The problem of the calissons, by rewriting*

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Abstract

We show each of four confluence techniques: random descent, proof orders for decreasing diagrams, bricklaying, and local undercutting, serves to solve the problem of the calissons.

Introduction The problem of the calissons as presented in [2] is to show that if a *box*, a regular hexagonal, can be filled with *calissons*, so named after certain diamond-shaped sweets, then in the resulting filled box the numbers of calissons in each of their 3 orientations are the same; in a formula $r = g = b$ for r, g and b the numbers of red, green and blue[1] calissons in the box. For instance, for a box B with sides of length 2, there are 4 calissons for each of the

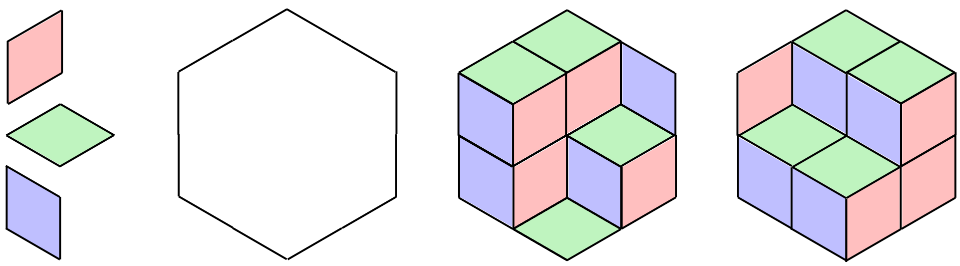

Figure 1: The Problem of the Calissons

3 orientations in both the filled B-boxes B_1 and B_2 in Figure 1; $r_i = g_i = b_i = 4$ for $i \in \{1, 2\}$.

The problem has received quite some attention since; the paper [2] currently has ≈150 citations. We refer the reader to that literature for descriptions, solutions, generalisations, applications and other discussions. The sole purpose here is to offer a rewriting perspective on the problem. We present four solutions, each based on a *confluence* technique.

Instead of requiring boxes to be equiangular hexagons that are also *equilateral* we relax the latter requirement to being *zonogonal*, i.e. to only having *opposite sides* of the same length. We show that if such a box is filled, then for each of their 3 orientations the number of calissons is always the same; if B_1 and B_2 fill the same box B, then $r_1 = r_2$, $g_1 = g_2$ and $b_1 = b_2$. This solves the original problem of the calissons since if B *is* equilateral, is a regular hexagon, the 3 numbers of calissons must in fact be the same as seen by rotational symmetry.

Looking at the filled boxes B_1 and B_2 in Figure 1 it's almost impossible not to see them as different stackings of small cubes inside a large *cube*. From that three-dimensional perspective, the generalisation considered here corresponds to stacking small cubes inside a large (rectangular) *cuboid*. That perspective suggests the number of calissons of a given orientation, is the product of the lengths of the sides of the hexagon parallel to the sides of calissons of that orientation; since B in Figure 1 'is' a cube with sides of length 2, the number of calissons of each type is $2 \times 2 = 4$ as indeed is the case for B_1 and B_2 in the figure.

[1] We assign colours to the orientations for convenient referencing and reasons of aesthetics.

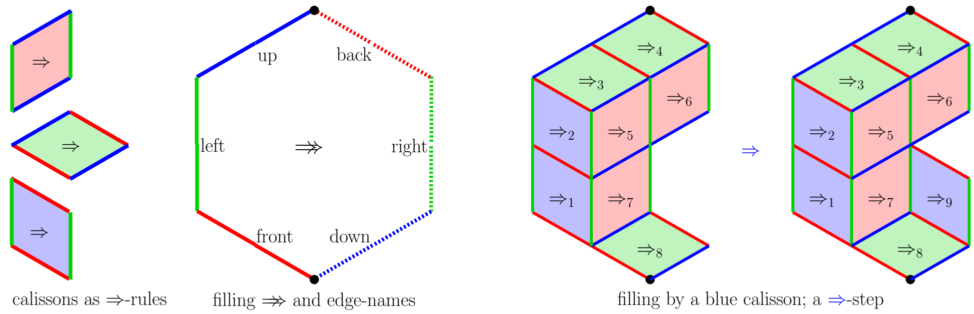

Figure 2: Solving the problem of the calissons by random descent

Random descent In the first approach to the problem by rewriting we view each calisson as a rewrite *rule* $\Rightarrow$ used to transform the left leg up–left–front of the box into its right leg back–right–down, see Figure 2. A *filling* is a $\Rightarrow$-reduction gradually transforming the *path* from the bullet $\bullet$ at the top to the $\bullet$ at the bottom of the box, into the dashed path, in the figure:

