Contents

1. Equivalence of reductions

Problem 1 *When are two reductions equivalence?*

Solution 2 *If they perform the same steps, possibly in a different order.*

1.1. Running example

TRS with rules

$$\begin{aligned} a &\to b \\ f(x,b) &\to g(x) \\ g(b) &\to c \end{aligned}$$

Two reductions from $f(a,a)$ to the term c.

$$\mathcal{R} \ : \ f(\tilde{a},a) \to f(b,\overline{a}) \to \underline{f}(b,\underline{b}) \to \underline{g}(\underline{b}) \to c$$

$$\mathcal{S} \ : \ f(a,\overline{a}) \to \underline{f}(a,\underline{b}) \to g(\tilde{a}) \to \underline{g}(\underline{b}) \to c$$

Reductions intuitively equivalent

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2. Distinguishing steps

2.1. Syntactic accidents

Ex 3 *TRS* $\mathcal{T}$ *has rule* $\varrho : f(x) \to x$.
Two steps from $f(f(a))$*:*

$$\underline{f}(f(a)) \to f(a) \qquad f(\underline{f}(a)) \to f(a)$$

Both steps give rise to the step $f(f(a)) \to_{\mathcal{T}} f(a)$ *in the underlying abstract rewriting system* $\to_{\mathcal{T}}$ *of* $\mathcal{T}$*: a syntactic accident.*

Problem 4 *Steps have no identity*

2. Distinguishing steps

2.1. Syntactic accidents

Ex 3 *TRS $\mathcal{T}$ has rule $\varrho : f(x) \to x$.*
Two steps from $f(f(a))$:

$$\underline{f}(f(a)) \to f(a) \qquad f(\underline{f}(a)) \to f(a)$$

Both steps give rise to the step $f(f(a)) \to_{\mathcal{T}} f(a)$ in the underlying abstract rewriting system $\to_{\mathcal{T}}$ of $\mathcal{T}$: a syntactic accident.

Problem 4 *Steps have no identity*

Solution 5 *Provide steps with identity*

2.2. Abstract rewriting systems

Def 6 *An abstract rewriting system, ARS for short, is a quadruple* $\langle A, \Phi, \mathsf{src}, \mathsf{tgt} \rangle$ *with*

1. A *set of objects*
2. Φ *set of steps*
3. $\mathsf{src}, \mathsf{tgt} : \Phi \to A$ *the source and target functions*

$$\text{object} \xleftarrow{\ \mathsf{src}\ } \text{step} \xrightarrow{\ \mathsf{tgt}\ } \text{object}$$

– $\to, \rightsquigarrow, \Rightarrow, \ldots$ range over ARSs

– $a, b, c, \ldots$ range over objects

– $\phi, \psi, \chi, \ldots$ range over steps

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2.3. Examples of ARSs

Exs 7 1. *The black hole ARS* $\circlearrowright$ *has a single object* $\bullet$ *and a single step from the object to itself.*

2. *For every natural number* n, *the ARS* $\rightarrow^n$ *has objects* $\bullet_1, \ldots, \bullet_n$ *and steps* $i+1 : \bullet_i \rightarrow^n \bullet_{i+1}$, *for every* i *such that* $1 \leq i$ *and* $i+1 \leq n$.

   Special cases: the empty ARS $\rightarrow^0$ *and the single-object ARS* $\rightarrow^1$

3. *The ARS* $\multimap\!\!\infty\!\!\rightarrow$ *is union of all* $\rightarrow^n$: *the infinite straight line* $\bullet_1 \rightarrow^1 \bullet_2 \rightarrow^2 \bullet_3 \rightarrow^3 \cdots$.

4. *The syntactic accident ARS* $\rightrightarrows$ *consists of two objects and two steps, as indicated.*

5. *The diamond ARS* $\Diamond$ *is defined in the obvious way*

2.4. Composites

Def 8 *The reflexive–transitive closure* $\longrightarrow^*$ *of an abstract rewriting system* $\longrightarrow = \langle A, \Phi, \mathsf{src}, \mathsf{tgt}\rangle$ *is:*

– A *is the set of objects.*

– *Steps with source, target:*

$$\frac{a \in A}{1_a : a \longrightarrow^* a} \quad \frac{\phi : a \longrightarrow b \in \Phi}{\phi : a \longrightarrow^* b} \quad \frac{\phi : a \longrightarrow^* b \quad \psi : b \longrightarrow^* c}{(\phi \cdot \psi) : a \longrightarrow^* c}$$

– *The empty step for an object* a *is* $1_a : a \longrightarrow^* a$

– $(\phi \cdot \psi)$ *is a composite step*

$(\phi \cdot \psi) \cdot \chi$ and $\phi \cdot (\psi \cdot \chi)$ are (distinct) steps of $\longrightarrow^*$

2.5. Reductions

Def 9 $\twoheadrightarrow$ *is* $\longrightarrow^*$ *modulo reduction identities*

$$
\begin{array}{rcl}
1 \cdot \phi & \approx & \phi \\
\phi \cdot 1 & \approx & \phi \\
(\phi \cdot \psi) \cdot \chi & \approx & \phi \cdot (\psi \cdot \chi)
\end{array}
$$

Steps, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{P}$, *of* $\twoheadrightarrow$ *are called (finite) reductions.*

Problem 10 *How to choose unique representatives?*

2.5. Reductions

Def 9 $\twoheadrightarrow$ *is* $\longrightarrow^*$ *modulo reduction identities*

$$\begin{array}{rcl} 1 \cdot \phi & \approx & \phi \\ \phi \cdot 1 & \approx & \phi \\ (\phi \cdot \psi) \cdot \chi & \approx & \phi \cdot (\psi \cdot \chi) \end{array}$$

Steps, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{P}$, *of* $\twoheadrightarrow$ *are called (finite) reductions.*

Problem 10 *How to choose unique representatives?*

Solution 11 *Orient equations into complete TRS:*

$$\begin{array}{rcl} 1 \cdot x & \Rightarrow & x \\ x \cdot 1 & \Rightarrow & x \\ (x \cdot y) \cdot z & \Rightarrow & x \cdot (y \cdot z) \end{array}$$

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2.6. Term rewriting systems

Def 12 *A* *term rewriting system*, *TRS for short,* $\mathcal{T}$ *is a structure* $\langle \Sigma, R \rangle$ *such that*

- Σ *is an alphabet*
- R *is an abstract rewriting system having terms over* Σ *(and variables) as objects*

$$\text{term} \xleftarrow{\text{lhs}} \text{rule} \xrightarrow{\text{rhs}} \text{term}$$

$\varrho, \vartheta, \varsigma, \ldots$ range over rules

Problem 13 *How do generated steps look like?*

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2.6. Term rewriting systems

Def 12 *A term rewriting system, TRS for short,* $\mathcal{T}$ *is a structure* $\langle \Sigma, R \rangle$ *such that*

