

Confluence and orthogonality by residuation illustrated by the problem of the calissons

Abstract—We revisit the classical notions of confluence and orthogonality from the perspective of rewrite systems equipped with a residuation, modelled as 1-algebras. The perspective allows to smoothly connect confluence and orthogonality to various fields, e.g. to (least) upper bounds in order theory, (least) common multiples in algebra, and (relative) pushouts in category theory. Taking inspiration from the connexions, we prove 3 confluence results: completeness of random descent for completeness, 3-confluence by 3-local confluence, and orthogonality by undercutting. We illustrate the results by examples from various fields, and show each provides a solution to the problem of the calissons.

Index Terms—rewriting, confluence, orthogonality, residuation, upper bound, common multiple, pushout, random descent, proof order, undercutting, calisson.

I. INTRODUCTION

The *raison d'être* of term rewriting [1]–[4] is the correspondence between *validity* $\mathcal{E} \models t = s$ in all models (algebras satisfying $\mathcal{E}$), *provable equality* $\mathcal{E} \vdash t = s$ in equational logic, and *convertibility* $t \xleftrightarrow{*}_{\mathcal{E}} s$ for $\mathcal{E}$ as a term rewrite system, given an equational theory $\mathcal{E}$ and equation $t = s$, correspondences known as Birkhoff's theorem and logicity [2], [4]. They extend to sub-equational logics. For example, dropping symmetry: *valid* $\mathcal{E} \models t \geq s$ iff *provably reachable* $\mathcal{E} \vdash t \geq s$ iff *reducible* $t \twoheadrightarrow_{\mathcal{E}} s$ [5], [6].

To decide whether terms t, s belong to the same $\mathcal{E}$ -equivalence class, a basic way to proceed is to define *unique representatives* for $\mathcal{E}$ -equivalence classes, and a *normalisation* function $\hat{\cdot}$ mapping terms to the unique representative of their class, after which one checks whether $\hat{t} = \hat{s}$. Completion [2], [7] automates both the choice of representatives and computing them, by constructing a term rewrite system $\mathcal{T}$ (having the same provable equality as $\mathcal{E}$) that is *complete*, i.e. terminating and confluent. By the above, this allows to reduce the decision problem to deciding whether t, s are *convertible* $t \xleftrightarrow{*}_{\mathcal{T}} s$, and answering it to checking whether t, s have a *valley*, a common reduct is found by $\mathcal{T}$ -normalising t, s to their *normal forms*.

The backdrop for the confluence results in this paper (mentioned in the abstract) is that one can view a valley $\varsigma: t \twoheadrightarrow_{\mathcal{T}} \cdot \leftarrow_{\mathcal{T}} s$ showing that t, s have a common reduct, as obtained from the conversion $\varrho: t \xleftrightarrow{*}_{\mathcal{T}} s$ by means of a rewrite process itself $\varrho \twoheadrightarrow \varsigma$ referred to as *2-rewriting*. Since this is a rewrite process *on* conversions and reductions, it requires conversions and reductions to be first-class citizens.

Though one finds various accounts of 2-rewriting in the literature, cf. e.g. [8]–[13], our perspective on it developed in Sec. II is novel in that we base it on 1-algebras, algebras that have rewrite systems as carrier. 1-residual 1-algebras will be our motivating example, as *residuation* is seen to be at the basis of the process of 2-rewriting a conversion into a valley. This process, known as *tiling* with *diagrams* [9], is a staple of rewrite theory since [8], [14], and its formalisation forms the backdrop for our results. We develop it for rewrite systems where objects are not assumed to have (term, ...) structure.

To make our notions and results evident in a background-free way, we showcase them on the problem of the calissons [15]. The problem is easy to visualise and so will be our solutions.¹ We focus solely on the rewriting perspective on the problem and refer the interested reader to the rich literature on (non-rewriting) descriptions, solutions, and discussions.

Example 1. *The problem of the calissons is to show that if a box B , a regular hexagonal, can be filled with calissons (named after certain diamond-shaped sweets) then in a resulting filled box the numbers of calissons in each of the 3 orientations are the same. A first observation is that it*

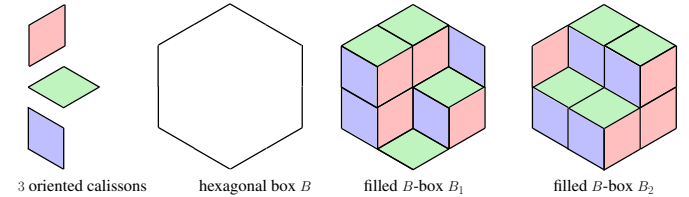


Fig. 1. The Problem of the Calissons

suffices to show that all filled B -boxes have the same spectrum (r, g, b) , for r, g and b the numbers of red, green and blue² calissons, since then we conclude by rotational symmetry that $r = g = b$. Indeed, for the filled B -boxes B_1 and B_2 in Figure 1 we have $r_i = g_i = b_i = 4$ for $i \in \{1, 2\}$.

II. REWRITE SYSTEMS, 1-ALGEBRAS, RESIDUATION

We present a novel perspective on rewrite systems and their properties of interest, and present our running examples: $\lambda\beta$ -calculus, (positive) braids, and calissons. The presentation will

¹<https://calissons-hexagon-solver.netlify.app> visualises one of them.

²We assign colours to the orientations for convenient referencing.

be at the basis of our results in Sec. III and IV. We assume familiarity with rewriting [1]–[4], [16] and the λ -calculus [4], [17], and some knowledge of braids [12]. We base ourselves on the classical notion of a rewrite system from [8], [4, Ch. 8]:

Definition 1. A rewrite system is a quadruple $\langle \mathcal{O}, \mathcal{S}, \text{src}, \text{tgt} \rangle$ with $\mathcal{O}$ and $\mathcal{S}$ collections of objects and steps and source, target maps src, tgt from steps to objects. We write $\phi : a \rightarrow b$ or $a \rightarrow_\phi b$ to denote ϕ is a step from a to b . An object from which there is no $\rightarrow$ -step is in normal form.

We use arrow-like notations like $\rightarrow, \rightrightarrows, \twoheadrightarrow$ to denote rewrite systems, which come equipped with the usual notion of morphism, and we write $\rightarrow \subseteq \rightrightarrows$ to denote that $\rightarrow$ is a sub-system of $\rightrightarrows$ [4, Def. 8.2.5]. By their structure being basic, rewrite systems go under various names, e.g. quivers, multidigraphs and pre-categories. Rewrite systems have steps as first-class citizens [8], [11], [18], [4, Chs. 8, 9], and are to be contrasted with rewrite *relations* as in [2], [17], [4, Chs. 1–7], binary endorelations on a set. A rewrite relation is obtained from a rewrite system $\rightarrow$ by forgetting about the identity of steps; a is related to b if there exists a step from a to b in $\rightarrow$. The following two standard rewrite systems serve as running examples. We focus on their steps, absent when presented as rewrite relations.

We recapitulate the presentation of the $\lambda\beta$ -calculus [17] as a higher-order pattern rewrite system (PRS) from [4, Ex. 11.2.22(ii)], see [19] for more on PRSs. To render steps of a PRS as first-class citizens, they are defined as *terms* over the signature extended with *rule-symbols* [4, Ch. 8], [20].

Example 2. The untyped λ -calculus is a PRS having a higher-order signature Λ of function symbols $\{\text{abs} : (\text{term} \rightarrow \text{term}) \rightarrow \text{term}, \text{app} : \text{term} \rightarrow (\text{term} \rightarrow \text{term})\}$ and a signature B of rule-symbol(s) $\{\beta : (\text{term} \rightarrow \text{term}) \rightarrow (\text{term} \rightarrow \text{term})\}$. The left-hand side (lhs) and right-hand side (rhs) of the rule-symbol β are:

$$\beta : MN.\text{app}(\text{abs}(x.M(x)), N) \rightarrow MN.M(N)$$

for $M : \text{term} \rightarrow \text{term}$, $N : \text{term}$ and $\text{lhs}(\beta), \text{rhs}(\beta) : (\text{term} \rightarrow \text{term}) \rightarrow (\text{term} \rightarrow \text{term})$, the type of β . The multistep rewrite system $\rightrightarrows_\beta$ has (simply-typed) terms over Λ as objects and terms over $(\Lambda \cup B)$ as steps. The source and targets maps from steps to objects are induced by the (type-preserving) functions mapping the rule-symbol β to its left-hand side ($\text{lhs}(\beta)$) respectively right-hand side ($\text{rhs}(\beta)$). The (single) step rewrite system $\rightarrow_\beta$ is the sub-system of $\rightrightarrows_\beta$ obtained by restricting multisteps to having exactly 1 occurrence of β .

For example, $\beta(x.x, \text{app}(\mathbf{I}, y))$ and $\text{app}(\mathbf{I}, \beta(x.x, y))$, where $\mathbf{I} := \text{abs}(x.x)$, both are $(\Lambda \cup B)$ -terms and in fact steps, as β occurs once in them. By syntactic accident both witness $\text{app}(\mathbf{I}, \text{app}(\mathbf{I}, y)) \rightarrow_\beta \text{app}(\mathbf{I}, y)$. The $(\Lambda \cup B)$ -term $\beta(x.x, \beta(x.x, y))$ is a multistep by the rule-symbol β occurring twice in it. It witnesses $\text{app}(\mathbf{I}, \text{app}(\mathbf{I}, y)) \rightrightarrows_\beta y$.

One might expect that making steps explicit would render the presentation more involved. But in fact, the presentation of the λ -calculus in Ex. 2 is more succinct than usual [17] owing to modelling it as a PRS and using multisteps [4], [20]: both

λ -terms and β -steps are terms over (simply-typed) signatures, affording economy. Rule-symbols allow to reuse *terms* for defining (multi)steps and to naturally distinguish between the 2 steps in Ex. 2, without having to resort to external means [17, Ex. 11.2.10] like inference rules [17, Def. 3.1.5], needed there because there $\rightarrow_\beta$ is a rewrite *relation*. Similarly, the multistep in Ex. 2 corresponds to *complete development* $\mathbf{I}(\mathbf{I}y) \rightrightarrows_\beta y$, requiring external means to be presented in [17].

We recapitulate the presentation of (positive) braids [12], [21]–[23] as a rewrite system [24]–[26], [4, Sec. 8.9].

Example 3. The multistep rewrite system $\rightrightarrows_B$ on ℓ strands

- has as objects braids, words over the Artin generators σ_i for $1 \leq i < \ell$, modulo $\equiv$ generated by:

$$\begin{aligned} \sigma_i \sigma_j \sigma_i &\equiv \sigma_j \sigma_i \sigma_j & \text{if } |i - j| = 1 \\ \sigma_i \sigma_j &\equiv \sigma_j \sigma_i & \text{if } |i - j| > 1 \end{aligned}$$

- multisteps $(w, v) : w \rightrightarrows_B v$ for any braid w and prefix-modulo $\equiv$ v of the fundamental word Δ_ℓ (so v is a simple braid), where $\Delta_1 := \varepsilon$ and $\Delta_{n+1} := \Delta_n \sigma_n \dots \sigma_1$. The (single) step rewrite system $\rightarrow_B$ is the sub-system of $\rightrightarrows_B$ obtained by restricting multisteps to having 1 generator.

For example, for $\ell \geq 3$ the steps $\sigma_1 \sigma_2 \rightarrow \sigma_1 \sigma_2 \sigma_1$ and $\sigma_2 \sigma_1 \rightarrow \sigma_2 \sigma_1 \sigma_2$ have the same target, $\sigma_1 \sigma_2 \sigma_1 \equiv \sigma_2 \sigma_1 \sigma_2$, so constitute a valley, and $\varepsilon \rightrightarrows \sigma_1 \sigma_2$ is a multistep, but $\varepsilon \rightarrow \sigma_1 \sigma_1$ isn't.

Properties of rewrite systems are expressed in terms of their objects $a, b, c, \dots$ and steps. Useful properties of steps ϕ, ψ are that they are *co-initial* if $\text{src}(\phi) = \text{src}(\psi)$, *composable* if $\text{tgt}(\phi) = \text{src}(\psi)$, *co-final* if $\text{tgt}(\phi) = \text{tgt}(\psi)$, and *parallel* to each other if both co-initial and co-final. For rewrite systems the operations (producing steps) that will be of interest here, with constraints on sources and targets as specified in Fig. 2, are the following four:

- 1) *residuation* ϕ / ψ on co-initial ϕ, ψ ;
- 2) *loop* 1_a (we usually omit the subscript for legibility);
- 3) *composition* $\phi \cdot \psi$ on composable ϕ, ψ ;
- 4) *reverse* ϕ^{-1} .