Definition 1. *Filling* $\Rightarrow$ is the string rewrite system (SRS) over the alphabet $\{\textcolor{red}{\rule{1.5em}{3pt}}, \textcolor{green}{\rule{1.5em}{3pt}}, \textcolor{blue}{\rule{1.5em}{3pt}}\}$ of edges and rewrite rules $\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}} \textcolor{red}{\Rightarrow} \textcolor{green}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}}$, $\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}} \textcolor{green}{\Rightarrow} \textcolor{red}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}}$, $\textcolor{green}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}} \textcolor{blue}{\Rightarrow} \textcolor{red}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}$.

A possible filling F_1 from $\textcolor{blue}{\rule{3em}{3pt}}\textcolor{green}{\rule{3em}{3pt}}\textcolor{red}{\rule{3em}{3pt}}$ to $\textcolor{red}{\rule{3em}{3pt}}\textcolor{green}{\rule{3em}{3pt}}\textcolor{blue}{\rule{3em}{3pt}}$ is $\textcolor{blue}{\Rightarrow}\textcolor{blue}{\Rightarrow}\textcolor{green}{\Rightarrow}\textcolor{green}{\Rightarrow}\textcolor{red}{\Rightarrow}\textcolor{red}{\Rightarrow}\textcolor{red}{\Rightarrow}\textcolor{green}{\Rightarrow}\textcolor{blue}{\Rightarrow}\textcolor{green}{\Rightarrow}\textcolor{blue}{\Rightarrow}\textcolor{red}{\Rightarrow}$; it corresponds to the filled box B_1 above. *Other* fillings for the same filled box B_1 are possible, but does any filled box B have *some* filling? Any *partial* filling F of B ends in a path P allowing a filling step. If *none* of them would yield a partial filling of B again, then there could not be any occurrence of $\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}}$ in P by the assumptions. Moreover, a rightmost occurrence of $\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}$ in P would then be filled in B by a pair of green,blue calissons and to the right of the blue one only other such could occur, giving a contradiction (to B being a filled box). Analogous reasoning pertains to a leftmost occurrence of $\textcolor{red}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}$ in P, showing there's always *some* way to let filling make progress toward B. For instance, Figure 2 depicts that after the first 8 filling steps of F_1, progress toward B_1 is made by the further $\textcolor{blue}{\Rightarrow}$-step (not by the, also possible, $\textcolor{red}{\Rightarrow}$-step!).

Having established adequacy of the modelling, the problem of the calissons resurfaces as a *quantitative* confluence question:

Does random descent hold for measure $\textcolor{red}{\Rightarrow} \mapsto (\textcolor{red}{1}, \textcolor{green}{0}, \textcolor{blue}{0})$, $\textcolor{green}{\Rightarrow} \mapsto (\textcolor{red}{0}, \textcolor{green}{1}, \textcolor{blue}{0})$, $\textcolor{blue}{\Rightarrow} \mapsto (\textcolor{red}{0}, \textcolor{green}{0}, \textcolor{blue}{1})$?

Recall [9, 10, 13, 11] that a rewrite system having *random descent* (RD) means that for any object that is normalising, rewrites to *a* normal form, we have (i) maximally rewriting the former *always* ends in the latter, and (ii) that all such rewrite sequences have the same *measure*.

In this case, filling $\Rightarrow$ is seen to be normalising (WN) for the same reason that sorting is; the $\Rightarrow$-rules simply sort edges into red–green–blue order, cf. [10, Example 7], showing that filling *does* result in a filled box.

Also RD is easily seen to hold: *Because* the (only) critical peak $\textcolor{green}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}} \textcolor{red}{\Leftarrow} \textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}} \textcolor{blue}{\Rightarrow} \textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}$ is joinable as $\textcolor{green}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}} \textcolor{green}{\Rightarrow} \textcolor{green}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}} \textcolor{blue}{\Rightarrow} \textcolor{red}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}} \textcolor{red}{\Leftarrow} \textcolor{red}{\rule{1.5em}{3pt}}\textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}} \textcolor{green}{\Leftarrow} \textcolor{blue}{\rule{1.5em}{3pt}}\textcolor{red}{\rule{1.5em}{3pt}}\textcolor{green}{\rule{1.5em}{3pt}}$ and both legs have measure $(\textcolor{red}{1}, \textcolor{green}{1}, \textcolor{blue}{1})$, *ordered local confluence* (OWCR) holds entailing RD by [13, Lem. 24].