- Σ *is an alphabet*
- R *is an abstract rewriting system having terms over* Σ *(and variables) as objects*

$$\text{term} \xleftarrow{\text{lhs}} \text{rule} \xrightarrow{\text{rhs}} \text{term}$$

$\varrho, \vartheta, \varsigma, \ldots$ range over rules

Problem 13 *How do generated steps look like?*

Solution 14 *Proof terms*

2.7. Proof terms

Def 15 *The proof term alphabet is the (disjoint) union of Σ, the set of rule symbols of R, and $\{\cdot\}$, where $\cdot$ is the binary composition symbol. The underlying ARS $\geq_{\mathcal{T}}$*

- *Objects are terms over Σ.*
- *Steps are proof terms defined by*

$$\frac{\phi_1 : s_1 \geq t_1 \quad \dots \quad \phi_n : s_n \geq t_n}{f(\phi_1, \dots, \phi_n) : f(s_1, \dots, s_n) \geq f(t_1, \dots, t_n)} \text{ (repl)}$$

$$\frac{\phi_1 : s_1 \geq t_1 \quad \dots \quad \phi_n : s_n \geq t_n}{\varrho(\phi_1, \dots, \phi_n) : l(s_1, \dots, s_n) \geq r(t_1, \dots, t_n)} \text{ (rule)}$$

$$\frac{\phi : s \geq t \quad \psi : t \geq u}{(\phi \cdot \psi) : s \geq u} \text{ (trans)}$$

2.8. Examples of proof terms

If $\varrho : f(x) \to x$, $\varrho(f(a))$ and $f(\varrho(a))$ both witness $f(f(a)) \geq f(a)$:

$$\dfrac{\dfrac{\dfrac{}{a : a \geq a}\,\text{(replacement)}}{f(a) : f(a) \geq f(a)}\,\text{(replacement)}}{\varrho(f(a)) : f(f(a)) \geq f(a)}\,\text{(rule)}$$

$$\dfrac{\dfrac{\dfrac{}{a : a \geq a}\,\text{(replacement)}}{\varrho(a) : f(a) \geq a}\,\text{(rule)}}{f(\varrho(a)) : f(f(a)) \geq f(a)}\,\text{(replacement)}$$

No syntactic accident!

2.9. Running example: proof term

TRS with named rules

$$
\begin{aligned}
\varrho &: \quad a \longrightarrow b \\
\vartheta &: \quad f(x, b) \longrightarrow g(x) \\
\varsigma &: \quad g(b) \longrightarrow c
\end{aligned}
$$

Proof terms for reductions

$$
\begin{aligned}
\mathcal{R} &= \underline{f(\varrho, a) \cdot f(b, \varrho)} \cdot \vartheta(b) \cdot \varsigma \\
&: \quad f(\tilde{a}, a) \longrightarrow_{\phi_1} f(b, \overline{a}) \longrightarrow_{\phi_2} \underline{f}(b, \underline{b}) \longrightarrow_{\phi_3} \underline{g}(\underline{b}) \longrightarrow_{\phi_4} c \\
\mathcal{S} &= f(a, \varrho) \cdot \vartheta(a) \cdot g(\varrho) \cdot \varsigma \\
&: \quad f(a, \overline{a}) \longrightarrow_{\psi_1} \underline{f}(a, \underline{b}) \longrightarrow_{\psi_2} g(\tilde{a}) \longrightarrow_{\psi_3} \underline{g}(\underline{b}) \longrightarrow_{\psi_4} c
\end{aligned}
$$

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2.10. Common restrictions on proof terms

TRS with rule $\varrho : f(x, y) \to x$, constants a, b, c, d

- Multi-step: no transitivity
  $\varrho(\varrho(a, b), c) : f(f(a, b), c) \mathrel{\circ\!\!\!\!\longrightarrow} a$
- Parallel step: no nested rule symbols
  $f(\varrho(a, b), \varrho(c, d)) : f(f(a, b), f(c, d)) \mathrel{+\!\!\!\!\!\!\longrightarrow} f(a, c)$
- Ordinary step: exactly one rule symbol
- Reduction: concatenation of ordinary steps
  $f(\varrho(a, b), c) \cdot \varrho(a, c)$
- Term: no rule symbols, no transitivity
  $f(a, c)$

3. Permutation equivalence

Solution 16 *Equivalence generated by permutations on proof terms.*

Ex 17 *(idea)* $[2, 1, 0]$ *and* $[0, 1, 2]$ *are equivalent:*

$$[2, \underline{1, 0}] \cong [\underline{2, 0}, 1] \cong [0, \underline{2, 1}] \cong [0, 1, 2]$$

by repeatedly permuting adjacent members

3.1. Permutation identities

$$
\begin{aligned}
1 \cdot \phi &\approx \phi \\
\phi \cdot 1 &\approx \phi \\
(\phi \cdot \psi) \cdot \chi &\approx \phi \cdot (\psi \cdot \chi) \\
f(\vec{\phi}) \cdot f(\vec{\psi}) &\approx f(\phi_1 \cdot \psi_1, \ldots, \phi_n \cdot \psi_n) \\
\varrho(\phi_1, \ldots, \phi_n) &\approx l(\phi_1, \ldots, \phi_n) \cdot \varrho(t_1, \ldots, t_n) \\
\varrho(\phi_1, \ldots, \phi_n) &\approx \varrho(s_1, \ldots, s_n) \cdot r(\phi_1, \ldots, \phi_n)
\end{aligned}
$$

– first three : reduction identities (monoid)

– first four : structural identities ($\equiv$)

– all : permutation identities ($\cong$)

Problem 18 *How to decide equivalence?*

3.2. Structural equivalence example

$$\varrho : \; a \;\longrightarrow\; b$$

3.2. Structural equivalence example

$$\varrho : \; a \;\longrightarrow\; b$$

$$\begin{aligned} g(\varrho, a) \cdot g(b, \varrho) \;&:\; g(a, a) \geq g(b, b) \\ g(a, \varrho) \cdot g(\varrho, b) \;&:\; g(a, a) \geq g(b, b) \end{aligned}$$

$$\begin{aligned} g(\varrho, a) \cdot g(b, \varrho) &\equiv g(\varrho \cdot b, a \cdot \varrho) \\ &\equiv g(\varrho, \varrho) \\ &\equiv g(a \cdot \varrho, \varrho \cdot b) \\ &\equiv g(a, \varrho) \cdot g(\varrho, b) \end{aligned}$$

