ obtained by LD for the local peak $b \leftarrow a \rightarrow c$ where $a \rightarrow b \rightarrow^* z$. If the conversion is empty, then we conclude by the main IH for b . Otherwise, the conversion has shape $b \leftrightarrow^* d \leftrightarrow c$ for some d . If $d \rightarrow c$ then c is z -complete as one-step reduct of d , which is z -complete by the sub IH. Otherwise, $d \leftarrow c$ and c is z -complete by the main IH, which applies since c reduces to z via d , and $|z| - |c| < |z| - |a|$ by c being on the conversion and LD. $\square$

The proposition is the conversion-version of Klop’s result $\text{Inc} \ \& \ \text{WCR} \ \& \ \text{WN} \Rightarrow \text{SN}$,³ in the same way that Winkler & Buchberger’s Lemma is the conversion-version of Newman’s Lemma, cf. [2]. The proposition strengthens Klop’s result in two ways: it only asks for LD (a *conversion* above the source) instead of Inc & WCR (a *valley* above the source), and moreover it is *local* (assuming only $\text{WN}(a)$) instead of global (assuming WN) so applies also to rewrite systems having objects without normal form. In the nomenclature introduced in [3], LD is a *sufficient* condition for *uniform* completeness, i.e. that all objects convertible to normal form are *complete* (SN & CR). The rewrite system $\rightarrow$ defined by $n + 1 \rightarrow n$ for all $n \in \mathbb{N}$ shows that LD is not a *necessary* condition for (uniform) completeness.

Acknowledgment We thank Anupam Das for asking for a simple proof of Klop’s result. The simple proof in this appendix was obtained by ‘cut-elimination’ on its proof presented on the previous page.

³That follows since $\text{Inc} \ \& \ \text{WCR} \implies \text{LD}$ as noted, and WN and SN mean $\text{WN}(a)$ respectively $\text{SN}(a)$ for all a .