Finally, to see that by having answered the quantitative confluence question in the affirmative we have solved the problem of the calissons, note that the measure given counts the respective numbers of red, green and blue steps while filling. Hence the triple of numbers of red, green and blue of calissons in a filled box, its *spectrum* [4], *is* the measure of its filling. Indeed, the measure $(\textcolor{red}{4}, \textcolor{green}{4}, \textcolor{blue}{4})$ of the filling F_1 is the same as the spectrum of the filled box B_1.

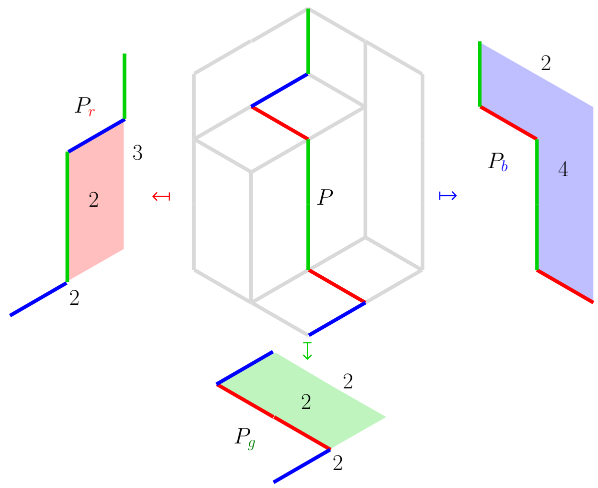

Figure 3: Volume of path P as areas of 3 projections P_r, P_g, P_b (by 'forgetting' colours)

Proof orders for decreasing diagrams Above we proceeded by (weak) normalisation (WN) and ordered local confluence (OWCR). Here we proceed instead by termination (SN) and local confluence (WCR) of filling $\Rightarrow$, as suggested by WN & OWCR $\iff$ SN & WCR [11].

Since the single critical peak of $\Rightarrow$ was shown to be joinable already, yielding WCR, it suffices to show termination of $\Rightarrow$ by a measure from which the numbers of calissons can be retrieved. To realise the idea that was given at the bottom of the first page we make use of the area measure on conversions, introduced for measuring decreasing diagrams in [5, Example 3].

Definition 2. The *area* measure of a conversion comprising ℓ forward and r backward steps, is a triple (ℓ, a, r) measuring how many *square* tiles a are needed to complete the conversion into a valley. The *volume* of a path P is the triple (r, g, b) of *area* measures of the conversions P_r, P_g and P_b obtained from P by forgetting respectively the red, green and blue edges.

Referring the reader to [5, Example 3] for formal details, we illustrate the definition by means of the path P given by as depicted in Figure 3. Then P_r is the conversion $\to\leftarrow\to\to\leftarrow$ obtained by forgetting the -edges in P and orienting the - and -edges in opposite directions, yielding area $(3,2,2)$, P_g is the conversion $\leftarrow\to\to\leftarrow$ having area $(2,2,2)$ and P_b the conversion $\leftarrow\to\leftarrow\leftarrow\to$ with area $(3,4,2)$.

We claim that if V is the volume (r, g, b) of the initial path P of a filling F of box B, then the spectrum of the box is the triple V^2 of second components of V. For instance, the volume of the initial path for box B in Figure 1 is $((2,4,2),(2,4,2),(2,4,2))$, and indeed its triple $(4,4,4)$ of second components is the spectrum of B comprising 4 calissons of each colour.

To prove the claim we prove the property that for any filling step of a given colour only the second component of that colour is decremented in the volume, with the areas of the other colours being unchanged. This suffices, since for a filling yielding a filled box, the volume of its final path Q has second components that are all 0, since 'forgetting' then yields valleys: Q_r has shape $\twoheadrightarrow\twoheadleftarrow$, Q_g has shape $\twoheadrightarrow\twoheadleftarrow$ and Q_b shape $\twoheadrightarrow\twoheadleftarrow$. To see the property holds observe that a filling step of a given colour swaps *adjacent* edges of the other colours, so leaves the areas of *those other* colours unchanged, but decrements that of the *given* colour. Indeed, the filling $\Rightarrow$-step in Figure 2 transforms into and volume $((2,1,2),(2,1,2),(2,2,2))$ into $((2,1,2),(2,1,2),(2,1,2))$, decrementing (only) the blue area.