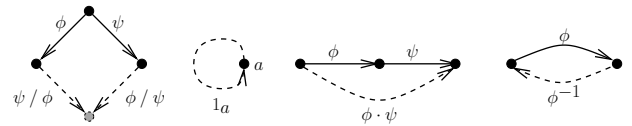


Fig. 2. Operations: 1 residuation, 2 loop, 3 composition, 4 reverse

The other 3 operations being standard, we comment on and exemplify residuation ϕ / ψ (pronounced *phi after psi*; cf. Ex. 24). Early uses of residuation in rewriting are in [8], [14], later ones in [4], [11]–[13], [18], [27]–[31]. Residuation is natural; it relates to composition as monus relates to addition on natural numbers; cf. Ex. 24. It is of central interest here as it witnesses confluence. More precisely, it is obtained by Skolemising the *diamond* property [2], [4], [8] expressing that for all co-initial ϕ, ψ , there exist ψ', ϕ' such that $\phi / \psi \rightrightarrows \phi' \cdot \psi'$, where we use ϕ / ψ to denote ϕ, ψ, ψ', ϕ' constitute a diamond: $\text{src}(\phi) = \text{src}(\psi)$ &

$\text{tgt}(\phi) = \text{src}(\psi') \ \& \ \text{tgt}(\psi) = \text{src}(\phi') \ \& \ \text{tgt}(\psi') = \text{tgt}(\phi')$. Though Skolemising gives *a priori* rise to two operations $\backslash, /$ on steps, corresponding to witnessing existence of the pair ψ', ϕ' via the pair of steps $\phi \backslash \psi, \phi / \psi$, one operation suffices:

Lemma 1. *We may assume the two Skolem-functions $\backslash, /$ for $\forall \phi \psi. \exists \psi' \phi'. \phi \diamond_{\psi'}^{\psi} \phi'$ to be involutive: $\psi \backslash \phi = \phi / \psi$.*

Proof. Let $\phi / \psi := \phi \backslash \psi$ if $\phi \preceq \psi$ and $\psi \backslash \phi$ otherwise, for some total order $\preceq$ on steps (existence of $\preceq$ may be assumed using the axiom of choice). Then $\phi \diamond_{\psi}^{\psi} \phi / \psi$ for all ϕ, ψ , with $/$ involutive per construction. $\square$

This validates using the single residuation $/$ in Fig. 2, and entails residuation is not only natural but unavoidable for confluence: a rewrite system has a residuation iff its rewrite relation has the diamond property in the sense of [2], [4], [8]. On top of that, residuation affords economy of proving:

Example 4. *The (step) rewrite systems of Exs. 2 and 3 do not have the diamond property; consider co-initial β -steps from $\text{app}(\omega, \text{app}(\mathbf{I}, y))$ where $\omega := \text{abs}(x.\text{app}(x, x))$, respectively the co-initial braid steps $\varepsilon \rightarrow \sigma_1$ and $\varepsilon \rightarrow \sigma_2$. The multistep rewrite system $\Rightarrow_{\beta}$ in Ex. 2 does have the diamond property: residuation $/$ may be defined recursively, with interesting (non-congruence) cases, for multisteps $\varrho, \varsigma, \zeta, \xi, \vartheta, \omega$:*

ϑ	ω	ϑ / ω
$\text{app}(\text{abs}(x.\varrho), \varsigma)$	$\beta(x.\zeta, \xi)$	$(x.\varrho / \zeta) (\varsigma / \xi)$
$\beta(x.\varrho, \varsigma)$	$\beta(x.\zeta, \xi)$	$(x.\varrho / \zeta) (\varsigma / \xi)$
$\beta(x.\varrho, \varsigma)$	$\text{app}(\text{abs}(x.\zeta), \xi)$	$\beta(x.\varrho / \zeta, \varsigma / \xi)$

The constraints on sources, targets required of residuation (Fig. 2) are verified by induction on multisteps.³ Also the braid multistep rewrite system $\Rightarrow_{\beta}$ has the diamond property, shown by induction on ℓ , cf. [24, Thm. 20].

Remark 1. *The recursive definition of residuation $/$ on β -multisteps is an internal version (for multisteps) of the external (for what are often called parallel steps in the λ -calculus literature) one [17, Lem. 3.2.6] due to Tait & Martin-Löf. It exploits lifting the formalisation of objects as terms over a simply-typed signature as in higher-order abstract syntax (HOAS [32]), to steps; substitutions are represented internally (as simply-typed β -redexes $(x.\varrho) (\varsigma)$) rather than externally (as $\varrho[x:=\varsigma]$ [17]).*

$\text{assoc}(\phi, \psi, \chi)$	$:$	$(\phi \cdot \psi) \cdot \chi$	$=$	$\phi \cdot (\psi \cdot \chi)$
$l\text{-unit}(\phi)$	$:$	$1 \cdot \phi$	$=$	ϕ
$r\text{-unit}(\phi)$	$:$	$\phi \cdot 1$	$=$	ϕ
invol-id	$:$	1^{-1}	$=$	1
$\text{anti-auto}(\phi, \psi)$	$:$	$(\phi \cdot \psi)^{-1}$	$=$	$\psi^{-1} \cdot \phi^{-1}$
$\text{invol}(\phi)$	$:$	$(\phi^{-1})^{-1}$	$=$	ϕ

Fig. 3. 1-laws of 1-involutive 1-monoids (dagger categories)

³ The most efficient way to proceed is to recursively define an auxiliary commutative operation $\sqcup$ such that $\text{tgt}(\varrho \sqcup \varsigma) = \text{tgt}(\varrho / \varsigma)$.

Thus far, the operations in Fig. 2 do not have any properties other than the constraints imposed on sources and targets. To express and axiomatise properties of them, we employ 1-algebras,⁴ algebras having rewrite systems as carriers. We let $1\text{-}X$ denote the 1-algebra pendant of an algebraic notion X . An algebra induces a 1-algebra on a singleton carrier with the objects of the former as steps. The main 1-algebras of interest are the 1-residual 1-algebras in Sec. IV. Here we already identify the 1-algebras that are part of the backdrop, needed for rewriting itself, whose operations are required to satisfy the 1-laws in Fig. 3 (in as far as applicable).

Example 5. (i) a 1-involutoid⁵ is a 1-algebra $\langle \rightarrow, {}^{-1} \rangle$;
(ii) a 1-monoid (category) is a 1-algebra $\langle \rightarrow, 1, \cdot \rangle$;
(iii) a 1-involutive 1-monoid is a 1-algebra $\langle \rightarrow, 1, \cdot, {}^{-1} \rangle$.

These three 1-algebras are relevant to rewriting since standard rewrite systems $\leftrightarrow, \rightarrow, \xrightarrow{*}$ arise from them as free constructions on $\rightarrow$. Formally, assume $\rightarrow := \langle \mathcal{O}, \mathcal{S}, \text{src}, \text{tgt} \rangle$ to be a rewrite system, and use lowercase letters $\phi, \psi, \chi, v, \dots$ from the end of the Greek alphabet to range over its steps. From the steps of $\rightarrow$ we can naturally, inductively build terms together with source and target maps on them using the operations $1, {}^{-1}$ and $\cdot$ in Fig. 2 as (1-function) symbols (of arities 0, 1 and 2). Following [4, Ch. 8], [20], [31], we refer to such terms as *proofterms*, as they are *terms* that, due to them having source and target maps, can be conceived also as *proofs* in (sub-)equational logic(s) induced by the steps. We use other lowercase Greek letters $\varrho, \varsigma, \zeta, \xi, \dots$ to range over proofterms. Proofterms being terms, they are amenable to being rewritten by the rules of a term rewrite system (TRS). Orienting the 1-laws in Fig. 3 from left to right as in Fig. 4

$\text{assoc}(\varrho, \varsigma, \zeta)$	$:$	$(\varrho \cdot \varsigma) \cdot \zeta$	$\Rightarrow$	$\varrho \cdot (\varsigma \cdot \zeta)$
$l\text{-unit}(\varrho)$	$:$	$1 \cdot \varrho$	$\Rightarrow$	ϱ
$r\text{-unit}(\varrho)$	$:$	$\varrho \cdot 1$	$\Rightarrow$	ϱ
invol-id	$:$	1^{-1}	$\Rightarrow$	1
$\text{anti-auto}(\varrho, \varsigma)$	$:$	$(\varrho \cdot \varsigma)^{-1}$	$\Rightarrow$	$\varsigma^{-1} \cdot \varrho^{-1}$
$\text{invol}(\varrho)$	$:$	$(\varrho^{-1})^{-1}$	$\Rightarrow$	ϱ

Fig. 4. Proofterm rewrite system, obtained by orienting 1-laws of Fig. 3

renders them not just term rewrite rules, but proofterm rewrite rules; they respect sources and targets, applying them rewrites proofterms into proofterms. In fact, for each item in Ex. 5 the sub-system of applicable rules is complete, terminating and confluent, easily proven⁶ by term rewrite tools [34, App. A].

Example 6. *Taking proofterms in $\Rightarrow$ -normal form as steps, with each operation defined by applying the corresponding symbol followed by $\Rightarrow$ -normalisation, gives rise to the following free 1-algebras corresponding to those of Ex. 5:*

(i) the rewrite carrier $\leftrightarrow$ of the free 1-involutoid $\langle \leftrightarrow, {}^{-1} \rangle$ has steps in $\leftarrow \cup \rightarrow$, where $\leftarrow$ comprises backward steps

⁴After that steps are 1-cells [8]; 1-algebras are *typed structures* in [33].

⁵For want of a better name.

⁶See https://termination-portal.org/wiki/Termination_Competition, <http://cl-informatik.uibk.ac.at/software/cocoweb/> for termination, confluence tools.

of shape ϕ^{-1} for ϕ a forward step in $\rightarrow$. The rewrite relations of $\leftarrow$ and $\leftrightarrow$ are the converse and symmetric closure of that of $\rightarrow$;

- (ii) the rewrite carrier $\rightarrow$ of the free 1-monoid $\langle \rightarrow, 1, \cdot \rangle$ (the path category [35]) has as elements right-branching trees of compositions of $\rightarrow$ -steps, (finite) reductions [14], [4, Def. 8.2.10]. The rewrite relation of $\rightarrow$ is the reflexive-transitive closure of that of $\rightarrow$;
- (iii) the rewrite carrier $\xleftrightarrow{*}$ of the free 1-involutive 1-monoid $\langle \xleftrightarrow{*}, 1, \cdot, {}^{-1} \rangle$ can be seen as having finite sequences of forward and backward $\rightarrow$ -steps, conversions [14] also known as zigzags. The rewrite relation of $\xleftrightarrow{*}$, convertibility, is the equivalence closure of $\rightarrow$'s rewrite relation.

Reductions / expansions arise as special conversions namely those only comprising forward / backward steps. A conversion is a peak if of shape $\leftarrow \cdot \rightarrow$ and a valley if of shape $\rightarrow \cdot \leftarrow$ [14].⁷

We recapitulate potentially infinite reductions.

Definition 2. Co-inductively define a reduction from a to be either the empty reduction ε_a from a or a pair (ϕ, ϱ) of a step $\phi : a \rightarrow b$ and a reduction ϱ from b . If a reduction is finite, then it is to some object, namely the target of its last step (or in case of the empty reduction from a just a).

As such reductions from a to b are in 1-1 correspondence with the finite reductions having a as source and b as target above, we conflate both. An object a is *terminating* (SN) if there are only finite reductions from a , and *normalising* (WN) if there is a reduction from it to normal form. Both notions extend objectwise to rewrite systems.

Remark 2. Modelling conversions via the free 1-involutive 1-monoid is just right. Though they could also be modelled via the 1-algebra $\langle \leftrightarrow, 1, \cdot \rangle$ of $\leftrightarrow$ -reductions or via the free 1-group (groupoid) $\langle \xleftrightarrow{*}, 1, \cdot, {}^{-1} \rangle$ obtained by normalising witnesses under the $\Rightarrow$ -rules of both Figs. 4 and 5, the former would be

$$\begin{array}{lll} l\text{-inv}(\varrho) & : & \varrho^{-1} \cdot \varrho \Rightarrow 1 \\ r\text{-inv}(\varrho) & : & \varrho \cdot \varrho^{-1} \Rightarrow 1 \\ l\text{-inv-}x(\varrho, \varsigma) & : & \varrho^{-1} \cdot (\varrho \cdot \varsigma) \Rightarrow \varsigma \\ r\text{-inv-}x(\varrho, \varsigma) & : & \varrho \cdot (\varrho^{-1} \cdot \varsigma) \Rightarrow \varsigma \end{array}$$

Fig. 5. Proofterm rewrite system extending Fig. 4 for 1-groups

too weak by missing out on that conversions can be reversed, whereas the latter would be too strong by identifying too much (a step preceded by its reverse is a peak, not nothing). For that reason, though 1-groups are of obvious interest [12], [36], we leave them for later, only remarking on them.