3.3. Deciding structural equivalence

Solution 19 *Completed structural identities*

$$
\begin{aligned}
1 \cdot x &\Rightarrow x \\
x \cdot 1 &\Rightarrow x \\
(x \cdot y) \cdot z &\Rightarrow x \cdot (y \cdot z) \\
f(x_1, \ldots, x_n) \cdot f(y_1, \ldots, y_n) &\Rightarrow f(x_1 \cdot y_1, \ldots, x_n \cdot y_n) \\
f(\vec{x}) \cdot (f(y_1, \ldots, y_n) \cdot z) &\Rightarrow f(x_1 \cdot y_1, \ldots, x_n \cdot y_n) \cdot z
\end{aligned}
$$

$$
\begin{aligned}
\overline{g(\varrho, a) \cdot g(b, \varrho)} &\Rightarrow g(\overline{\varrho \cdot b}, \overline{a \cdot \varrho}) \\
&\Rightarrow g(\varrho, \varrho) \\
&\Leftarrow g(\underline{a \cdot \varrho}, \underline{\varrho \cdot b}) \\
&\Leftarrow \underline{g(a, \varrho) \cdot g(\varrho, b)}
\end{aligned}
$$

3.4. Running example: permutation equivalence

$$\begin{aligned}
\varrho &: \quad a \longrightarrow b \\
\vartheta &: \quad f(x,b) \longrightarrow g(x) \\
\varsigma &: \quad g(b) \longrightarrow c
\end{aligned}$$

$$\begin{aligned}
\mathcal{R} &= \underline{f(\varrho,a)\cdot f(b,\varrho)}\cdot\vartheta(b)\cdot\varsigma \\
&\cong \underline{f(\varrho,\varrho)}\cdot\vartheta(b)\cdot\varsigma \\
&\cong f(a,\varrho)\cdot\underline{f(\varrho,b)\cdot\vartheta(b)}\cdot\varsigma \\
&\cong f(a,\varrho)\cdot\underline{\vartheta(\varrho)}\cdot\varsigma \\
&\cong f(a,\varrho)\cdot\vartheta(a)\cdot g(\varrho)\cdot\varsigma \\
&= \mathcal{S}
\end{aligned}$$

4. Standardization equivalence

Solution 20 *Complete permutation identities*

Rewrite rule $\varrho : l \to r$, proof terms $x_i : s_i \geq t_i$

$$\begin{aligned} l(\vec{x}) \cdot \varrho(t_1, \ldots, t_n) &\Rightarrow_{\mathsf{I}} \varrho(x_1, \ldots, x_n) \\ \varrho(x_1, \ldots, x_n) &\Rightarrow_{\mathsf{O}} \varrho(s_1, \ldots, s_n) \cdot r(x_1, \ldots, x_n) \end{aligned}$$

apply modulo structural equivalence
normal forms are standard reductions

Thm 21 *Standardization is complete modulo structural equivalence*

Difficult, but term rewriting techniques (critical pairs) available because of proof terms

4.1. Running example: standardization equivalence

$$\mathcal{R} \quad : \quad f(\tilde{a}, a) \to f(b, \overline{a}) \to \underline{f}(b, \underline{b}) \to \underline{g}(\underline{b}) \to c$$

4.1. Running example: standardization equivalence

$$\mathcal{R} \ : \ f(\tilde{a}, a) \rightarrow f(b, \overline{a}) \rightarrow \underline{f}(b, \underline{b}) \rightarrow \underline{g}(\underline{b}) \rightarrow c$$

$$\equiv$$

4.1. Running example: standardization equivalence

$$
\begin{aligned}
\mathcal{R} \ &: \ f(\tilde{a}, a) \rightarrow f(b, \overline{a}) \rightarrow \underline{f}(b, \underline{b}) \rightarrow \underline{g}(\underline{b}) \rightarrow c \\
\equiv \ \mathcal{P} &: \ f(a, \overline{a}) \rightarrow f(\tilde{a}, b) \rightarrow \underline{f}(b, \underline{b}) \rightarrow \underline{g}(\underline{b}) \rightarrow c
\end{aligned}
$$

4.1. Running example: standardization equivalence

$$\begin{aligned}
\mathcal{R} \quad : \quad & f(\tilde{a}, a) \to f(b, \overline{a}) \to \underline{f}(b, \underline{b}) \to \underline{g}(\underline{b}) \to c \\
\equiv \ \mathcal{P} : \ & f(a, \overline{a}) \to f(\tilde{a}, b) \to \underline{f}(b, \underline{b}) \to \underline{g}(\underline{b}) \to c \\
\Rightarrow &
\end{aligned}$$

4.1. Running example: standardization equivalence

$$
\begin{aligned}
\mathcal{R} \quad &: \quad f(\tilde{a}, a) \to f(b, \overline{a}) \to \underline{f}(b, \underline{b}) \to \underline{g}(\underline{b}) \to c \\
\equiv \ \mathcal{P} &: \ f(a, \overline{a}) \to f(\tilde{a}, b) \to \underline{f}(b, \underline{b}) \to \underline{g}(\underline{b}) \to c \\
\Rightarrow \ \mathcal{S} &: \ f(a, \overline{a}) \to \underline{f}(a, \underline{b}) \to g(\tilde{a}) \to \underline{g}(\underline{b}) \to c
\end{aligned}
$$

$\mathcal{S}$ is standard

5. Labelling equivalence

Problem 22 *What is labelling?*

Exs 23 1. *Ornithology: ringing birds*

2. *λ-calculus: Church-typing for typable λ-terms*

3. *Term rewriting: semantic labelling of a TRS*

4. *Chemistry: label reaction by some (stable) isotope:*

$$CH_3COOH + C_2H_5OH \rightarrow C_2H_5COOCH_3 + H_2O$$

Label the second O as ^{18}O, giving $CH_3CO^{18}OH$:

$$CH_3CO^{18}OH + C_2H_5OH \rightarrow C_2H_5COOCH_3 + H_2{}^{18}O$$

Solution 24 *Labelling: adjoining information in a behavior preserving way*

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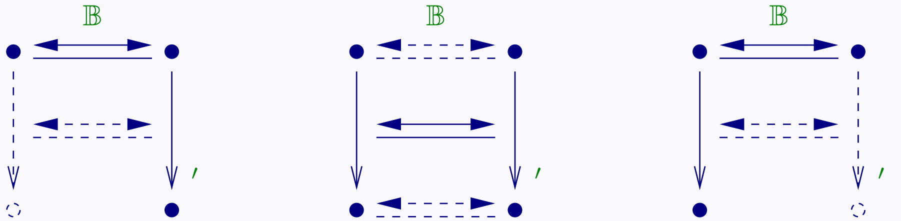
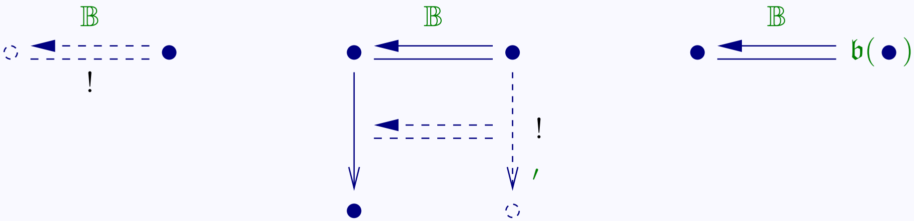
5.1. Bisimulation and labelling