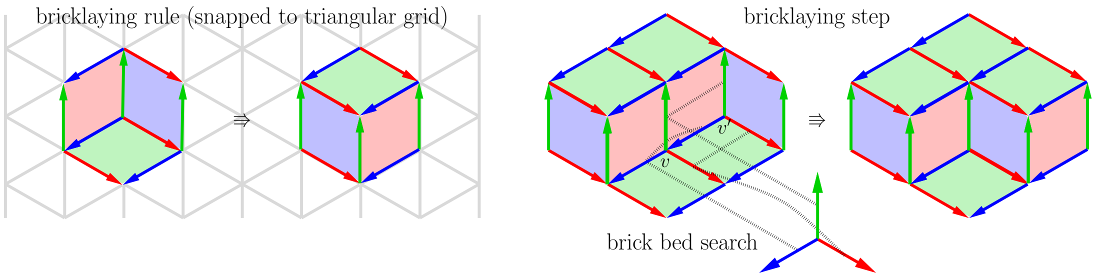

Figure 4: Solving the problem of the calissons by bricklaying

Bricklaying For the third approach to the problem of the calissons by rewriting we change the modelling; filled boxes are now the *objects* of a rewrite system $\Rrightarrow$ having *bricklaying* as *rule* [12] displayed on the left in Figure 4, allowing to locally rearrange the calissons in a box.

Definition 3. Linear combinations $r_r\binom{\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} + r_g\binom{1}{0} + r_b\binom{-\frac{1}{2}\sqrt{3}}{-\frac{1}{2}}$ give the *grid* of *vertices* for *natural number* scalars r_r, r_g, r_b, a *box* when restricting them to *real* intervals $[0,w]$, $[0,h]$, $[0,d]$ for *natural number* width w, height h, depth d, and *calissons* when further restricting two among w, h, d to 1 and the other (its *colour*) to 0; reversing this yields *edges*. Box, diamond and edge *occurrences* arise by translation. We suppress writing 'occurrence'. A family $\mathcal{D}_I$ of diamonds is a *tiling* (of a box B) if $\mathcal{D}_i \cap \mathcal{D}_j$ is a subset of some edge for $i \neq j$ (and $\bigcup \mathcal{D}_I \subseteq B$).

W.l.o.g. we analyse only the *discrete* problem where calissons occur at vertices, B at the origin, and $B = \bigcup \mathcal{D}_I$. By the spectrum obviously being invariant under $\Rrightarrow$, the problem of the calissons resurfaces as the confluence / uniqueness of normal form question:

$$\text{Does } \bigcup \mathcal{D}_I = B = \bigcup \mathcal{D}'_J \text{ for box } B, \text{ entail } \mathcal{D}_I,\ \mathcal{D}'_J \text{ have the same } \Rrightarrow\text{-normal form?}$$

Mapping calissons to their vertices and (coloured) edges, turns tilings into *bed-graphs* [12]: (i) every green tile is a tetragonal cycle of shape $\leftarrow\leftarrow\rightarrow\rightarrow$ and similarly for red and blue tiles; (ii) vertices have at most a single *in-*/*out*-edge of a given colour; (iii) there are no paths having edges of each of the 3 colours; (iv) every path $\rightarrow\rightarrow$ belongs to some tile and similarly for other colour-pairs; (v) if $\leftarrow a \rightarrow$ does not belong to a tile then a has a green in-edge and similarly for other colour-triples. Taking edges as vectors in the 3 colour-dimensions shows tilings even are *beds* [12], i.e. are *plane* bed-graphs under projection from viewpoint $\begin{pmatrix}\infty\\\infty\\\infty\end{pmatrix}$, as used in illustrations.

Let an *i-peak* for such a tiling $\mathcal{D}_I$ be a vertex in B having exactly i *out*-edges. Then $0 \leq i \leq 3$ by there being 3 colours, and we distinguish cases on whether or not there are 3-peaks:

If there are 3-peaks, then the bricklaying $\Rrightarrow$-rule is applicable to at at least one of them. This holds for any bed [12] as depicted in Figure 4: If v is a 3-peak but $\Rrightarrow$ does not apply, then by (v) it has an in-edge of colour c from v', which is a 3-peak by (iv) and *its* in-edge, if any, has colour c by (iii) from which we conclude by finiteness of tilings / monochrome paths in beds.