Each of the complete (sub-)TRSs for normalising proofterms above was obtained simply by orienting the 1-laws in Fig. 3. As for the group laws [7], this fails for 1-groups: The (derivable) rules $l\text{-inv-}x$ and $r\text{-inv-}x$ need to be adjoined [37], [38] to turn

⁷Reductions, expansions, peaks and valleys are referred to as positive, negative, negative-positive respectively positive-negative paths in [12]. Peaks are also called spans or branchings, and valleys co-spans or rewrite proofs.

$\Rightarrow$ into a complete proofterm rewrite system, by a process that one could call Knuth-Bendix 1-completion.

III. CONFLUENCE

The field of rewriting provides methods to establish properties of interest for the *reductions* and *conversions* of a rewrite system $\rightarrow$ via properties of its *steps* [1]–[4], [16]. Our property of interest will be *confluence*, the diamond property of (finite) reductions $\rightarrow$. It plays a pivotal rôle in various areas, e.g.:

Consistency of the equational theory of the λ -calculus, that not all equations are derivable or (Sec. I) that there are *some* λ -terms that are not $\xleftrightarrow{*}_\beta$ -related, follows from that there are distinct $\rightarrow_\beta$ -normal forms and confluence of $\rightarrow_\beta$ as shown in [14] (cf. Ex. 4), or rather the equivalent [8] Church-Rosser property (CR) was shown: $M \xleftrightarrow{*} N$ implies $M \rightarrow \cdot \leftarrow N$.

A *decision procedure* for an equational theory is then obtained when existence of *rewrite proofs* [10, p. 243], of valleys $t \rightarrow \cdot \leftarrow s$, is decidable. For instance, if $\rightarrow$ is moreover normalising or terminating, then one may proceed by normalising t, s to $\hat{t}, \hat{s}$ and checking $t \rightarrow \hat{t} \stackrel{?}{=} \hat{s} \leftarrow s$ [7].

Functional programming languages such as Haskell and Agda typically enforce / depend on confluence to guarantee programs are (partial) functions, results are unique [39], [40].

We are interested in *diagrammatic* confluence techniques, establishing confluence as a consequence of having some *diagrams* at one's disposal and *tiling* with them. (The diagrammatic perspective originates with [8, Section 6]).

Definition 3. A 2-rewrite system (cf. 2-polygraph [13], [41], [42]) is a rewrite system having the $\rightarrow$ -conversions as objects and a collection $\mathcal{D}$ of diagrams $D, E, F, \dots$ equipped with maps lhs, rhs to $\rightarrow$ -conversions such that $\text{lhs}(D)$ and $\text{rhs}(D)$ are parallel to each other. The steps of the 2-rewrite system are 2-multisteps, conversions over $(S \cup \mathcal{D})$ with their source / target map the (1-involutive 1-monoid) morphism induced by lhs / rhs on $\mathcal{D}$, where D^{-1} has $\text{lhs}(D)^{-1}$ as source and $\text{rhs}(D)^{-1}$ as target. A 2-multistep is a 2-step if exactly 1 diagram of $\mathcal{D}$ occurs in it, and is positive if no D^{-1} occurs.

We use double-shafted arrows to denote 2-rewrite systems, uppercase letters $\Phi, \Psi, X, \dots$ from the end of the alphabet for 2-steps, and other such $P, \Sigma, Z, \dots$ for 2-conversions. Diagrams can be viewed as 2-rules on conversions that are *closed under contexts* (of conversions; whiskering) to yield 2-steps. Such transformations pervade the rewriting literature, with *standardisation* (transforming reductions into more standard ones) and *tiling* (transforming conversions into valleys) being prominent [8], [9], [11], [14], [17], [43], [44] and tiling has recently found applications elsewhere, e.g. in Garside theory under the name of *reversing* [23, Ch. 1], [12]. A diamond $\varrho \diamond \varsigma$ where $\varrho, \varsigma, \zeta, \xi$ are reductions gives rise to *two* diagrams: a vertical *tiling* diagram $\varrho^{-1} \cdot \varsigma \Downarrow \zeta \cdot \xi^{-1}$, cf. [45, Def. 2.4.13] and a horizontal *filling* diagram $\varrho \cdot \varsigma \Rightarrow \zeta \cdot \xi$, see Fig. 17 (left) for a *local* diagram / diamond, where ϱ, ς are single $\rightarrow$ -steps.

Example 7. We model filling a box with calissons by means of a 2-rewrite system $\Rightarrow$ on $\rightarrow$ -reductions, where both $\rightarrow$ -steps and $\Rightarrow$ -steps may come in three colours, as depicted in

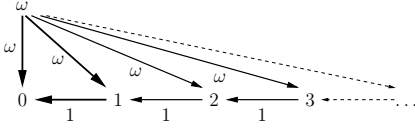


Fig. 7. Measures for the example in the proof of Thm. 3

$b \rightarrow_{n'} d \leftarrow_{m'} c$ with $n + m' \leq m + n'$. Peak random descent (PR) holds if for any reduction to normal form $a \rightarrow_m \hat{a}$ and reduction from a , the latter is finite of shape $a \rightarrow_n c$ and there is a reduction $c \rightarrow_k \hat{a}$ such that $m = n + k$.

OWCR puts no constraints on objects not convertible to normal form, e.g. $b \leftarrow b \leftarrow a \rightarrow c \rightarrow c$ is OWCR (so OWCR $\not\Rightarrow$ WCR), but objects that are convertible to normal form are complete. This is in fact a characterisation.

Theorem 3. $\rightarrow$ is PR for some measure iff $\rightarrow$ is uniformly complete, i.e. if all objects convertible to normal form are complete (SN & CR).

Proof. The idea, as illustrated for the example below in Fig. 7, is that in a directed acyclic graph (the SN & CR accessible sub-graph) ordinal-weights can be assigned to edges such that all paths from one node to another in it have the same measure, by topological sorting bottom-up, starting from the normal forms. E.g. consider the rewrite system having steps $a \rightarrow b_i$ and $b_{i+1} \rightarrow b_i$ for $i \in \mathbb{N}$. To deal with the system being non-FB a must get measure the supremum of $\{i + 1 \mid i \in \mathbb{N}\}$, i.e. ω , which thus is the measure of each step from it too. (Reductions $a \rightarrow b_0$ all have measure ω as $\omega \dot{+} n = \omega$; Ex. 9.) $\square$

We dubbed the property *uniform completeness* by analogy with the extant notion of *uniform termination*, expressing that all objects convertible to normal form are *terminating*, cf. [52].

Corollary 1. A rewrite system $\rightarrow$ is complete (SN & CR) iff it is normalising (WN) and ordered locally confluent (OWCR).

Remark 3. For systems that are confluent such as pure type systems (PTSs) this gives a new perspective on proving them SN. E.g. the Barendregt–Geuvers–Klop conjecture stating [53, Conj. 14.7.1] WN entails SN for PTSs is equivalent to that PTSs are OWCR, and to that PTSs are uniformly complete.

As first example we present [50, Ex. 7], as it is illustrative and basic yet interesting. It shows bubblesort and insertion sort to take *exactly* the same number of steps to sort a given list, where $\rightarrow$ swaps (adjacent) out-of-order elements. (So, e.g., termination of insertion sort follows from that of bubblesort).

Example 10. By Thm. 2 it suffices to show swapping to have OWCR, which follows by a critical peak analysis having $[c, b, a]$ as only interesting case, resolved by that the two legs $[c, b, a] \rightarrow [b, c, a] \rightarrow [b, a, c] \rightarrow [a, b, c]$ and $[c, b, a] \rightarrow [c, a, b] \rightarrow [a, c, b] \rightarrow [a, b, c]$ of its local confluence diamond have the same length, 3.

We solve the problem of the calissons in Exs. 1,7 by showing all spectra of fillings of a given box to be the same.

Example 11. Consider triples of natural numbers, equipped with pointwise zero, addition, and comparison. This gives a reduction measure when mapping $\Rightarrow, \Rightarrow, \Rightarrow$ -steps to respectively $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Given a grid, filling has OWCR as shown by a critical peak analysis having $\rightarrow \rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \rightarrow \Rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \rightarrow$ as only interesting case, resolved by $\rightarrow \rightarrow \rightarrow \Rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \rightarrow$. By Thm. 2 we conclude that $\Rightarrow$ has random descent, since both legs of the diamond have $(1, 1, 1)$ as measure / spectrum, entailing that all fillings of a given box have the same spectrum, solving the problem of the calissons (Exs. 1,7). Filling terminates as it is a ‘colouring’ of sorting in Ex. 10.

Example 12. OWCR holds for braids as one easily checks. For instance, though the peak of braids steps $\varepsilon \rightarrow \sigma_1$ and $\varepsilon \rightarrow \sigma_2$ cannot be completed into a diamond of steps (Ex. 4), 2 steps do suffice: $\sigma_1 \rightarrow \sigma_1 \sigma_2 \rightarrow \sigma_1 \sigma_2 \sigma_1 \equiv \sigma_2 \sigma_1 \sigma_2 \leftarrow \sigma_2 \sigma_1 \leftarrow \sigma_2$. Thm. 2 is of no immediate use, since no braid normalises; there are no normal forms. Still, it has sub-systems (e.g. simple braids Ex. 3) that are terminating where RD is useful.

The fourth example shows a powerful technique to reduce termination (SN) to normalisation (WN), originating with the λ -calculus [9], [54], to be a trivial consequence of random descent. The idea is that WN gives *some* upperbound and increasingness makes that rewriting squeezes objects below it. Recall [4, Def. 1.1.15(ii)] that a function $|\cdot| : A \rightarrow \mathbb{N}$ on the objects of a rewrite system $\rightarrow$ is *increasing* (Inc) if $a \rightarrow b \Rightarrow |a| < |b|$. Lifting $|\cdot|$ to a function on steps by $|\phi : a \rightarrow b| := |b| - |a|$, the former is increasing iff the latter is a reduction measure since $|a| < |b|$ iff $|b| - |a| \neq 0$. We refer to it as the *dif*-measure (induced by the function on objects). For a dif-measure $|\cdot|$ all reductions from a to b are commensurate, have the same measure $|b| - |a|$.

Example 13. WCR & Inc & WN $\Rightarrow$ SN [9, Cor. 5.19(ii)], [4, Thm. 1.2.3(iii)]. Local confluence (WCR) and increasingness (Inc) entail ordered local confluence (OWCR) as legs of local confluence diamonds are commensurate for the dif-measure. Normalisation (WN) and Thm. 2 entail termination (SN).

In the fifth example we reprove that in the λI -calculus any λ -term that is normalising is terminating [14, Thm. 2], where the λI -calculus is the sub-system $\rightarrow_{\beta I}$ of $\rightarrow_{\beta}$ obtained by restricting its objects by that for every abstraction $\text{abs}(x.M)$, x must have some free occurrence in M [17, Def. 2.2.2]. The idea is that though $\rightarrow_{\beta I}$ -steps need not increase the size of terms, they can be *made* so by inserting *inert* symbols.

Example 14. Adjoin $c : \text{term} \rightarrow \text{term}$ to the signature Λ and replace the rule-symbol β in B by βc^n , for all n , with rules:

$$\beta c^n : MN.\text{app}(c^n(\text{abs}(x.M(x))), N) \rightarrow MN.c^{n+1}(M(N))$$

This is an orthogonal PRS hence its rewrite system $\rightarrow_{\beta c}$ is confluent (CR) [19, Thm. 6.11], and so is its (reduction-closed) sub-system $\rightarrow_{\beta c I}$ (defined as above). One shows successively:

- (i) $\rightarrow_{\beta cI}$ is increasing (Inc) when measuring terms by the number of c -symbols, so has random descent by Ex. 13;
- (ii) The map (on both terms and steps) $c \mapsto x.x$ forgetting c gives a morphism h from $\rightarrow_{\beta c}$ to $\rightarrow_{\beta}$ and $\rightarrow_{\beta cI}$ to $\rightarrow_{\beta I}$;
- (iii) if $\hat{t}$ is in $\rightarrow_{\beta cI}$ and ϕ is a step from $h(t)$ in $\rightarrow_{\beta I}$, there exists a unique $\hat{\phi}$ from $\hat{t}$ in $\rightarrow_{\beta cI}$ such that $h(\hat{\phi}) = \phi$.