Bisimulation



both objects and steps are related

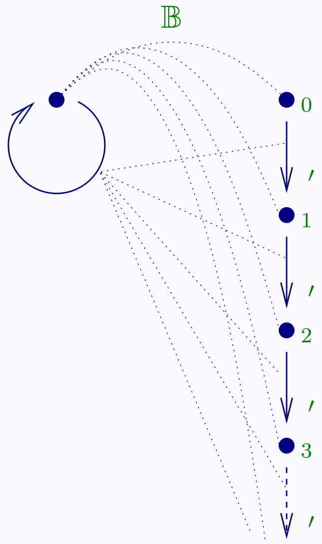
Labelling



labelling steps proceeds deterministically

5.2. Example: bisimulation/labelling

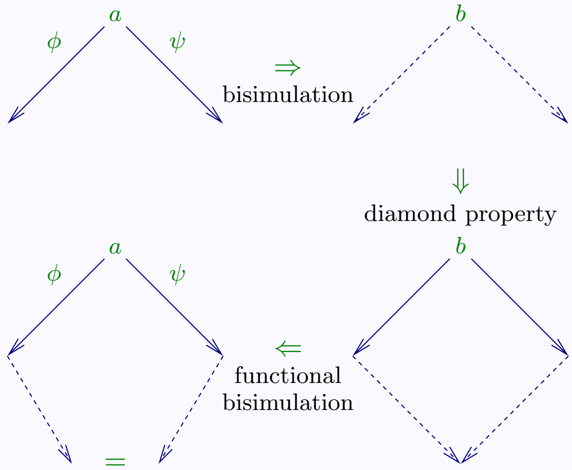
A bisimulation $\mathbb{B}$ between the black hole ARS $\circlearrowright$ and the infinite straight line ARS $-\!\infty\!\!\rightarrow$:



5.3. Transfer of properties along bisimulation/labelling

if $a \mathbb{B} b$ for some bisimulation $\mathbb{B}$, then

- a is normalizing (WN) iff b is normalizing,
- a is terminating (SN) iff b is terminating
- if $\mathbb{B}$ is a labelling, $\Diamond(b)$ implies $\Diamond(a)$



5.4. Term rewrite labelling

Term rewrite labelling $\mathfrak{B}$:

– Labelling $\mathbb{B}$ on its alphabets and rules

– initial labelling $\mathfrak{b}$ mapping terms to labelled terms

5.4. Term rewrite labelling

Term rewrite labelling $\mathfrak{B}$:

– Labelling $\mathbb{B}$ on its alphabets and rules

– initial labelling $\mathfrak{b}$ mapping terms to labelled terms

Stacks of natural numbers

rules push^n : $\top \to n(\top)$, pop^n : $n(\top) \to \top$

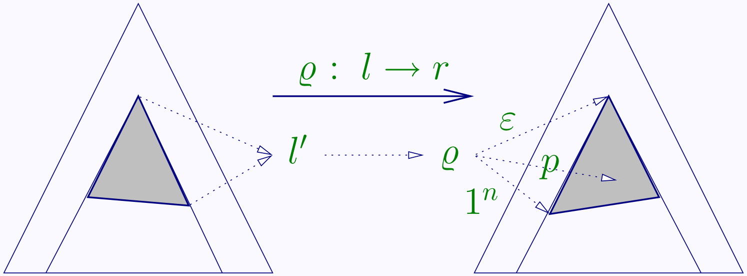
reduction $\top \to 5(\top) \to 5(1(\top)) \to 5(\top) \to 5(3(\top))$

label top of stack by height

rules push$_i^n$: $\top_i \to n(\top_{i+1})$, pop$_i^n$: $n(\top_{i+1}) \to \top_i$

reduction $\top_0 \to 5(\top_1) \to 5(1(\top_2)) \to 5(\top_1) \to 5(3(\top_2))$

5.5. Lévy labelling



record history of lhs in each symbol of rhs

rule: $\varrho : \; a \longrightarrow a$

reduction: $\mathcal{R} : \; a \longrightarrow a \longrightarrow a \longrightarrow a \longrightarrow \cdots$

Lévy labelled rules: $\varrho_{a^\varepsilon} : \; a^\varepsilon \longrightarrow \varrho^\varepsilon_{a^\varepsilon}$, $\varrho_{\varrho^\varepsilon_{a^\varepsilon}} : \; \varrho^\varepsilon_{a^\varepsilon} \longrightarrow \varrho^\varepsilon_{\varrho^\varepsilon_{a^\varepsilon}}$

Lévy labelled reduction: $a^\varepsilon \longrightarrow \varrho^\varepsilon_{a^\varepsilon} \longrightarrow \varrho^\varepsilon_{\varrho^\varepsilon_{a^\varepsilon}} \longrightarrow \varrho^\varepsilon_{\varrho^\varepsilon_{\varrho^\varepsilon_{a^\varepsilon}}} \longrightarrow \cdots$.

5.6. Labelling equivalence

Labelling equivalence: targets the same after any labelling

$$\begin{array}{lcl}\mathcal{R} & : & f(\tilde{a},a) \to f(b,\overline{a}) \to \underline{f}(b,\underline{b}) \to \underline{g}(\underline{b}) \to c \\ \mathcal{S} & : & f(a,\overline{a}) \to \underline{f}(a,\underline{b}) \to g(\tilde{a}) \to \underline{g}(\underline{b}) \to c\end{array}$$

Lévy labelling gives $\mathfrak{L}(\mathcal{R})$ and $\mathfrak{L}(\mathcal{S})$

$$f(\tilde{a},a) \to f(b^{\tilde{a}},a) \to f(b^{\tilde{a}},b^{a}) \to g^{f(x,b^{a})}(b^{\tilde{a}}) \to c^{g^{f(x,b^{a})}(b^{\tilde{a}})}$$

$$f(\tilde{a},a) \to f(\tilde{a},b^{a}) \to g^{f(x,b^{a})}(\tilde{a}) \to g^{f(x,b^{a})}(b^{\tilde{a}}) \to c^{g^{f(x,b^{a})}(b^{\tilde{a}})}$$

targets have same labelling, hence labelling equivalent

Note a not labelling equivalent to $a \to a$!