If there are no 3-peaks, then we have one large *brick* [12] generalising that in the rhs of the $\Rrightarrow$-rule in Figure 4. That is, at the top we *must* have a big green calisson composed of smaller such and *mutatis mutandis* the same for blue / red at the bottom–left / right. This holds in fact for any bed: Any $\leftarrow\rightarrow$-peak then *must* belong to a *green* tile since otherwise (v) and the above reasoning would give rise to a 3-peak contradicting the assumption. The big red, green, blue calissons share boundary paths and these 3 *rays* end up in the same *nexus*, the common reduct; this holds by monochrome paths being finite and (iii), with the former a consequence of the bed-graph being plane. We conclude by noting the 3 big calissons only depend on B.

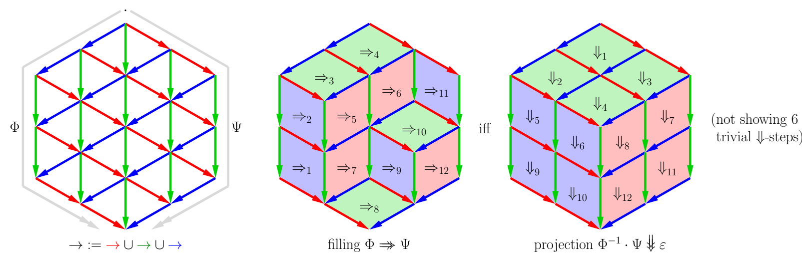

Figure 5: Solving the problem of the calissons by local undercutting; filling iff projection

Local undercutting Our fourth approach to the problem of the calissons by rewriting, inspired by [3, Proposition 4.16(4.18)], mixes the above modellings: We again view the grid for a box B as a rewrite system $\rightarrow := \rightarrow \cup \rightarrow \cup \rightarrow$ but with green $\rightarrow$-steps now oriented *downward* as displayed on the left in Figure 5. Calissons now are *diamonds* ${}^{\phi}_{\chi}\!\diamond{}^{\psi}_{\upsilon}$ inducing *filling* $\phi \cdot \chi \Rightarrow \psi \cdot \upsilon$ respectively *projection* $\phi^{-1} \cdot \psi \Downarrow \chi \cdot \upsilon^{-1}$ rules. Filling is modelled as $\Phi \Rrightarrow \Psi$ for left,right legs Φ, Ψ of B, and we claim it holds iff projecting Φ, Ψ is empty, i.e. iff $\Phi^{-1} \cdot \Psi \Downarrow \varepsilon$ with the spectra of filling and projection the same, from which we conclude as the spectrum of projection only depends on B, as seen before. Here we use $\Upsilon, \Phi, X, \Psi, \ldots$ to range over *conversions*, elements of the free (typed) involutive monoid over $\rightarrow$-steps $\upsilon, \phi, \chi, \psi, \ldots$, with $\cdot$ denoting *composition* and ${}^{-1}$ *reverse*; conversions are (possibly empty; ε) compositions of steps and reverse steps [5].

Definition 4. A *local undercutting*[2] (LUC) is a collection of diamonds $\mathcal{D}$ consisting of for every local peak $\phi^{-1} \cdot \psi$ at most one diamond of shape ${}^{\phi}_{X}\!\diamond{}^{\psi}_{\Upsilon}$ for reductions X, Υ such that: ${}^{\phi}_{\varepsilon}\!\diamond{}^{\phi}_{\varepsilon} \in \mathcal{D}$, and $(\phi \cdot X)^{-1} \cdot \psi \cdot \Upsilon \Downarrow \varepsilon$ if $\phi^{-1} \cdot \chi \cdot \chi^{-1} \cdot \psi \Downarrow X \cdot \Upsilon^{-1}$. $\mathcal{D}$ is spectrum-*preserving* if the spectra of the latter two projections are the same (they are unique by random descent [10] of $\Downarrow$).