Then to show a normalising term M of $\rightarrow_{\beta I}$ to be terminating, consider a $\rightarrow_{\beta I}$ -reduction ϱ from M to normal form N . If there were an infinite $\rightarrow_{\beta I}$ -reduction from M , then by (iii) also an infinite $\rightarrow_{\beta cI}$ -reduction from M (as $M = h(M)$). But also ϱ lifts to a $\rightarrow_{\beta cI}$ -reduction ϱ' from M and by (ii) to a term that is in $\rightarrow_{\beta cI}$ -normal form, contradicting (i).

Remark 4. The example and proofs (by inductions on steps) are generic and simple; they carry over to any orthogonal non-erasing term rewrite system, cf. [9, Sec. II.5] [55, Cor. 7].

In the final example the idea is adapted to infer termination (SN) from normalisation (WN) of the simply-typed λ -calculus [56]. Let $\rightarrow_{\beta}$ denote $\rightarrow_{\beta}$ restricted to simply-typed terms, the formalisation of which is largely immaterial here but cf. [57], [58]. The main obstacle to increasingness (Inc) is that contracting a redex $\beta(x.M, N)$ yields M if x is not free in the body M , thereby *erasing* the argument N . The idea to overcome this, as pioneered in [9], [59], cf. [54], [58], is to make β non-erasing by *memorising* the argument N .

Example 15. Adjoin a symbol $[\cdot, \cdot] : \text{term} \rightarrow \text{term} \rightarrow \text{term}$ for memory [9] to the signature Λ . Replace the rule-symbol β in B by $\beta[n]$, for all n , with rules:

$$\beta[n] : MN.\text{app}([\text{abs}(x.M(x)), \vec{P}], N) \rightarrow MN.[M(N), \vec{P}N]$$

where nested memory $[Q, \vec{P}^n]$ denotes $[[Q, \vec{P}^{n-1}], P_n]$ if $n > 1$ and Q if $n = 0$. This is an orthogonal PRS hence its rewrite system $\rightarrow_{\beta[]}$ is confluent (CR), and so is its (reduction-closed) simply-typed sub-system $\rightarrow_{\beta[]}$, where $[M, N]$ has type σ if M has type σ and N has type τ [58, Def. 3.3].

- (i) $\rightarrow_{\beta[]}$ is increasing (Inc), measuring terms by the number of $[\cdot, \cdot]$ -symbols, so has random descent by Ex. 13;
- (ii) The map $[\cdot, \cdot] \mapsto xy.x$ forgets the memory, gives a morphism h from $\rightarrow_{\beta[]}$ to $\rightarrow_{\beta}$ and $\rightarrow_{\beta[]}$ to $\rightarrow_{\beta}$;
- (iii) if $\hat{t}$ is in $\rightarrow_{\beta[]}$ and ϕ is a step from $h(t)$ in $\rightarrow_{\beta}$, there exists a unique $\hat{\phi}$ from $\hat{t}$ in $\rightarrow_{\beta[]}$ such that $h(\hat{\phi}) = \phi$.

Then if there were an infinite $\rightarrow_{\beta}$ -reduction from M , then by (iii) also an infinite $\rightarrow_{\beta[]}$ -reduction from M (as $M = h(M)$). But since $\rightarrow_{\beta[]}$ can be shown to be normalising in the same way as $\rightarrow_{\beta}$ (e.g. contracting an innermost redex of maximal type decreases the multiset of redex-types), this contradicts (i).

As for the previous example, the point is not the novelty of the result but the smoothness afforded by the combination of presenting a system as an (orthogonal) PRS, with random descent, allowing both for results about normalising *objects* (Ex. 14) and *systems* (Ex. 15), cf. Prop. 1 and Ex. 12 below.

B. Filling and tiling

We resume the study of filling and tiling, using Sec. III-A.

Example 16. We solve the problem of the calissons based on filling (Ex. 11), this time exemplifying proof orders. Using that filling is WCR (shown in Ex. 11) we show SN by the volume-proof order from which the spectrum can be read off. For a

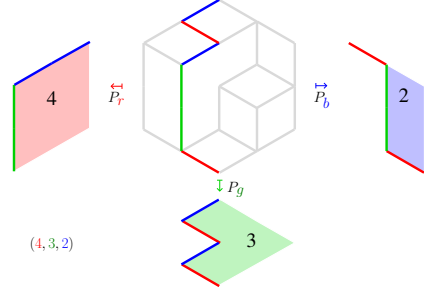


Fig. 8. Volume of P as areas of 3 projections P_r, P_g, P_b (forgetting colours)

partial filling ending in some path P (forgetting directions of steps), as visualised in Fig. 8, we let its volume be the triple (r, g, b) of area-measures (Ex. 8) of the three projections P_r, P_g and P_b of P . We only exemplify this for the path P given by depicted in Figure 8. Then P_r is having area $(2, 4, 2)$, P_g is with area $(2, 3, 2)$ and P_b is with area $(2, 2, 2)$. One verifies that for any filling step of a given colour (the 2nd component of) that colour is decremented, with the areas of the other colours unchanged. Since in the initial / final filling of the box all (2nd components of) areas are 4 / 0, we conclude the spectrum is $(4, 4, 4)$, independent of the filling.

We further investigate the properties of tiling $\Downarrow_{\mathcal{D}}$ with diagrams $\mathcal{D}$ for a rewrite system $\rightarrow$. Typically, diagrams in $\mathcal{D}$ are induced by local diamonds $\phi \Downarrow_{\zeta} \psi$ for $\rightarrow$ -steps ϕ, ψ and $\rightarrow$ -reductions ϱ, ς inducing tiling rules $\phi^{-1} \cdot \psi \Downarrow \varrho \cdot \varsigma^{-1}$, but *a priori* we do not restrict them. We allow them to have any of the shapes in Fig. 9, with the shapes top / bottom-left being

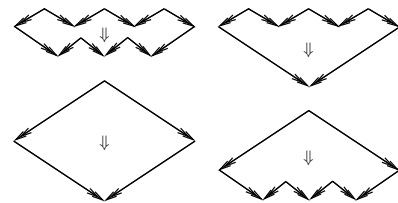


Fig. 9. Some common diagram shapes $\begin{smallmatrix} \text{lhs} \\ \text{rhs} \end{smallmatrix}$

the most general (conversion to conversion) / specific (peak to valley, i.e. a diamond). Throughout we use $\Downarrow$ to denote a 2-rewrite system having a collection $\mathcal{D}$ of diagrams (2-rules) for a rewrite system $\rightarrow := \langle \mathcal{O}, \mathcal{S}, \text{src}, \text{tgt} \rangle$.

We showed (Ex. 8) some known confluence results could be rephrased as termination-of-tiling results, and that in a sense any such can be rephrased that way (Thm. 1), for *appropriate* sets of diagrams $\mathcal{D}$. But what about *arbitrary* such?

Example 17 ([8, Fig. 3]). Let for $\phi : a \rightarrow a'$, $\phi' : a' \rightarrow a$, $\psi : a \rightarrow b$, $\chi : a' \rightarrow c$, the tiling diagrams in $\mathcal{D}$ be those of

the local diamonds $\psi \diamond_{\phi \cdot \psi}^{\phi}$ and $\phi' \diamond_{\phi \cdot \chi}^{\phi}$, their involutive versions $\phi \diamond_{\phi \cdot \psi}^{\psi}$ and $\chi \diamond_{\phi \cdot \chi}^{\phi'}$, and the trivial ones $\chi \diamond_{\epsilon}^{\chi}$ for each step χ . Then $\mathcal{D}$ is a full, deterministic, involutive collection of local diamonds, but tiling $\Downarrow$ does not terminate; as visualised in

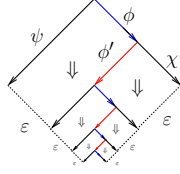


Fig. 10. Infinite descent of tiling in Ex. 17

Fig. 10 it leads to an infinite descent for $b \psi \leftarrow a \rightarrow_{\phi} a' \rightarrow_{\chi} c$.

Typically, there may be several tiling rules applicable to a given conversion. For example, in the conversion $\varrho : b \psi \leftarrow a \rightarrow_{\phi} a' \phi \leftarrow a \rightarrow_{\psi} b$ the two local peaks $b \psi \leftarrow a \rightarrow_{\phi} a'$ and $a' \phi \leftarrow a \rightarrow_{\psi} b$ occur, allowing for two different tiling steps. The first observation is that strategy is irrelevant when tiling, if the only choice is between such occurrences not between what tiling rule to apply then. If some tiling descends into a valley, then tiling randomly descends into it [8, p. 226].

Proposition 1. *If $\mathcal{D}$ is deterministic, then positive (Def. 3) tiling has RD, and tiling has RD if $\mathcal{D}$ is also involutive.*

We present a proof of Prop. 1 as it is simple, showcasing Thm. 2. The proposition is below also in its contrapositive form, to conclude that *no* tiling of a given conversion terminates because there is *some* infinite tiling of it.

Proof. We show OWCR for positive tiling by analysing the critical peaks between tiling steps $\Phi : \varrho \Downarrow \varsigma$ and $\Psi : \varrho \Downarrow \zeta$. This is trivial: If Φ, Ψ apply to the same occurrence of a local peak in ϱ , then by $\mathcal{D}$ being deterministic and $\Downarrow$ being positive they are *the same step*, so ς is the common reduct reached in 0 $\Downarrow$ -steps in both cases. Otherwise, by the shape $\leftarrow \cdot \rightarrow$ of local peaks, they apply to *non-overlapping* occurrences, hence we get $\Downarrow$ -steps $\Psi' : \varsigma \Downarrow \xi$ and $\Phi' : \zeta \Downarrow \xi$ for some ξ reached in 1 $\Downarrow$ -step either way. For tiling in the presence of an involutive $\mathcal{D}$ the reasoning is the same, but now making use of that if D and E^{-1} could be applied to the same occurrence of a local peak so $\text{lhs}(D) = \text{lhs}(E)^{-1}$, then by $\mathcal{D}$ being deterministic and involutive also $D = E^{-1}$ and $\text{rhs}(D) = \text{rhs}(E)^{-1}$. $\square$

Example 18. *If the sets of diagrams in Ex. 8 are not just full as assumed there but also deterministic and involutive, then tiling $\Downarrow$ is SN & CR by Prop. 1. The same holds for OWCR and local Dyck diagrams [51, Def. 16] for random descent, but for positive tiling as such diagrams are a priori not involutive.*

Under the conditions of the proposition, $\multimap$ -residuation can be suitably *lifted* from having diamonds for peaks of $\rightarrow$ -steps only, to all peaks of $\multimap$ -steps ($\rightarrow$ -reductions), by tiling.

Definition 5. *Let $\mathcal{D}$ be a deterministic collection of tiling diagrams obtained from local diamonds $\phi \diamond_{\psi \cdot \phi}^{\psi}$ for residuations*

$\backslash, /$ mapping $\rightarrow$ -steps to $\multimap$ -reductions⁹ Define $\varrho \backslash \varsigma := \zeta$ and $\varrho / \varsigma := \xi$ if $\varrho^{-1} \cdot \varsigma \Downarrow \zeta \cdot \xi^{-1}$ for some valley, and let both $\varrho \backslash \varsigma$ and ϱ / ς be undefined otherwise.

Though the valley in Def. 5 need not exist, e.g. in case tiling does not terminate (Ex. 17) or gets stuck in a conversion that is not yet a valley because $\mathcal{D}$ is not full, Prop. 1 guarantees that if it does it is unique, so $\backslash, /$ are indeed (partial) functions. Also, involutive residuations (cf. Lem. 1) lift: if $\psi \backslash \phi = \phi / \psi$ for all $\rightarrow$ -peaks ϕ, ψ , then $\varsigma \backslash \varrho \simeq \varrho / \varsigma$ for all $\multimap$ -peaks ϱ, ς , where $\simeq$ is Kleene-equality expressing that either both sides denote and are equal, or that neither side denotes.