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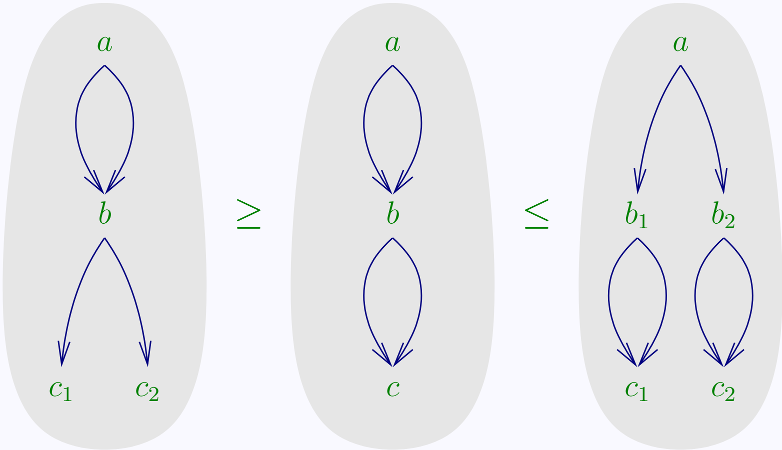
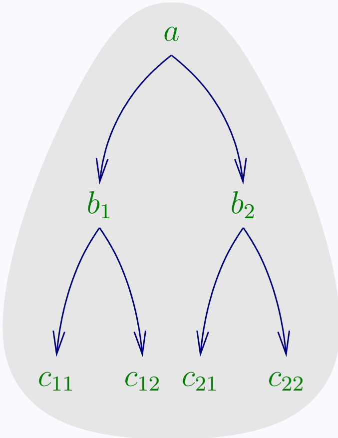
5.7. History order

History order $\mathfrak{A} \leq \mathfrak{B}$ if exists $\mathfrak{C}$, such that $\mathfrak{B}$ is the composition of $\mathfrak{A}$ and $\mathfrak{C}$



5.8. Maximality of Lévy labelling

Maximal with respect to history order.



Thm 25 *Lévy labelling is maximal among all term rewriting labellings*

Proof by tracing a connexion property

6. Projection equivalence

Problem 26 *What is projection?*

Solution 27 *Taking the residual*

$[2, 1, 0]$ and $[0, 1, 2]$ equivalent?

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6. Projection equivalence

Problem 26 *What is projection?*

Solution 27 *Taking the residual*

$[2, 1, 0]$ and $[0, 1, 2]$ equivalent?

$[0, 1, 2] \rightsquigarrow_2 [0, 1, \not{2}] \rightsquigarrow_1 [0, \not{1}, \not{2}] \rightsquigarrow_0 [\not{0}, \not{1}, \not{2}]$

$[2, 1, 0] \rightsquigarrow_0 [2, 1, \not{0}] \rightsquigarrow_1 [2, \not{1}, \not{0}] \rightsquigarrow_2 [\not{2}, \not{1}, \not{0}]$

6. Projection equivalence

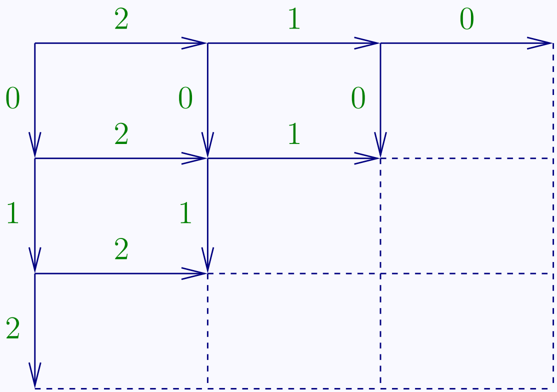
Problem 26 *What is projection?*

Solution 27 *Taking the residual*

$[2,1,0]$ and $[0,1,2]$ equivalent?

$[0,1,2] \rightsquigarrow_2 [0,1,\cancel{2}] \rightsquigarrow_1 [0,\cancel{1},\cancel{2}] \rightsquigarrow_0 [\cancel{0},\cancel{1},\cancel{2}]$

$[2,1,0] \rightsquigarrow_0 [2,1,\cancel{0}] \rightsquigarrow_1 [2,\cancel{1},\cancel{0}] \rightsquigarrow_2 [\cancel{2},\cancel{1},\cancel{0}]$



6.1. Examples of residual systems

Exs 28 *1. Natural numbers with cutoff-subtraction*

2. Sets with set-difference

3. Multiset with multiset difference

4. Braids

5. Stack numbers

6. Term rewriting: multi-step, parallel steps, proof terms (not ordinary steps, because of replication)

6.2. Residual Systems

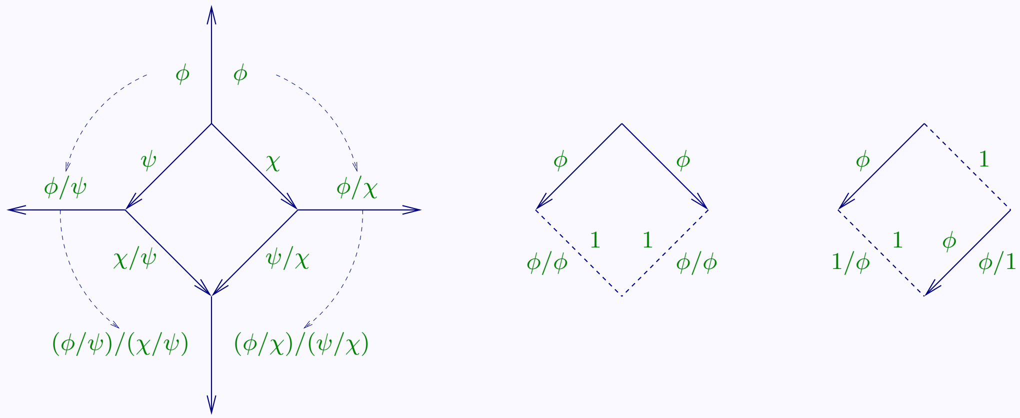
Def 29 *A residual system is a triple* $\langle \rightarrow, 1, / \rangle$*:*

- $\rightarrow$ *is an abstract rewriting system*
- 1 *(unit) is a function from objects to steps such that* $\mathsf{tgt}(1_a) = a = \mathsf{src}(1_a)$,
- $/$ *(residuation) is a function from pairs of co-initial steps to steps such that* $\mathsf{tgt}(\phi) = \mathsf{src}(\psi/\phi)$ *and* $\mathsf{tgt}(\phi/\psi) = \mathsf{tgt}(\psi/\phi)$