Calissons induce a spectrum-preserving LUC after adjoining ${}^{\phi}_{\varepsilon}\!\diamond{}^{\phi}_{\varepsilon}$; per definition of $\rightarrow$ the only non-trivial case is $(\leftarrow \cdot \rightarrow \cdot \leftarrow \cdot \rightarrow) \Downarrow^3 (\rightarrow \cdot \rightarrow \cdot \leftarrow \cdot \leftarrow)$ for which we indeed have $(\leftarrow \cdot \leftarrow \cdot \leftarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow) \Downarrow^6 \varepsilon$, and both projections have spectrum $(1, 1, 1)$. To prove the claim, it suffices that *filling iff projection* for spectrum-preserving LUCs, cf. Figure 5 right. To enable proving it *by inductions*, we rephrase filling using the notion of *foliage*[3] imaged on the left in Figure 6: a cyclic conversion $Z = Z_1 \cdot \ldots \cdot Z_n$ of length n, together with reductions Ξ_i for $0 \leq i \leq n$ with $\Xi_0 = \varepsilon = \Xi_n$, and fillings $\Xi_{i-1} \Rrightarrow Z_i \cdot \Xi_i$ if Z_i is a step and $Z_i^{-1} \cdot \Xi_{i-1} \Rrightarrow \Xi_i$ if Z_i is a reverse step.

Theorem 1. *If $\rightarrow$ is terminating and $\mathcal{D}$ LUC, then there is a foliage for conversion Z iff $Z \Downarrow \varepsilon$. If $\mathcal{D}$ moreover is spectrum-preserving then the foliage and the projection have the same spectra.*

See the appendix for a proof. Here we conclude by observing a filling $\Phi \Rrightarrow \Psi$ for left,right legs Φ, Ψ of a box B gives rise to a foliage for $\Phi^{-1} \cdot \Psi$ with the same spectrum, and *vice versa*.

Remark. The diagrammatic perspective originates with *Newman's II-Lemma* [9, Section 6]: If $\rightarrow$ is terminating, there is a diamond in $\mathcal{D}$ for every local peak, and $[\![\,]\!]$ is a typed involutive monoid homomorphism to a typed group mapping diamonds in $\mathcal{D}$ to 0, then $[\![\,]\!]$ maps every conversion cycle to 0. *Proof.* For any conversion Z there is a valley $X \cdot \Upsilon^{-1}$ with $[\![Z]\!] = [\![X \cdot \Upsilon^{-1}]\!]$, by enriching Newman's Lemma with that $[\![\,]\!]$ maps diamonds in $\mathcal{D}$ to 0. Hence if Z is a conversion *cycle*, say on a, then $[\![\Phi^{-1} \cdot Z \cdot \Phi]\!] = [\![\varepsilon]\!]$ for Φ a reduction from a to normal form, so $[\![Z]\!] = 0$.□ What Newman's Lemma is to the Critical Peak Lemma [6, Lemma 2.4] is Newman's II-Lemma to Squier's Finite Derivation Type method [14, 1]; it ought to be better-known.

[2]It expresses *cut*-elimination (transitivity-elimination) replacing two diamonds by a single one *under* them.
[3]Originally introduced for proving [12, Theorem 4].

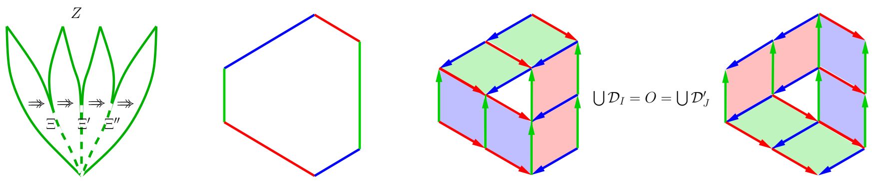

Figure 6: foliage (left), non-fillable box (middle), non-convex non-⇛-convertible (right)

Conclusion This note illustrates the power of modern confluence techniques: The first three provided solutions out of the box. The fourth, inspired by [3, Proposition 4.16(4.18)], is novel.

The equiangular hexagonal B in the middle in Figure 6 is not zonogonal; filling gets stuck. Still, as shown on the right, the spectra of both tilings $\mathcal{D}_I, \mathcal{D}'_J$ of B for $\bigcup \mathcal{D}_I = O = \bigcup \mathcal{D}'_J$ (a triangle is 'missing') *are* the same [4]. We leave it to future research to investigate whether the techniques presented here can be appropriately adapted (we expect the first and fourth can).