Example 19. *Let $\mathcal{D}$ be as in Def. 5 and moreover full. Then tiling in Ex. 8 (the diamond property, Newman's Lemma, decreasing diagrams) and in Thm. 1, induces a (total) residuation for reduction $\multimap$ (preserving being involutive).*

Proposition 2. *Under the conditions of Def. 5, for all $\multimap$ -reductions $\varrho, \varsigma, \zeta$ (matching up appropriately):*

$$\begin{aligned} \varrho / \epsilon &= \varrho & \varrho / (\varsigma \cdot \zeta) &\simeq (\varrho / \varsigma) / \zeta \\ \epsilon \backslash \varrho &= \varrho & (\varrho \cdot \varsigma) \backslash \zeta &\simeq \varsigma \backslash (\varrho \backslash \zeta) \\ \varrho \backslash \epsilon &= \epsilon & \varrho \backslash (\varsigma \cdot \zeta) &\simeq (\varrho \backslash \varsigma) \cdot ((\varrho / \varsigma) \backslash \zeta) \\ \epsilon / \varrho &= \epsilon & (\varrho \cdot \varsigma) / \zeta &\simeq (\varrho / \zeta) \cdot (\varsigma / (\varrho \backslash \zeta)) \end{aligned}$$

Proof. By tiling as Fig. 11, using Prop. 1 to factorise tiling, e.g., tiling for $\varrho / (\varsigma \cdot \zeta)$ through tiling for ϱ / ς . $\square$

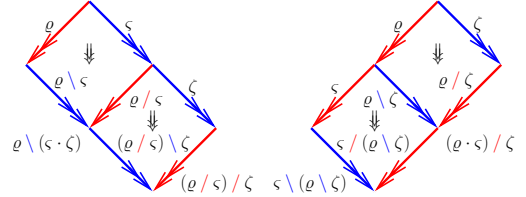


Fig. 11. Composition laws for residuation by tiling

Prop. 2 justifies *defining* lifting residuation from steps to reductions by *recursion*, having as base cases peaks of steps or empty reductions, and as (mutually) recursive clauses:

$$\begin{aligned} (\phi \cdot \varrho) / (\psi \cdot \varsigma) &:= ((\phi / \psi) \cdot (\varrho / (\phi \backslash \psi))) / \varsigma \\ (\phi \cdot \varrho) \backslash (\psi \cdot \varsigma) &:= \varrho \backslash ((\phi \backslash \psi) \cdot ((\phi / \psi) \backslash \varsigma)) \\ (\phi \cdot \varrho) / (\psi \cdot \varsigma) &:= ((\phi / \psi) \cdot (\varrho / (\psi / \phi))) / \varsigma \end{aligned}$$

where ϕ, ψ range over steps and ϱ, ς over reductions, with the third clause the single clause (other than for empty reductions) needed for an involutive residuation $/$, justified by $(\phi \cdot \varrho) / (\psi \cdot \varsigma) \simeq ((\phi \cdot \varrho) / \psi) / \varsigma \simeq ((\phi / \psi) \cdot (\varrho / (\psi / \phi))) / \varsigma$. Vice versa, since the laws give rise to a (this) tiling strategy, the recursive definition is the least (w.r.t. function as sets of pairs) extension of $/$ satisfying them; cf. [12, II Lem. 4.32].

IV. ORTHOGONALITY

Following [4, Ch. 8], we view *orthogonality* of rewrite systems as a strengthening of confluence by axiomatising residuation, in such a way that orthogonality relates to confluence as

⁹We assume neither that $\backslash, /$ be involutive, nor $\phi \backslash \phi = \epsilon$ or $\phi / \phi = \epsilon$.

least upperbounds to upperbounds [8], least common multiples to common multiples [12], and pushouts to co-cones [25]. Similarly to how diagrammatic confluence was based on 2-D tiling, we will base orthogonality on 3-D bricklaying.

Example 20. For the 3rd rewriting approach to the problem of the calissons we change its modelling. We view filled boxes now from a 3-D perspective, as beds on which bricks are laid, and in keeping with that orient the *green* arrows in the grid $\rightarrow$ (Fig. 6) upward. The bricklaying rewrite system $\Rightarrow$ has as objects filled boxes, and the rule $\Rightarrow$, allowing to rearrange triples of calissons forming hexagons, one can think of as bricklaying, visualised in Fig. 12¹ By the spectrum being invariant under

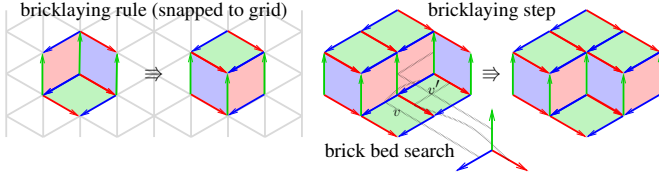


Fig. 12. 3-D calissons: bricklaying $\Rightarrow$ -rule (left) and $\Rightarrow$ -step (right)

$\Rightarrow$, the problem has a positive answer if all filled boxes have the same normal form. Observe that a filled box B is in $\Rightarrow$ -normal form iff it has no 3-peaks, triples ϕ, ψ, χ of $\rightarrow$ -steps, co-initial in the grid. This follows by ‘searching the bed’ (Fig. 12 right): if that 3-peak could not be rewritten, its source v has some in-edge, say from v' , which then (by B being hexagonal) is the source of another 3-peak. Repeating this we must end up with a 3-peak that can be $\Rightarrow$ -rewritten (by finiteness of B); We conclude by noting that if there is no 3-peak, then B is filled as one big brick, with all calissons of the same colour grouped together (cf. the rhs of $\Rightarrow$), that only depends on the box. This holds since each 2-peak, say $\leftarrow \cdot \rightarrow$, must then belong to a calisson of the other colour (here *green*; if not, its source would have an in-edge from a 3-peak).

The lhs and rhs of the bricklaying $\Rightarrow$ -rule in Ex. 20 combine into a brick, analogous to how the lhs (peak) and rhs (valley) of a tiling $\Rightarrow$ -rule combine into a diamond. When does local tiling gives rise to bricklaying rules? It typically does not.

Example 21. For the 3-peak from $\text{app}(\omega, \text{app}(\omega, \text{app}(\mathbf{I}, y)))$ (cf. Ex. 4), tiling the three peaks, followed by tiling the resulting three ($\rightarrow$ -)peaks succeeds, but does not give a brick; the resulting reductions are not (pairwise) the same (see [4, Fig. 8.53] for the analogous TRS example).

Similarly, for the 3-peak of braid steps from ε to $\sigma_1, \sigma_2, \sigma_3$ tiling (of first the three $\rightarrow$ -peaks and then the three $\rightarrow$ -peaks), succeeds but does not yield a brick (cf. e.g. [24, Exc. 61.2]).

To have diamonds that do give rise to bricks, whose reductions match up, is expressed by the *cube law* [60, Lem. 2.1.1], [17, Fig. 12.3] [23, Fig. 2.1], see Fig. 13 right. We rephrase its axiomatisation [4], [25], [27], [29] in terms of 1-algebras.

Definition 6. A residual 1-algebra (1-ra) is a 1-algebra $\langle \rightarrow, 1, / \rangle$ such that (1)–(4) hold; Fig. 13. A residual 1-algebra

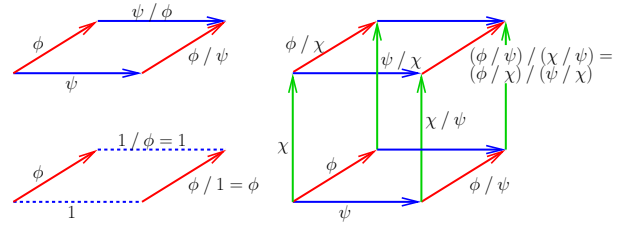


Fig. 13. 1-ra laws with cube: $(\phi / \psi) / (\chi / \psi) = (\phi / \chi) / (\psi / \chi)$

with composition (1-rac) extends this to $\langle \rightarrow, 1, /, \cdot \rangle$ such that also (5)–(7) hold; Fig. 11.

$$\phi / 1 = \phi \quad (1)$$

$$\phi / \phi = 1 \quad (2)$$

$$1 / \phi = 1 \quad (3)$$

$$(\phi / \psi) / (\chi / \psi) = (\phi / \chi) / (\psi / \chi) \quad (4)$$

$$\chi / (\phi \cdot \psi) = (\chi / \phi) / \psi \quad (5)$$

$$(\phi \cdot \psi) / \chi = (\phi / \chi) \cdot (\psi / (\chi / \phi)) \quad (6)$$

$$1 \cdot 1 = 1 \quad (7)$$

Example 22. Multisteps of both the λ -calculus and braids with their residuations (Ex. 4) constitute 1-ras as can be checked by inductions (The auxiliary $\sqcup$ operation on β -multisteps of Footnote 3 is helpful here too; it allows to decompose a cube into 2 prisms [30].)

Based on this example, we say a structured rewrite system is *orthogonal* if its multisteps constitute a 1-ra, where *multisteps* capture the idea of performing *multiple* (single) steps in *parallel* (to be contrasted with *many-steps* that may perform many in *sequence*). For instance, orthogonal PRSs in the syntactic sense of [19] are orthogonal in this sense.

Example 23. The construction of integers as pairs of natural numbers that in turn are constructed as sequences of bits, generalises [12] to the construction of steps of a 1-group (groupoid of fractions) of valleys of steps of a 1-rac, that in turn are reductions of a 1-ra. A prototypical instance is the construction in [12] of the braid group from the 1-rac of positive braids, and those in turn (cf. Sec. IV-A) from the 1-ra of simple braids (i.e. the multistep 1-ra in Ex. 22). For valleys to compose, the construction relies on confluence.

Example 24. ras, 1-ras with a singleton-object rewrite system, abound. For instance, bits / natural numbers with cut-off subtraction, (multi)sets with (multi)set difference, the positive naturals with cut-off division (cancelling factors that do occur). Also conditional probabilities can be understood through residuation. Taking the event $A \mid B$ as notation for a step from B to $A \cap B$, leaving B implicit if it's the sample space Ω (Fig. 14 left) makes laws (1)–(4) hold (by being a semilattice). Bayes' Theorem $P(A) \cdot P(B \mid A) = P(A \cap B) = P(B \cap A) = P(B) \cdot P(A \mid B)$ is then nothing but a map P from it to fractions (of cardinalities; Fig. 14 right). This allows to

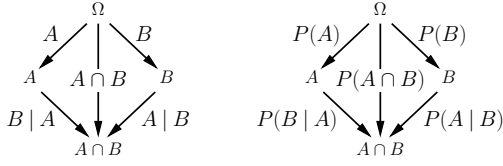


Fig. 14. Bayes' Theorem as an ra-morphism P from events to fractions

separate probabilities from events; $P((A | B) | (C | B)) = P((A | C) | (B | C))$ makes sense and is true.

In a 1-ra(c) there's a *natural* order on co-initial steps given by $\phi \preceq \psi := \phi / \psi = 1$. That need not be a partial order, but if not then quotienting out $\preceq \cap \succeq$ yields a 1-ra(c) again where it is [4, Lem. 8.7.25(iii) and 8.7.41(ii)]. By having a residuation $/$, the rewrite system $\rightarrow$ of a 1-ra has the diamond property (Fig. 2). In a 1-rac the diamonds constructed are universal, in the sense mentioned at the start of the section.

Theorem 4 ([27], [29], [44]). $\langle \rightarrow, 1, \cdot \rangle$ is a 1-monoid (a category) that is left-cancellative (each χ is epi: for all ϕ, ψ , if $\chi \cdot \phi = \chi \cdot \psi$ then $\phi = \psi$), gaunt (isomorphisms are 1) and has pushouts (in the standard categorical sense) iff $\langle \rightarrow, 1, /, \cdot \rangle$ is a 1-rac whose natural order is a partial order, where $\phi / \psi := \phi'$ for every peak ϕ, ψ and its pushout valley ψ', ϕ' .

Remark 5. Beware that the pushout of the two steps $\text{app}(\mathbf{I}, \text{app}(\mathbf{I}, y)) \rightarrow_{\beta} \text{app}(\mathbf{I}, y)$ in Ex. 2, is not the empty valley ε despite both steps having the same target $\text{app}(\mathbf{I}, y)$. The pushout comprises twice the step $\text{app}(\mathbf{I}, y) \rightarrow_{\beta} y$. The reason is that the same target $\text{app}(\mathbf{I}, y)$ was reached in the peak by syntactic accident [60, p. 34], but that the residuation $/$ constructs common reducts reached by the same work.