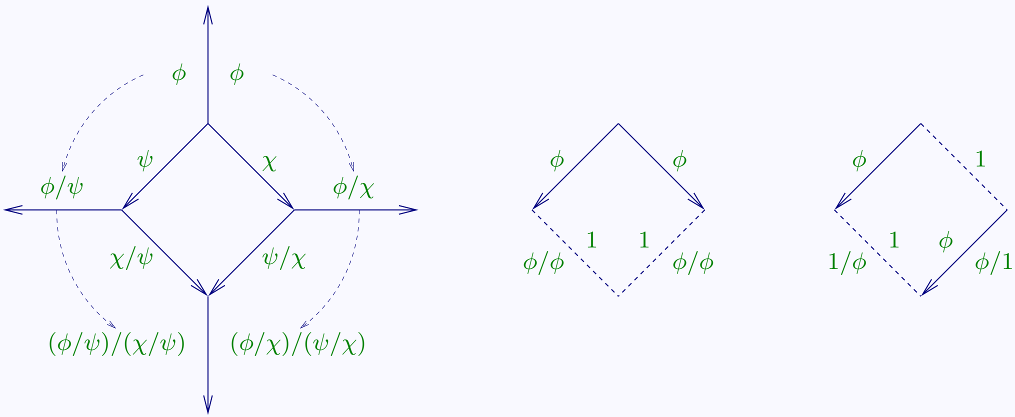
satisfying the residual identities:

$$
\begin{array}{rcl}
(\phi/\psi)/(\chi/\psi) & \approx & (\phi/\chi)/(\psi/\chi) \\
\phi/\phi & \approx & 1 \\
\phi/1 & \approx & \phi \\
1/\phi & \approx & 1
\end{array}
$$

6.3. Residual identities



6.3. Residual identities



Ex 30 $1/\phi = (1/1)/(\phi/1) = (1/\phi)/(1/\phi) = 1$

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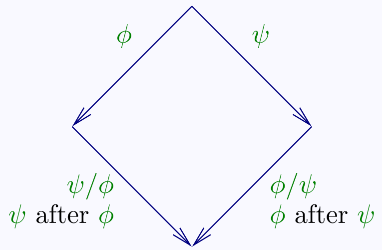
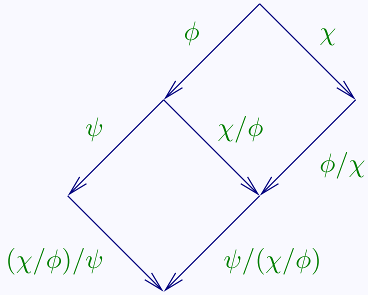
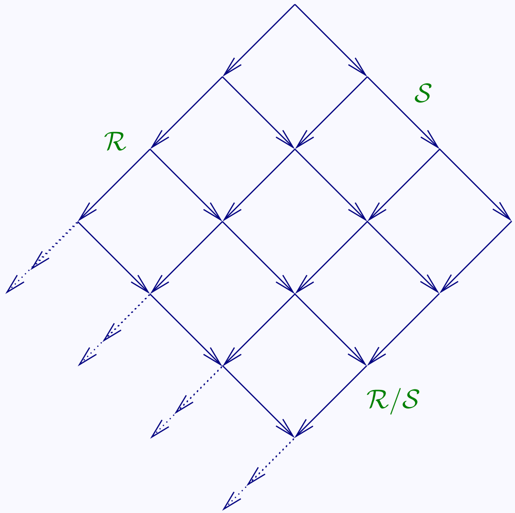
6.4. Residuals of steps, composites and reductions







6.5. Projection order

The projection order $\lesssim$ and the corresponding projection equivalence $\simeq$ are defined by

$$\phi \lesssim \psi \quad \text{if } \phi/\psi = 1$$
$$\phi \simeq \psi \quad \text{if } \phi \lesssim \psi \text{ and } \psi \lesssim \phi$$

for co-initial steps ϕ and ψ

projection order is quasi-order (need not be partial order!)

residuation monotonic in first (not necessarily antitonic in second)

projection equivalence congruence for the operations

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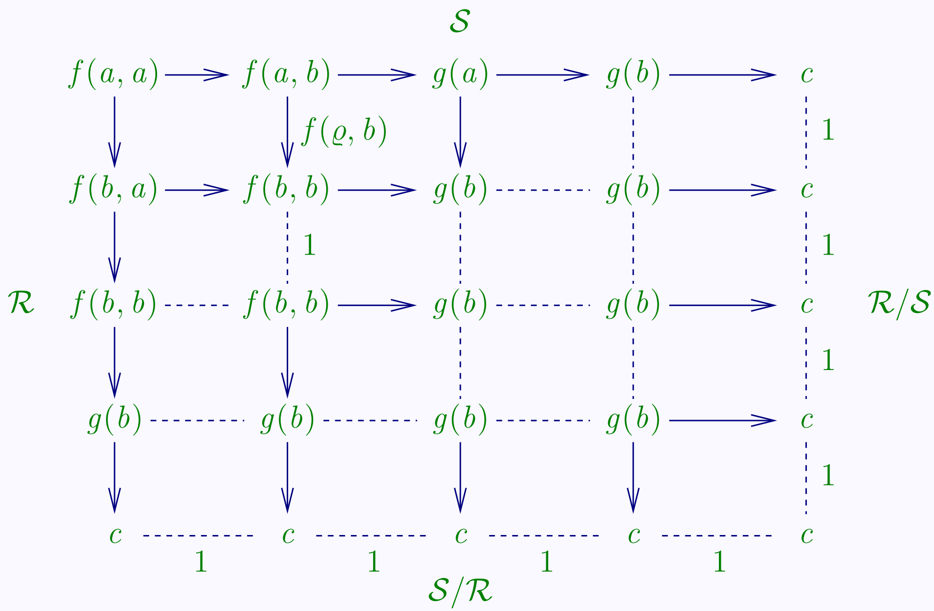
6.6. Term Residual Systems

Residual operation $/$:

$$
\begin{aligned}
f(\vec{\phi})/f(\vec{\psi}) &= f(\phi_1/\psi_1, \ldots, \phi_n/\psi_n) \\
\varrho(\vec{\phi})/l(\vec{\psi}) &= \varrho(\phi_1/\psi_1, \ldots, \phi_n/\psi_n) \\
l(\vec{\phi})/\varrho(\vec{\psi}) &= r(\phi_1/\psi_1, \ldots, \phi_n/\psi_n) \\
\varrho(\vec{\phi})/\varrho(\vec{\psi}) &= r(\phi_1/\psi_1, \ldots, \phi_n/\psi_n) \\
\chi/(\phi \cdot \psi) &= (\chi/\phi)/\psi \\
(\phi \cdot \psi)/\chi &= \phi/\chi \cdot \psi/(\chi/\phi) \\
\phi/\psi &= \mathsf{tgt}(\phi) \to_{\#} \# \quad \text{otherwise}
\end{aligned}
$$