Acknowledgment. *Jan Willem Klop brought the problem of the calissons, it being amenable to rewrite techniques, and 'piling = tiling' (Theorem 1, cf. [7][15, Chapter 8]) to my attention.*

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Appendix In the proof of Theorem 1 we *measure* a foliage for conversion Z by the multiset of pairs where the ith object a (*height*) of Z is paired with the number of $\Rrightarrow$-root-steps in $Z_i^{-1} \cdot \Xi_{i-1} \Rrightarrow \Xi_i \Rrightarrow Z_{i+1} \cdot \Xi_{i+1}$ if a is the apex of a local peak and 0 otherwise (*width*). The multiset extension of the lexicographic product of $\leftarrow^+$ and $<$ well-foundededly orders measures.

Proof of Theorem 1. We prove the if-direction by induction on the number of steps p in $Z \Downarrow^p \varepsilon$, cf. [12, Theorem 4]. If $p = 0$, we trivially conclude as $Z = \varepsilon$. Otherwise, for some ${}^{\phi}_{X}\!\diamond^{\psi}_{\Upsilon}$ we have $Z = Z^l \cdot \phi^{-1} \cdot \psi \cdot Z^r$ and $Z \Downarrow Z'$ and $Z' \Downarrow^{p-1} \varepsilon$ for $Z' = Z^l \cdot X \cdot \Upsilon^{-1} \cdot Z^r$. By the IH there is a foliage for Z' (with spectrum that of $Z' \Downarrow^{p-1} \varepsilon$); its subconversions Z^l, Z^r combined with prefixing ϕ to the last reduction of Z^l then give a foliage for Z (with spectrum that of $Z \Downarrow^p \varepsilon$).

We prove the only–if-direction by induction on the measure of the foliage for Z and cases on Z. If Z is a valley, then by definition of foliage $Z = \varepsilon$ using that the legs of diamonds in $\mathcal{D}$ are non-empty, and we conclude. Otherwise, Z has shape $Z^\ell \cdot \phi^{-1} \cdot \psi \cdot Z^r$ with $\phi \cdot \Xi_{i-1} \Rrightarrow \Xi_i \Rrightarrow \psi \cdot \Xi_{i+1}$ and we distinguish cases on the width w of the apex of the local peak.

If $w = 0$ then $\phi = \psi$ and $\Xi_{i-1} \Rrightarrow \Xi_{i+1}$. Then $Z \Downarrow Z'$ for $Z' := Z^\ell \cdot Z^r$ by LUC. Replacing[4] Ξ_{i-1} by Ξ_{i+1} in the foliage for Z^ℓ renders Z' a foliage. We conclude by the IH for Z'.

If $w = 1$ then $\phi \cdot \Xi_{i-1} \Rrightarrow \phi \cdot X \cdot \Xi' \Rightarrow \psi \cdot \Upsilon \cdot \Xi' \Rrightarrow \psi \cdot \Xi_{i+1}$ for some diamond ${}^{\phi}_{X}\!\diamond^{\psi}_{\Upsilon} \in \mathcal{D}$ and some Ξ', where the displayed $\Rrightarrow$ do not have head-steps. Then $Z \Downarrow Z'$ for $Z' := Z^\ell \cdot X \cdot \Upsilon^{-1} \cdot Z^r$. Replacing (cf. footnote 4) Ξ_{i-1} by $X \cdot \Xi'$ in the foliage for Z^ℓ and replacing Ξ_{i+1} by $\Upsilon \cdot \Xi'$ in the foliage for Z^r, renders Z' a foliage again. We conclude by the IH for Z'.

If $w > 1$ then $\phi \cdot \Xi_{i-1} \Rrightarrow \phi \cdot \Phi \cdot \Xi' \Rightarrow \chi \cdot \Psi \cdot \Xi' \Rrightarrow \chi \cdot X \cdot \Xi'' \Rightarrow \psi' \cdot \Upsilon \cdot \Xi'' \Rrightarrow \psi \cdot \Xi_{i+1}$ for some diamonds ${}^{\phi}_{\Phi}\!\diamond^{\chi}_{\Psi}, {}^{\chi}_{X}\!\diamond^{\psi'}_{\Upsilon} \in \mathcal{D}$ and some Ξ', Ξ'', where the first two displayed horizontal reductions do not have head-steps (we may but need not have $\psi' = \psi$). The second induces a foliage for the peak $(\Psi \cdot \Xi')^{-1} \cdot X \cdot \Xi''$ to which the IH applies (by its apex being reached via χ), yielding $(\Psi \cdot \Xi')^{-1} \cdot X \cdot \Xi'' \Downarrow\!\!\!\Downarrow \varepsilon$. By random descent for $\Downarrow$, this projection factors as $\Psi^{-1} \cdot X \cdot \Downarrow\!\!\!\Downarrow X' \cdot \Psi'^{-1}$ and $\Xi'^{-1} \cdot X' \cdot \Psi'^{-1} \cdot \Xi'' \Downarrow\!\!\!\Downarrow \varepsilon$ for some reductions X', Ψ'.