A. Bricklaying

Despite $\rightarrow$ -steps of a 1-ra not being closed under composition, they can be used to (freely) generate a 1-rac on $\rightarrow$ [27], [29], [61] with the intuition that $\rightarrow$ -cubes compose (in 3-D) to form $\rightarrow$ -bricks, i.e. $\rightarrow$ -cubes: Tiling with the diamonds for the residuation on $\rightarrow$ of a 1-ra gives rise to a residuation (Ex. 19) on $\rightarrow$ that trivially satisfies (1)–(3) and (7), and satisfies (5) and (6) by Prop. 2. Also the cube law (4) can be shown to hold by tiling [4, Lem. 8.4.47] [62] [12]. Here we show it as a consequence of tiling with *decreasing local bricks*.

Definition 7. Let $\mathcal{D}$ be a full, deterministic, involutive collection of tiling diagrams obtained from local diamonds $\phi \diamond_{\varrho}^{\psi} \varsigma$ of $\rightarrow$ -steps ϕ, ψ and $\rightarrow$ -reductions ϱ, ς . We say the 2-rewrite system $\Downarrow$ is 3-confluent, if for every 3-peak of $\rightarrow$ -reductions tiling its 6 peaks yields (the 6 faces of) a $\rightarrow$ -brick, locally 3-confluent if this holds for 3-peaks of $\rightarrow$ -steps, and decreasing if all diagrams in $\mathcal{D}$ are [4], [45], for some well-founded order $\preceq$ on steps. A brick is decreasing if its 6 faces are.

By Ex. 19, if the *local* diamonds are decreasing, then tiling the 6 peaks of any 3-peak results in 6 faces that all are decreasing diagrams, but without requiring 3-confluence these need not constitute a brick.

Theorem 5. If $\Downarrow$ is locally 3-confluent and decreasing, then it is 3-confluent, yielding decreasing bricks.

Proof. By induction on the multiset sum of the *lexicographic maximum measures* [45] of the $3 \rightarrow$ -reductions in a 3-peak, well-foundedly ordered by the multiset extension $\preceq_{mul}$ of $\preceq$, with the induction steps visualised in Fig. 15. $\square$

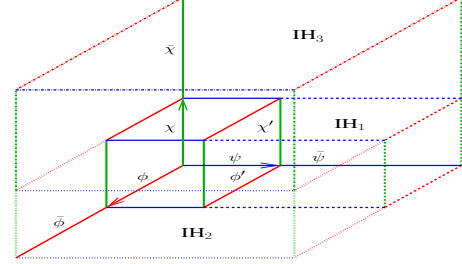


Fig. 15. 3 induction steps in the proof of Thm. 5

Since $\rightarrow$ -cubes are trivially decreasing, cube (3) holds for $\rightarrow$ in the generated 1-rac when taking as diamonds $\phi \diamond_{\varrho}^{\psi} \varsigma$.

Example 25. For every countable confluent $\rightarrow$, there exist a 1-rac on $\rightarrow$. Setting $\mathcal{D}$ to the local diamonds in Thm. 1, one checks that all local diamonds are decreasing and that local 3-confluence holds, which is sufficient by the above.

B. Undercutting

The methods developed above show, for reductions, that every peak could be completed into a diamond by a valley that is least. Here we are interested in determining this for individual peaks, even for rewrite systems that need not be confluent. The idea is analogous to *bounded-complete* partial orders, and to categories having *relative* pushouts [63]. Concretely, the *undercutting* criterion put forward below is inspired by [12, Proposition 4.16(4.18)]. A prototypical example is hole-filling of contexts; the holes in $f(g(\square), \square)$ and $f(\square, h(\square))$ can be filled in a most general way to $f(g(\square), h(\square))$, but in $f(\square)$ and $g(\square)$ they cannot (confluence fails as both are reachable from $\square$); see Fig. 16. Throughout, we assume to have a deterministic

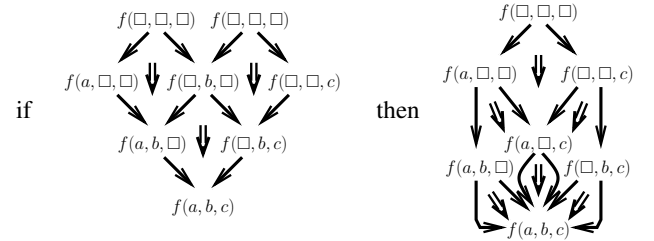


Fig. 16. Undercutting / semi-lattice for hole-filling

collection $\mathcal{D}$ of local diamonds $\phi \diamond_{\varrho}^{\psi} \varsigma$ for $\rightarrow$ -steps ϕ, ψ and $\rightarrow$ -reduction ϱ, ς , inducing the 2-rewrite systems of *filling* $\Rightarrow$ on $\rightarrow$ -reductions with rules $\phi \cdot \varrho \Rightarrow \psi \cdot \varsigma$, and *tiling* $\Downarrow$ on $\rightarrow$ -conversions with rules $\phi^{-1} \cdot \psi \Rightarrow \varrho \cdot \varsigma^{-1}$.

Definition 8. $\mathcal{D}$ is undercutting (UC) if it has trivial $\phi \diamond \phi$ and if $P : \phi^{-1} \cdot \chi \cdot \chi^{-1} \cdot \psi \Downarrow \zeta \cdot \xi^{-1}$ then $\Sigma : \zeta^{-1} \cdot \phi^{-1} \cdot \psi \cdot \xi \Downarrow \varepsilon$ with $\Sigma \sqsubseteq P$. $\mathcal{D}$ is semi-lattice (SL) if $\mathcal{D}$ is UC and involutive.

Here $\sqsubseteq$ compares the peak-multisets of 2-reductions ($\Downarrow$ or $\Rightarrow$) in the (well-founded) multiset extension of $\bullet \ll$ (based itself on some well-founded order on step-labels; cf. Ex. 8(iii)), where the peak-multiset morphism maps a 2-reduction to its multiset of local peaks, induced by mapping a diamond to the singleton of its local peak, but trivial diamonds to $\square$.

Recalling that $\Downarrow$ denotes tiling, the intuition for undercutting is that it captures cut-elimination (between the positive χ in $\phi^{-1} \cdot \chi$ and the negative χ in $\chi^{-1} \cdot \psi$ to yield $\phi^{-1} \cdot \psi$), and replacing two diagrams (the ones for $\phi^{-1} \cdot \chi$ and $\chi^{-1} \cdot \psi$) by a single one (for $\phi^{-1} \cdot \psi$) under them.

Theorem 6. $\varrho \Rightarrow \varsigma$ iff $\varrho^{-1} \cdot \varsigma \Downarrow \varepsilon$ for ϱ a reduction parallel to ς , and $\mathcal{D}$ UC.

Proof. We first reformulate filling: $\varrho \Rightarrow \varsigma$ iff there is a

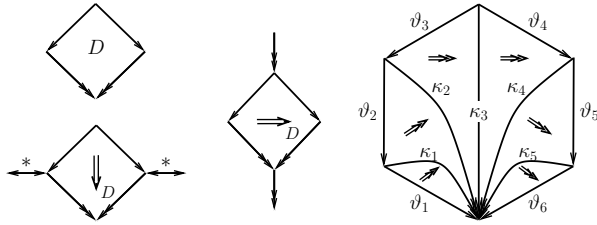


Fig. 17. Diamond D with $\Downarrow_D, \Rightarrow_D$ (left); foliage with $\Rightarrow$ -reductions (right)

foliage for $\varrho^{-1} \cdot \varsigma$, see Fig. 17 (right), where a foliage for a cyclic conversion $\vartheta = \vartheta_1 \cdot \dots \cdot \vartheta_n$ comprising n steps ϑ_i , has reductions κ_i for $0 \leq i \leq n$ with $\kappa_0 = \varepsilon = \kappa_n$, and reductions $P_i : \kappa_{i-1} \Rightarrow \vartheta_i \cdot \kappa_i$ if ϑ_i is a forward step, and $P_i : \vartheta_i^{-1} \cdot \kappa_{i-1} \Rightarrow \kappa_i$ if ϑ_i is a backward step, for $1 \leq i \leq n$. Then the only-if direction of the theorem is trivial, using the same diamonds as in the $\Downarrow$ tiling to construct the $\Rightarrow$ -reductions in the foliage. The if-direction is by induction first on $\sqsubseteq$ but then on the *width* of peaks in the cyclic conversion measuring how often UC could be applied, showing the tiling to be $\sqsubseteq$ -related to the foliage. $\square$

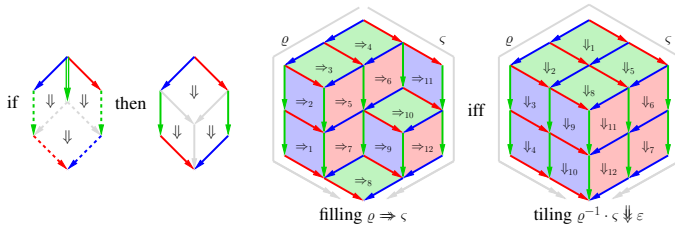


Fig. 18. Undercutting (left) and filling = tiling (right) for calissons

Example 26. In the 4th solution to the problem of the calissons, we return to its modelling in Ex. 7. Since UC holds for the diamonds employed (Fig. 18 left), $\sqsubseteq$ holds for the trivial order as both sides of UC have 3 diamonds, and both

sides also have one diamond of each colour so the spectrum is preserved, we conclude using Thm. 6 by that the tiling yields a filled box that only depends on the box (Fig. 18 right), so all spectra are the same.

Similar reasoning pertains to hole-filling (even SL holds). The correspondence of Thm. 6 is an important theme in the literature, e.g. both in [12] and [24] (Sec. 5), but also in the λ -calculus and term rewriting.

Example 27. For the λ -calculus the correspondence is known as that between permutation and projection equivalence, cf. [9, Thm. I.10.2.6], [60]. To infer this by Thm. 6 one must find a suitable order on steps such that the $\sqsubseteq$ -condition on UC holds. This is complicated by that $\rightarrow_\beta$ is not terminating. However, for its Hyland–Wadsworth labelling [17], [60] (orthogonal as a PRS), every local confluence diamond (computed by residuals, not by syntactic accidents) lifts to a labelled one, and any filling $\varrho \Rightarrow \varsigma$ (proof of permutation equivalence of reductions ϱ, ς) lifts to a HW-labelled filling where labels are bounded. This then gives a termination measure (via the maximal label in it, [60, Prop. 1.5.6]) allowing to verify UC and to turn the HW-labelled filling into a HW-labelled tiling. We then obtain a tiling simply by forgetting the labels, least because any other tiling will factor through it.

Also braids are not terminating, but one can proceed similarly, exploiting that they have random descent for the length measure, that to have a filling $\varrho \Rightarrow \varsigma$ entails ϱ, ς have the same length. Even stronger, UC can be shown to hold for all Artin–Tits [12, Ch. IX.1.2] (with order $\sqsubseteq$ based on right-Noetherianity), allowing one to reason by contraposition to conclude elements not to have common multiples: if some tiling of two elements ϱ, ς does not terminate, then no tiling does (by random descent), hence no least common multiple exists, hence no common multiple exists; cf. [12, Ex. II.4.28], e.g. providing the standard solution to the paint pot problem [64].

V. CONCLUSION

Building on the known confluence techniques of *decreasing diagrams* [34], [65] and *random descent* [8], [51], we presented three new confluence / orthogonality results and gave (potential) application(area)s in both rewriting and 2-rewriting. Surprisingly, each technique provided a simple solution to the problem of the calissons. To give an *algebraic* account of these results and in particular of tiling / residuation on which we based them, we resorted to introducing 1-algebras, algebras having rewrite systems as carrier, because by the very nature of rewriting we are interested in systems in which sequential compositions of steps need not exist.

We expressed bricklaying $\Rightarrow$ as a rewrite system on beds, analogously to the 2-rewrite systems on conversions, but only for the simple example of the calissons. The general case concerns *planar* graph rewriting, so seems challenging in the light of [66].

a) *Acknowledgments:* Jan Willem Klop suggested the problem of the calissons as testbed for rewriting, and its modelling as a confluence problem (per group e-mail (2024)).

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APPENDIX

A. Rewrite systems, 1-algebras, residuation

Remark 6 (On the notation for PRSs, Ex. 2). *The rendering of the $\lambda\beta$ -calculus as a higher-order pattern rewrite system*

in [4, Ex. 11.2.22(ii)] as employed here, is obtained from the same example as rendered in [19, Example 3.4] by dropping the λ from abstractions $\lambda x.M$ in the underlying substitution calculus [45], [67], and by not using free variables in (left- and right-hand sides of) rules but explicitly abstracting from them.