6.7. Running example: projection equivalence



$$\mathcal{R} \; : \; f(\tilde{a},a) \to f(b,\overline{a}) \to \underline{f}(b,\underline{b}) \to \underline{g}(\underline{b}) \to c$$

$$\mathcal{S} \; : \; f(a,\overline{a}) \to \underline{f}(a,\underline{b}) \to g(\tilde{a}) \to \underline{g}(\underline{b}) \to c$$

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6.8. Running example: projection equivalence (ctd)

$$
\begin{aligned}
\mathcal{R}/\underline{\mathcal{S}} &= \underline{\mathcal{R}/(f(a,\varrho)\cdot\mathcal{S}_1)}\\
&= (\underline{\mathcal{R}}/f(a,\varrho))/\mathcal{S}_1\\
&= \underline{(f(\varrho,a)\cdot\mathcal{R}_1)/f(a,\varrho)}/\mathcal{S}_1\\
&= (\underline{f(\varrho,a)/f(a,\varrho)}\cdot(\mathcal{R}_1/\underline{f(a,\varrho)/f(\varrho,a)}))/\mathcal{S}_1\\
&= (f(\underline{\varrho/a},\underline{a/\varrho})\cdot(\mathcal{R}_1/f(\underline{a/\varrho},\underline{\varrho/a})))/\mathcal{S}_1\\
&= (f(\varrho,b)\cdot(\underline{\mathcal{R}_1}/f(b,\varrho)))/\mathcal{S}_1\\
&= (f(\varrho,b)\cdot\underline{(f(b,\varrho)\cdot\mathcal{R}_2)/f(b,\varrho)})/\mathcal{S}_1\\
&= (f(\varrho,b)\cdot\underline{f(b,\varrho)/f(b,\varrho)}\cdot(\mathcal{R}_2/\underline{f(b,\varrho)/f(b,\varrho)}))/\mathcal{S}_1\\
&= (f(\varrho,b)\cdot 1\cdot\underline{\mathcal{R}_2/1})/\mathcal{S}_1\\
&= (f(\varrho,b)\cdot 1\cdot\mathcal{R}_2)/\mathcal{S}_1
\end{aligned}
$$

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6.9. Well-definedness of projection

Termination non-obvious because calls itself, e.g.

$$(\phi \cdot \psi)/\chi = \phi/\chi \cdot \psi/(\chi/\phi)$$

Confluence non-obvious because of critical pairs, e.g.

$$((x \cdot y)/z)/w \Leftarrow (x \cdot y)/(z \cdot w) \Rightarrow (x/(z \cdot w)) \cdot (y/((z \cdot w)/x)$$

Solution: semantic labelling

6.10. Laws for term residual systems

$$
\begin{array}{rcl@{\qquad}rcl}
\phi/\phi & = & 1 & (\phi/\psi)/(\chi/\psi) & = & (\phi/\chi)/(\psi/\chi) \\
\phi/1 & = & \phi & \chi/(\phi\cdot\psi) & = & (\chi/\phi)/\psi \\
1/\phi & = & 1 & (\phi\cdot\psi)/\chi & = & (\phi/\chi)\cdot(\psi/(\chi/\phi)) \\
\phi\cdot 1 & \simeq & \phi & (\phi\cdot\psi)\cdot\chi & \simeq & \phi\cdot(\psi\cdot\chi) \\
1\cdot\phi & \simeq & \phi & \phi\sqcup\psi & = & \phi\cdot(\psi/\phi) \\
1\sqcup\phi & \simeq & \phi & \chi/(\phi\sqcup\psi) & = & (\chi/\phi)/(\psi/\phi) \\
\phi\sqcup\phi & \simeq & \phi & (\phi\sqcup\psi)/\chi & = & (\phi/\chi)\sqcup(\psi/\chi) \\
\phi\sqcup\psi & \simeq & \psi\sqcup\phi & \phi\cdot(\psi\sqcup\chi) & \simeq & (\phi\cdot\psi)\sqcup(\phi\cdot\chi) \\
(\phi\sqcup\psi)\sqcup\chi & \simeq & \phi\sqcup(\psi\sqcup\chi) & \phi\cdot\psi\simeq\phi\cdot\chi & \Rightarrow & \psi\simeq\chi
\end{array}
$$

Join $\phi \sqcup \psi$ is defined as $\phi \cdot (\psi/\phi)$

7. Equivalences are the same

Problem 31 *How to show correspondence between permutation and labelling?*

Solution 32
- *labelling preserves permutation equivalence*
- *Lévy labelling allows for reconstruction up to permutation equivalence*

7.1. Reconstruction for ARSs

Unwinding of an ARS: record the complete history in the objects

Thm 33 (Reconstruction Theorem) *There exists a reconstruction map* $\mathfrak{R}$ *from unwound objects to ordinary reductions such that for any reduction* $\mathcal{R}$ *and any object* a' *in the unwinding*

$$\mathfrak{R}(a') = \mathcal{R} \textit{ iff } a' = \mathsf{tgt} \circ \mathfrak{U}(\mathcal{R})$$

Unwinding : ARS as Lévy labelling : TRS

7.2. Reconstruction for TRSs

Problem 34 *For term rewriting systems:*

- *order of steps is not recorded*
- *erased parts cannot be reconstructed*

Solution 35 *Reconstruct needed prefix up to permutation*

Thm 36 *There exists a reconstruction function* $\mathfrak{R}$ *which has the property that for any extracted reduction* $\mathcal{R} \triangleright Q$ *and any Lévy labelling* Q' *of its target*

$$\mathfrak{R}(Q') = \mathcal{R} \triangleright Q \quad \textit{iff} \quad Q' = \mathrm{tgt} \circ \mathfrak{L}(\mathcal{R} \triangleright Q) \qquad (1)$$

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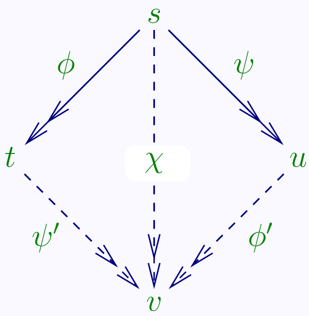
8. Parallel Proofs Lemma for orhogonal TRSs



for given coinitial ϕ, ψ there exist proof terms $\psi' : t \geq v$, $\chi : s \geq v$ and $\phi' : u \geq v$ without error symbols, such that

1. The common reduct v is found via permutation equivalent ways, $\phi \cdot \psi' \cong \chi \cong \psi \cdot \phi'$, and χ is a minimal proof term in the permutation order having this property.

2. The targets of the Lévy labellings (formally: liftings) of the ways are identical, $\mathsf{tgt}(\mathfrak{L}(\phi \cdot \psi')) = \mathsf{tgt}(\mathfrak{L}(\chi)) = \mathsf{tgt}(\mathfrak{L}(\psi \cdot \phi'))$, and $\mathsf{tgt}(\mathfrak{L}(\chi))$ rewrites to any other Lévy labelled term having this property.