The former $\Downarrow\!\!\!\Downarrow$ combined with two $\Downarrow$-steps for the diamonds gives $\phi^{-1} \cdot \chi \cdot \chi^{-1} \cdot \psi' \Downarrow\!\!\!\Downarrow \Phi \cdot X' \cdot (\Upsilon \cdot \Psi')^{-1}$ for which LUC entails $(\phi \cdot \Phi \cdot X')^{-1} \cdot \psi \cdot \Upsilon \cdot \Psi' \Downarrow\!\!\!\Downarrow \varepsilon$. The if-direction then yields a foliage for it, so $\phi \cdot \Phi \cdot X' \Rrightarrow \psi \cdot \Upsilon \cdot \Psi'$ having exactly 1 head-step (by a diamond for ϕ, ψ).

For the latter $\Downarrow\!\!\!\Downarrow$ the if-direction yields a foliage so $\Xi' \Rrightarrow X' \cdot \hat{\Xi}$ and $\Psi' \cdot \hat{\Xi} \Rrightarrow \Xi''$ for some $\hat{\Xi}$.

Combining both shows that $\phi \cdot \Phi \cdot \Xi' \Rrightarrow \psi' \cdot \Upsilon \cdot \Xi''$ using a single head-step, instead of the two before. Hence we conclude by the IH for the same Z but with this alternative foliage. ☐

Definition 5. *local semi-lattice* (LSL) is LUC with *commutativity* of $\mathcal{D}$: ${}^{\phi}_{\Phi}\!\diamond^{\psi}_{\Psi} \in \mathcal{D}$ iff ${}^{\psi}_{\Psi}\!\diamond^{\phi}_{\Phi} \in \mathcal{D}$.

Observe that to establish LUC it suffices to consider triples ϕ, ψ, χ where $\phi \neq \chi \neq \psi$ since if, say, $\phi = \chi$ then the assumption simplifies to $\phi^{-1} \cdot \psi \Downarrow\!\!\!\Downarrow X \cdot \Upsilon^{-1}$, which is seen to entail the conclusion $(\phi \cdot X)^{-1} \cdot \psi \cdot \Upsilon \Downarrow\!\!\!\Downarrow \varepsilon$ using that peaks between a step and itself were assumed trivial. LSL allows to also assume $\phi \neq \psi$, since if $\phi = \psi$ then $\phi^{-1} \cdot \chi \cdot \chi^{-1} \cdot \phi \Downarrow\!\!\!\Downarrow X \cdot \Upsilon^{-1}$ entails $X = \Phi = \Upsilon$ for ${}^{\phi}_{\Phi}\!\diamond^{\chi}_{\Psi}, {}^{\chi}_{\Psi}\!\diamond^{\phi}_{\Phi} \in \mathcal{D}$ so $(\phi \cdot X)^{-1} \cdot \phi \cdot \Upsilon \Downarrow\!\!\!\Downarrow \varepsilon$, using trivial peaks have trivial diamonds.

This resumes our attempts [7][15, Chapter 8] at a theory of *orthogonality* for rewriting and algebra: LSL holds for the $\lambda\beta$-calculus [8] (*local cube*) and for positive braids [3, Example 4.20]. Though neither $\to_\beta$ nor braids (Artin's σ_i) are terminating, that can be brought about (by *finiteness of family developments* [15] respectively *right-Noetherianity* [3]) making Theorem 1 applicable. From a rewriting / order perspective Theorem 1 aims at showing that *permutation equivalence* = *projection equivalence* [15] / reductions constitute a *semi-lattice* [8] (whence LSL). Contrapositively, it enables showing reductions do *not* have a common reduct (no *upperbound*) by showing projection of their peak does *not* terminate (no *least* upperbound) [3, Example 4.28].

[4] If $Z^\ell = \varepsilon$ replacing is not allowed but not needed: then $\Xi_{i-1} = \varepsilon = \Xi_{i+1}$ as legs of diamonds are non-ε.