Remark 7 (On steps as first-class citizens / terms). *The economy afforded by having steps as first-class citizens, in particular as (proof)terms over a signature (including rule-symbols), extends to proving properties, cf. Ex. 4 ff. For instance, PRSs being closed under signature extensions obviates the need for restating / reproving for variations like labelling, cf. [17, Def. 11.1.2].*

Having steps as first-class citizens in term rewriting enables reasoning algebraically about geometric notions for them. This applies to various operations on them like residuation [8], [14] (see below), to critical peaks (based on that patterns occurring in a term constitute a finite distributive lattice [49]), and to orthogonality (Sec. IV), and is heavily exploited in recent Isabelle formalisations¹⁰ of TRS-confluence results [31].

B. Confluence

Remark 8 (On adequacy of the modelling in Ex. 7). *Let F be a partial filling of B , i.e. $a \Rightarrow$ -reduction from $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ to some $\rightarrow$ -reduction P , having calissons as in B . Then we claim further progress can be made toward B . Key to the claim is that if $\rightarrow \rightarrow$ occurs in P then B has a green calisson, and otherwise if $\rightarrow \rightarrow$ occurs in P , then for the rightmost such either it is matched, i.e. occurs in a sub-reduction of P of shape $\rightarrow \rightarrow \rightarrow \rightarrow$ and then B either has a red-calisson for $\rightarrow \rightarrow$ or a blue calisson for $\rightarrow \rightarrow$, or it is unmatched and then B has a red-calisson for it by boxes being convex. Each case allows some further $\Rightarrow$ -step toward B .*

Remark 9 (On the proof of Thm. 1). *We obtained Thm. 1 in 2008. The compact proof presented here builds on the recent advances of [68], [69], showing that steps can in fact be mapped to belonging to the spanning $\rightarrow$ -tree (0) or not (1), i.e. to a bit.*

C. Random descent

Proof of Thm. 3. For the only-if-direction first note that the assumption implies that all objects convertible to some normal form are complete. Thus $\leftarrow$ is a well-founded order on them, and we will exploit it to define a measure on the steps in conversions to normal form. We measure steps in the reduction monoid of the ordinals with 0 and flipped ordinal sum $\dot{+}$.

We construct measuring functions both for steps and objects, with the measure of an object being based on the measures of all its reductions to normal form. We first partition the objects into those that are convertible to some normal form, and those that are not. We measure the latter, and steps between them, arbitrarily, by 1. An object a of the former is measured by the supremum of the successors of the measures of all b such

¹⁰Cf. operations on proofterms in <http://informatik-protem.uibk.ac.at/Okui/>.

that $a \rightarrow b$. This is well-defined by well-foundedness of $\leftarrow$. In turn, each such step $a \rightarrow b$ is measured by the ordinal γ such that $\gamma \dot{+} \beta = \alpha$, where α, β are the measures of a, b . γ exists and is non-0 per construction.

We claim peak random descent then holds. Note that it suffices to verify it for *finite* peaks $\hat{a} \leftarrow_{\alpha} a \rightarrow_{\beta} b$ with $\hat{a}$ in normal form; an infinite reduction from a would contradict uniform termination. We prove that then $\alpha = \beta \dot{+} \gamma$ with γ the measure of b , by induction on the length of the peak, distinguishing cases on the direction of its last step. For the empty peak, it is trivial as normal forms have measure 0. Otherwise, $\hat{a} \leftarrow_{\alpha'} a \rightarrow_{\beta'} b' \leftrightarrow b$, where $\alpha' = \beta' \dot{+} \gamma'$ holds with γ' the measure of b' by the induction hypothesis. Let δ be the measure of $b' \leftrightarrow b$. If $b' \rightarrow b$, then $\alpha = \alpha' = \beta' \dot{+} \gamma' =^{(\dagger)} \beta' \dot{+} (\delta \dot{+} \gamma) = (\beta' \dot{+} \delta) \dot{+} \gamma = \beta \dot{+} \gamma$ where $(\dagger)$ holds since γ and γ' are the measures of b and b' and δ the measure of $b' \rightarrow b$ so $\gamma' = \delta \dot{+} \gamma$. If $b' \leftarrow b$, then $\alpha = \delta \dot{+} \alpha' = \delta \dot{+} (\beta' \dot{+} \gamma') = \delta \dot{+} (\beta \dot{+} \gamma') =^{(*)} \beta \dot{+} (\delta \dot{+} \gamma') = \beta \dot{+} \gamma$, using for $(*)$ that this case can only happen when $\beta = 0 = \beta'$ (while constructing the left leg of the peak, its right leg is empty).

For the if-direction, we claim that a rewrite system $\rightarrow$ is uniformly complete iff all objects convertible to a normal form are terminating and reduce to that normal form. Suppose the claim holds.

To show all objects convertible to a normal form are terminating, one may proceed by induction on the length of the conversion to the normal form, and by cases on its first step $a \leftrightarrow b$ with b terminating by the IH. If the step is of shape $a \leftarrow b$, then we conclude by the IH. If the step is of shape $a \rightarrow b$ then we conclude by PR for the reduction from a to normal form (via b).

That objects convertible to a normal form reduce to that normal form, follows by induction on the number of peaks in the conversion to normal form, cf. [51, p. 32:3].

It remains to show the claim. For the only-if-direction, suppose $\rightarrow$ is uniformly complete. An object convertible to normal form a , then is complete hence certainly terminating, so reduces to some normal form b . Moreover, all objects on the conversion between a, b are complete, so $a = b$. For the if-direction, suppose all objects convertible to a normal form are terminating and reduce to that normal form. Then if an object a is convertible to some normal form b , it is terminating, and moreover the reductions in a peak from a are convertible to b too, so must reduce to it, showing confluence of a . $\square$

Example 28 (Another example of an application of RD). Ind & Inc $\Rightarrow$ SN [54] [9, Lem. 5.17(i)] [4, Thm. 1.2.3(ii)], where $\rightarrow$ is inductive (Ind) if for all $a_0 \rightarrow a_1 \rightarrow \dots$, $\exists a \in A$ such that $a_i \rightarrow a$ for all $i \in \mathbb{N}$. By Inc all reductions from a_0 to a are commensurate, so there can only be finitely many a_i by addition on the natural numbers being strictly monotonic in its first argument (this would fail for addition of ordinals).

Example 29 (Example associativity-TRS is OWCR, but not for the length measure; shown by (1-)algebraic means). Con-

sider the term rewrite rule for associativity:¹¹ $\varrho(x, y, z) : xyz \rightarrow x(yz)$. It is linear and it has a single critical peak which may be completed into a local confluence diagram with legs $xyzw \rightarrow x(yz)w \rightarrow x(yzw) \rightarrow x(y(zw))$ and $xyzw \rightarrow xy(zw) \rightarrow x(y(zw))$. To show OWCR, observe the length measure does not work as the legs have different lengths. Measuring a step contracting $\varrho(t, s, r)$ by twice the number of leaves of t does: both legs then have the same measure: $2n+2n+2m = 2(n+m)+2n$ with n, m the number of leaves of t, s . For non-overlapping peaks ordered local confluence follows from that counting the number of leaves in a term yields a model, i.e. is invariant under ϱ . Since the bullet function of [70, Definition 32] induces a normalising strategy [70, Lemma 35(Extensive)] $\rightarrow$ is complete by Cor. 1.

An algebraic way of defining the measure may be obtained by employing proofterms to represent reductions resulting, e.g., in representing the legs of the diagram as $\varrho(x, y, z)w \cdot (\varrho(x, yz, w) \cdot x\varrho(y, z, w))$ and $\varrho(xy, z, w) \cdot \varrho(x, y, zw)$. Then the measure is defined by a 1-algebra, i.e. an algebra for proof terms, building on an algebra, i.e. an algebra on terms. In the algebra, computing the number of leaves, we interpret variables by assigning 1 to them and interpret @ as addition. The 1-algebra, computing the sum of the numbers of leaves in the first argument of each ϱ -redex contracted in a reduction, builds on that by interpreting variables as 0, @ and $\cdot$ as addition, and ϱ as the value of its first argument (a term).

Remark 10 (On WN iff SN for $\lambda\beta I$ -terms). • The result in Ex. 14 and its proof carry over immediately to any orthogonal (first- or higher-order) non-erasing (pattern) term rewriting system, saturating left-hand sides of rules with c -symbols and prefixing right-hand sides by their sum plus one. For higher-order term rewrite systems (PRSs) one has to be careful though with the notion of erasingness, since for systems of order 3 or higher it might well be that though a step is not erasing a subterm, it is critical in the sense that it transforms a non-terminating term into a terminating one [55, Ex. 2].

- The idea to make rewrite systems increasing by carrying extra information along is a special case of the (70+ years) old idea of labelling [4, Sec. 8.4], with the most prominent labellings of the λ -calculus being Lévy-labelling and Hyland–Wadsworth labelling [17, Ch. 14].
- The above proof steps amount to verifying that $\rightarrow_{\beta c}$ is a rewrite labelling of $\rightarrow_{\beta}$ and $\rightarrow_{\beta c I}$ of $\rightarrow_{\beta I}$, in the sense of [4, Def. 8.4.5] for the initial labelling mapping a term to itself, establishing a bisimulation between M and $h(M)$. Even stronger, the labelling is a term rewrite labelling in that it is induced by labelling rules (in this case only the β -rule) [4, Def. 8.4.26].
- A slightly more general $\lambda\beta I$ -calculus is obtained by not restricting the terms but only the proofterms, i.e. requiring x to occur in M for a subterm $\beta(x.M, N)$. The same

¹¹In applicative notation, using association to the left for the implicit infix application symbol @.

effect could be obtained by restricting the substitution calculus [45], [67] of PRSs to $\lambda\beta I$.

Remark 11 (Random descent of bricklaying $\Rightarrow$ for calissons). For the same reason as for $\Downarrow$, bricklaying $\Rightarrow$ for calissons has random descent (' $\Rightarrow$ -redexes do not have overlap').

D. Bricklaying

Proof of Thm. 5. If one of the reductions of the 3-peak is empty, then the faces adjacent to it are trivial by the assumption that faces are obtained by tiling, and the face orthogonal to it is decreasing, so the latter can be combined with two further trivial faces, to yield a 3-valley completing the 3-peak into a brick that is trivially seen to be decreasing.

Otherwise consider the local brick for the 3-peak formed by the initial steps ϕ, ψ, χ of the reductions $\phi \cdot \bar{\phi}, \psi \cdot \bar{\psi}, \chi \cdot \bar{\chi}$; it is decreasing by assumption. We first show that laying such a brick *decreases* the measure of each of the 3 resulting 3-peaks, as visualised in Fig. 19 (for the bottom-right 3-peak).

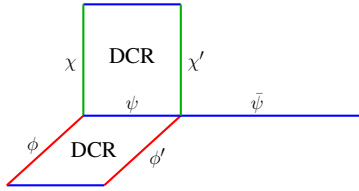


Fig. 19. Decrease in measure of 3-decreasingness

We show that $|\phi'| \uplus |\bar{\psi}| \uplus |\chi'| \prec_{mul} |\phi| \uplus |\psi \cdot \bar{\psi}| \uplus |\chi|$ with ϕ', χ' the reductions opposite of ϕ, χ , by generalising the idea and proof of [45, Lem. 2.3.19], with each line justified below.

$$\begin{aligned}
& |\phi'| \uplus |\bar{\psi}| \uplus |\chi'| = \\
& = (|\phi'| \uplus |\bar{\psi}| \uplus |\chi'|) \cap \gamma\psi \uplus ((|\phi'| \uplus |\bar{\psi}| \uplus |\chi'|) - \gamma\psi) \quad (8) \\
& \prec_{mul} |\psi| \uplus ((|\phi'| \uplus |\bar{\psi}| \uplus |\chi'|) - \gamma\psi) \quad (9) \\
& = |\psi| \uplus (|\phi'| - \gamma\psi) \uplus (|\bar{\psi}| - \gamma\psi) \uplus (|\chi'| - \gamma\psi) \quad (10) \\
& = |\psi \cdot \phi'| \uplus (|\bar{\psi}| - \gamma\psi) \uplus (|\chi'| - \gamma\psi) \quad (11) \\
& \preceq_{mul} |\phi| \uplus |\psi| \uplus (|\bar{\psi}| - \gamma\psi) \uplus (|\chi'| - \gamma\psi) \quad (12) \\
& = |\phi| \uplus (|\bar{\psi}| - \gamma\psi) \uplus |\psi \cdot \chi'| \quad (13) \\
& \preceq_{mul} |\phi| \uplus (|\bar{\psi}| - \gamma\psi) \uplus |\chi| \uplus |\psi| \quad (14) \\
& = |\phi| \uplus |\psi \cdot \bar{\psi}| \uplus |\chi| \quad (15)
\end{aligned}$$

The justifications given refer to results in [45].