3. The ways $\phi \cdot \psi'$, χ and $\psi \cdot \phi'$ yield identical (parallel) standard reductions.

4. The three ways can be computed using the residual and join operations of the associated residual system: $\psi' \simeq \psi / \phi$, $\chi \simeq \phi \sqcup \psi$ and $\phi' \simeq \phi / \psi$.

9. λ-calculus with end-of-scope

Observations

- λx is like opening an 'x-bracket'
- no corresponding closing bracket!

Proposal

- Adjoin closing 'x-brackets' $\reflectbox{$\lambda$}x$
  (adbmal, unbind, end-of-scope)
- $\reflectbox{$\lambda$}x.M$ closes matching λx: x is 'free' in $\lambda x.\reflectbox{$\lambda$}x.x$
- Can be nested $\underbrace{\lambda x.\overbrace{\lambda x.\reflectbox{λ}x}.\reflectbox{λ}x}.x$ (x free again)
- Proper nesting: $\lambda x.\lambda y.\reflectbox{λ}x.\reflectbox{λ}y.M$ not allowed
  (better: $\reflectbox{$\lambda$}x$ implictly closes λy)

10. β-reduction

Observation: $\not\lambda$'s in between $@$ and λ

- $(\lambda x.\not\lambda x.M)N$ should reduce to M
- $(\not\lambda y.\lambda x.x)y$ should not reduce to $\not\lambda y.y$ (but y)
- $(\not\lambda y.\lambda x.z)y$ should not reduce to z (but $\not\lambda y.z$)

Where should end-of-scopes go?

- search for matching x
- in case of x: remove end-of-scopes, put argument
- in case of $\not\lambda x$: put end-of-scopes, remove argument

$(\not\lambda X.\lambda x.M)N \to M[X, x{:=}N, \Box]$:

- X remembers the end-of-scopes
- third argument: stack used for matching

11. Formalization in Coq

Axiom 1 *Assume a parameter set* $\mathcal{V}$ *of infinitely many variable names for which equality is decidable.*

– $x = y \vee x \neq y$ *for all* $x, y : \mathcal{V}$

– $\exists x : \mathcal{V}.\, x \notin X$ *for all* $X : \mathit{list}(\mathcal{V})$

```
Parameter name : Set.
Axiom eq_dec : (x,y:name){x=y}+{~x=y}.
Axiom inf_many_names :
  (l:(list name)){a:name|~(In a l)}.
```

Def 37 *Substitution* $M[X, x{:=}N, Y]$ *is defined by:*

$$
\begin{aligned}
y[X, x{:=}N, Y] &= y, \text{ if } y \in Y \\
y[X, x{:=}N, Y] &= \reflectbox{λ} Y.N, \text{ if } y \notin Y, x = y \\
y[X, x{:=}N, Y] &= \reflectbox{λ} Y.\reflectbox{λ} X.y, \text{ if } y \notin Y, x \neq y \\
(\lambda y.M)[X, x{:=}N, Y] &= \lambda y.M[X, x{:=}N, yY] \\
(\reflectbox{λ} y.M)[X, x{:=}N, \Box] &= \reflectbox{λ} X.M, \text{ if } x = y \\
(\reflectbox{λ} y.M)[X, x{:=}N, \Box] &= \reflectbox{λ} X.\reflectbox{λ} y.M, \text{ if } x \neq y \\
(\reflectbox{λ} y.M)[X, x{:=}N, zY] &= \reflectbox{λ} y.M[X, x{:=}N, Y], \text{ if } y = z \\
(\reflectbox{λ} y.M)[X, x{:=}N, zY] &= (\reflectbox{λ} y.M)[X, x{:=}N, Y], \text{ if } y \neq z \\
(M_1 M_2)[X, x{:=}N, Y] &= M_1[X, x{:=}N, Y] M_2[X, x{:=}N, Y]
\end{aligned}
$$

Lem 1 *Closed substitution lemma: if* $\langle X'xZ\rangle s$, $\langle Y'yZ'Z\rangle t$ *and* $\langle YZ'Z\rangle u$, *then:*

$$s[Y'yZ', x{:=}t, X'][Y, y{:=}u, X'Y'] \\ = s[Y'YZ', x{:=}t[Y, y{:=}u, Y'], X']$$

```
Lemma closed_subst_bal :
 (s,t,u:term;X',Y,Y',Z,Z':(list name);x,y:name)
 (bal (conc X' (cons x Z)) s)
  -> (bal (conc Y' (conc (cons y Z') Z)) t)
  -> (bal (conc Y (conc Z' Z)) u)
   -> (subst Y (conc X' Y') (subst (conc Y'
            (cons y Z')) X' s x t) y u)
     = (subst (conc Y' (conc Y Z')) X' s x
             (subst Y Y' t y u)).
```

Lem 2 *Open substitution lemma: if* $\langle X'xY'yZ\rangle s$, $\langle XY'yZ\rangle t$ *and* $\langle YZ\rangle u$, *then:*

$$s[X, x{:=}t, X'][Y, y{:=}u, X'XY'] = s[Y, y{:=}u, X'xY'][X, x{:=}t[Y, y{:=}u, XY'], X']$$

```
Lemma open_subst_bal :
 (s,t,u:term;X,X',Y,Y',Z:(list name);x,y:name)
  (bal (conc X' (conc (cons x Y')(cons y Z))) s)
   -> (bal (conc X (conc Y' (cons y Z))) t)
    -> (bal (conc Y Z) u)
     -> (subst Y (conc X' (conc X Y'))
         (subst X X' s x t) y u)
         = (subst X X' (subst Y
         (conc X' (cons x Y')) s y u) x
         (subst Y (conc X Y') t y u)).
```

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12. Results

- Confluence of β without α!
- α needed to remove $\not\lambda$'s:
  $\lambda x.\not\lambda x.x \rightarrow_{\alpha} \lambda y.\not\lambda y.x \rightarrow_{\mathsf{forget}} \lambda y.x$
- Confluence of ordinary β modulo α
- Proofs in Coq (also for de Bruijn indices)

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13. Related/current work

Related work

– refines Guillaume, David, Bird, Patterson: de Bruijn ($\lambdabar$ = Successor, variable = zero)

– Kesner, Di Cosmo, ...: with labels (commutativity, α)

– Explicit weakening

Current work

– Push $\lambdabar$'s locally : $\lambdabar x.\lambda x.M \to \lambda_1 x.\lambdabar x.M$

– inverses (reopen scope...)

– optimal implementation (brackets, no scopes)

– explicit substitution (lemmas as rules) (CR, PSN)