- (8) Split 1.4.6(7d)
- (9) Noise Reduction 1.4.11(4), $\psi \neq \varepsilon$
- (10) Distribute 1.4.6(8d)
- (11) Compose 2.3.13(2) for ϕ'
- (12) DCR for ϕ, ψ
- (13) Compose 2.3.13(2) for χ'
- (14) DCR for ψ, χ
- (15) Compose 2.3.13(2) for $\bar{\psi}$

Thus (since $|\phi| \preceq_{mul} |\phi \cdot \bar{\phi}|$ and $|\chi| \preceq_{mul} |\chi \cdot \bar{\chi}|$) the IH applies allowing to complete the 3-peak with reductions ϕ', ψ, χ' into a decreasing brick (IH₁ in Fig. 15). Since

decreasing diagrams compose [45, Lem. 2.3.17], and also the $\Downarrow$ -tiling of their faces composes by Prop. 2, so do decreasing bricks, entailing (*mutatis mutandis*) the IH applies successively from front to back (along ϕ , IH₂), and bottom to top (along χ , IH₃). $\square$

Compared to the proof of [45, Lem. 2.3.19] the main difference is that ψ is used *repeatedly* above, like a catalyst, in the justification given. This is needed to deal with noise in *both* ϕ' and χ' introduced into them by the decreasing diagrams of ψ with *both* ϕ and χ .

Remark 12 (On Ex. 25). For decreasingness it suffices to order all $\rightarrow$ -steps ($\rightarrow$ -steps belonging to the spanning forest) in the proof of Thm. 1 below all other steps of ϕ .

To see that local 3-confluence holds for a 3-peak (ϕ, ψ, χ) , note that the three $\rightarrow$ -peaks are, per construction, completed by tiling to three $\rightarrow\Rightarrow$ -valleys ending in the least upperbound in the tree of each pair. Then the tiling of each of the 3 $\rightarrow\Rightarrow$ -peaks ends up in the least upperbound in the tree of the triple, which is unique, using that $\rightarrow$ is deterministic.

E. Undercutting

Remark 13 (On checking UC / SL). To establish UC it suffices to consider triples ϕ, ψ, χ where $\phi \neq \chi \neq \psi$ since if, say, $\phi = \chi$ then the assumption simplifies to $\phi^{-1} \cdot \psi \Downarrow \zeta \cdot \xi^{-1}$, which is seen to entail the conclusion $\zeta^{-1} \cdot \phi^{-1} \cdot \psi \cdot \xi \Downarrow \varepsilon$ using that peaks between a step and itself were assumed trivial. As observed in [12] SL allows to also assume $\phi \neq \psi$, since if $\phi = \psi$ then $\phi^{-1} \cdot \chi \cdot \chi^{-1} \cdot \phi \Downarrow \zeta \cdot \xi^{-1}$ entails $\zeta = \varrho = \xi$ for $\phi \diamond_{\zeta}^{\chi}, \chi \diamond_{\xi}^{\phi} \in \mathcal{D}$ so $\zeta^{-1} \cdot \phi^{-1} \cdot \phi \cdot \xi \Downarrow \varepsilon$.

Proof of Theorem 6. We first establish the equivalence between filling and foliage. Let $\varrho = \varrho_1 \dots \varrho_n$ and $\varsigma = \varsigma_1 \dots \varsigma_m$ be reductions comprising n steps ϱ_i respectively m steps ς_j , with ϱ parallel to ς . Then $\vartheta := \varrho^{-1} \cdot \varsigma$ is a cyclic conversion.

For the if-direction, let κ_i be the $\rightarrow$ -reductions of the foliage for the cyclic conversion ϑ as in the definition. By definition of foliage and closure of $\Rightarrow$ -steps under contexts (whiskering) then $\varrho \Rightarrow \kappa_n \Rightarrow \varsigma$; Fig. 17.

For the only-if-direction, we distinguish cases on n being 0 or not. If n is 0, then so is m and $\varrho \Rightarrow \varsigma$ must be the empty filling on an empty $\rightarrow$ -reduction, by the assumption that $\mathcal{D}$ comprises local confluence diagrams only whose legs are *non-empty* reductions, hence $\Rightarrow$ can only be applied to non-empty $\rightarrow$ -reductions; Fig. 17. If n is positive, then ϱ is non-empty and by the assumption on $\mathcal{D}$ again, all $\Rightarrow$ -reducts of ϱ along $\varrho \Rightarrow \varsigma$ are non-empty. Defining $\kappa_i := \varrho_{n-(i-1)} \dots \varrho_n$ for $1 \leq i \leq n$ and $\kappa_i := \varsigma_{(i+1)-n} \dots \varsigma_m$ for $n+1 \leq i \leq n+m$ then gives rise to a foliage for ϑ : All $\Rightarrow$ -reductions for the foliage may be set to empty $\Rightarrow$ -reductions, except that to show $\kappa_n = \varrho \Rightarrow$ -reduces to $\varsigma_1 \cdot \kappa_{n+1} = \varsigma_1 \cdot \varsigma_2 \dots \varsigma_m = \varsigma$ we use the assumptions that $\varrho \Rightarrow \varsigma$ and that ς is non-empty (so has a first step).

We now prove the if-direction of the theorem (for foliages) by induction on the number of tiling steps p in $\vartheta \Downarrow^p \varepsilon$. If $p = 0$, we trivially conclude as $\vartheta = \varepsilon$. Otherwise, we can

write $\vartheta = \vartheta^l \cdot \phi^{-1} \cdot \psi \cdot \vartheta^r$ for some $\phi \diamond_{\xi}^{\psi} \in \mathcal{D}$ and conversions ϑ^l, ϑ^r , and there is a tiling step $\vartheta \Downarrow \vartheta'$ to $\vartheta' = \vartheta^l \cdot \zeta \cdot \xi^{-1} \cdot \vartheta^r$, such that $\vartheta' \Downarrow^{p-1} \varepsilon$. By the IH there is a foliage for ϑ' ; its subconversions ϑ^l, ϑ^r combined with prefixing ϕ to the last reduction of ϑ^l then give a foliage for ϑ . Note that the peak-multisets of the $\Downarrow$ - and $\Rightarrow$ -reductions (of the foliage) are the same.

Finally, we prove the only-if-direction (again for foliages instead of fillings) by induction on the lexicographic product of $\sqsubseteq$ on the foliage for ϑ , next $\leq$ on the length of ϑ , and finally the multiset extension of $\leq$ for the multiset of widths (see below) of local peaks, and by cases on ϑ . We strengthen it to that the peak-multiset of the tiling is $\sqsubseteq$ than that of (the multiset sum of the fillings in) the foliage. We may assume the foliage has no $\Rightarrow$ -steps for trivial peaks. If ϑ is a valley, then by definition of foliage $\vartheta = \varepsilon$ using that the legs of diamonds in $\mathcal{D}$ are non-empty, and we conclude. Otherwise, ϑ has shape $\vartheta^l \cdot \phi^{-1} \cdot \psi \cdot \vartheta^r$ with $\phi \cdot \kappa_{i-1} \Rightarrow \kappa_i \Rightarrow \psi \cdot \kappa_{i+1}$ and we distinguish cases on the *width*, the number of $\Rightarrow$ -root-steps in the $\Rightarrow$ -reductions.

If $w = 0$ then $\phi = \psi$ and $\kappa_{i-1} \Rightarrow \kappa_{i+1}$. Then $\vartheta \Downarrow \vartheta'$ for $\vartheta' := \vartheta^l \cdot \vartheta^r$ by UC. Replacing¹² κ_{i-1} by κ_{i+1} in the foliage for ϑ^l renders ϑ' a foliage. We conclude by the IH for ϑ' (decrease in second component).

If $w = 1$ then $\phi \cdot \kappa_{i-1} \Rightarrow \phi \cdot \zeta \cdot \kappa' \Rightarrow \psi \cdot \xi \cdot \kappa' \Rightarrow \psi \cdot \kappa_{i+1}$ for some diamond $\phi \diamond_{\xi}^{\psi} \in \mathcal{D}$ and some κ' , where the displayed $\Rightarrow$ do not have head-steps. Then $\vartheta \Downarrow \vartheta'$ for $\vartheta' := \vartheta^l \cdot \zeta \cdot \xi^{-1} \cdot \vartheta^r$. Replacing (cf. footnote 12) κ_{i-1} by $\zeta \cdot \kappa'$ in the foliage for ϑ^l and replacing κ_{i+1} by $\xi \cdot \kappa'$ in the foliage for ϑ^r , renders ϑ' a foliage again. We conclude by the IH for ϑ' (decrease in first component).

If $w > 1$ then $\phi \cdot \kappa_{i-1} \Rightarrow \phi \cdot \varrho \cdot \kappa' \Rightarrow \chi \cdot \zeta \cdot \kappa' \Rightarrow \chi \cdot \zeta \cdot \kappa'' \Rightarrow \psi' \cdot \xi \cdot \kappa'' \Rightarrow \psi \cdot \kappa_{i+1}$ for some diamonds $\phi \diamond_{\xi}^{\psi}, \chi \diamond_{\zeta}^{\psi} \in \mathcal{D}$ and some κ', κ'' , where the first two displayed $\Rightarrow$ -reductions do not have head-steps (we may but need not have $\psi' = \psi$). The second induces a foliage for the peak $\kappa'^{-1} \cdot \zeta^{-1} \cdot \zeta \cdot \kappa''$ to which the IH applies (decrease in first component, because the removed peaks are non-trivial) yielding $\kappa'^{-1} \cdot \zeta^{-1} \cdot \zeta \cdot \kappa'' \Downarrow \varepsilon$. By random descent for $\Downarrow$ (Prop. 1), this projection factors as $\zeta^{-1} \cdot \zeta \cdot \Downarrow \zeta' \cdot \zeta'^{-1}$ and $\kappa'^{-1} \cdot \zeta' \cdot \zeta'^{-1} \cdot \kappa'' \Downarrow \varepsilon$ for some reductions ζ', ζ' (with the peak-multisets of the factors multiset-summing up to that of their composition).

The former $\Downarrow$ combined with two $\Downarrow$ -steps for the diamonds gives $\phi^{-1} \cdot \chi \cdot \chi^{-1} \cdot \psi' \Downarrow \varrho \cdot \zeta' \cdot \zeta'^{-1} \cdot \xi^{-1}$ for which UC entails $\zeta'^{-1} \cdot \varrho^{-1} \cdot \phi^{-1} \cdot \psi \cdot \xi \cdot \zeta' \Downarrow \varepsilon$ (with $\sqsubseteq$ -peak multiset). The if-direction then yields a foliage for it (with the same peak multiset), so $\phi \cdot \varrho \cdot \zeta' \Rightarrow \psi \cdot \xi \cdot \zeta'$ having exactly 1 head-step (by a diamond for ϕ, ψ).

For the latter $\Downarrow$ the if-direction yields a foliage so $\kappa' \Rightarrow \zeta' \cdot \hat{\kappa}$ and $\zeta' \cdot \hat{\kappa} \Rightarrow \kappa''$ for some $\hat{\kappa}$.

Combining both shows that $\phi \cdot \varrho \cdot \kappa' \Rightarrow \psi' \cdot \xi \cdot \kappa''$ using a single head-step, instead of the two before. Hence we conclude by the IH for the same ϑ but with this alternative foliage

(decrease in the third component, if not already in the first). $\square$

F. Haskell

Haskell code exhibiting recursive definition of residuation for reductions, based on residuation for steps, for the particular case of the paint pot problem [64]. For instance, trying to compute the residual of the list of other generators after the first does not terminate (it should not terminate [12]).

```
paintpot = [(1,2), (1,3), (1,4), (1,5)]
graph = paintpot
nr = foldr (\(i,j) m -> max (max i j) m) 1 graph
resstp i j = if (i==j) then [] else
  if elem (i,j) graph || elem (j,i) graph
  then [i,j] else [i]
resred [] v = []
resred u [] = u
resred (i:u) (j:v) = resred ((resstp i j) ++
  (resred u (resstp j i))) v
```

¹² If $\vartheta^l = \varepsilon$ replacing is not allowed but not needed: then $\kappa_{i-1} = \varepsilon = \kappa_{i+1}$ as legs of diamonds are non- ε .