Multi-redexes and multi-treks induce residual systems

least upper bounds and left-cancellation up to homotopy

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Abstract

Residual theory in rewriting goes back to Church, Rosser and Newman at the end of the 1930s. We investigate an axiomatic approach to it developed in 2002 by Melliès. He gave four axioms (SD) *self-destruction*, (F) *finiteness*, (FD) *finite developments*, and (PERM) *permutation*, showing that they entail two key properties of reductions, namely having (i) *least upper bounds* (*lubs*) and (ii) *left-cancellation*.[1] These properties are shown to hold *up to* the equivalence generated by identifying the legs of local confluence diagrams inducing the same residuation, which corresponds to Lévy's permutation equivalence. Melliès in fact presented two sets of axioms, one for *redexes* as in classical residual theory and another more general one for *treks*. We show his results factor through the theory of *residual* systems we introduced in 2000, in that any rewrite system satisfying the four axioms (for redexes or treks) can be enriched to a residual system such that (i) and (ii) follow from the theory of residual systems. We exemplify the axioms are sufficient but not necessary.

Proofs omitted in this abstract can be found in the appendix of [18].

1 Residual systems

We are interested in the theory of computation based on rewriting. As this requires to have computations as first-class citizens, we use rewrite *systems* [14],[20, Def. 8.2.2] (not rewrite *relations*), whose *steps* have *sources* and *targets*. We recapitulate *residual* systems [20, Def. 8.7.2].

Definition 1. *A* residual *system (RS)* $\langle \to, 1, / \rangle$ *comprises a rewrite system* $\to$ *and a* residual *function* $/$ *having* 1 *as* unit*:* 1 *is a function from objects to steps such that* $\mathsf{tgt}(1_a) = a = \mathsf{src}(1_a)$ *and for co-initial steps* ϕ, ψ, χ*, the* residual *identities (1)–(3) in Tab. 1 must be satisfied. The* projection *order* $\lesssim$ *is defined by* $\phi \lesssim \psi$ *if* $\phi/\psi = 1$ *for co-initial steps* ϕ, ψ*.*

The projection order $\lesssim$ is a quasi-order [20, Lem. 8.7.23] inducing *projection* equivalence $\simeq \,:=\, \lesssim \cap \gtrsim$. Examples of rewrite systems that can be equipped with residual structure abound.

Example 1. *For the following rewrite systems* $\to$*, residual structure is obtained from the proof of the diamond property for an appropriate rewrite system that is between* $\to$ *and its reflexive–transitive closure: i) the* $\lambda\beta$*-calculus induces a residual system by the Tait–Martin-Löf proof that* $\geq_1$ *has the diamond property [1]; ii)* β*-steps in the* linear $\lambda\beta$*-calculus have the diamond property themselves; iii)* parallel *steps* $\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+\;$/multisteps $\multimap$ *in orthogonal first/higher-order term rewrite systems [8, 20, 2]; iv) positive braids with* parallel crossings *of strands [20, Sect. 8.9].*

Here we show multi-redexes and multi-treks as in Melliès' axiomatic residual theory naturally induce residual systems, entailing the results of [13] *via* the theory of residual systems [20]. We use $\phi, \psi, \chi, \ldots$ and $\gamma, \delta, \epsilon, \ldots$ to range over steps respectively reductions. We denote *finite*

[1] Instead of the order-theoretic setting employed here, Melliès employs a category-theoretic setting and the corresponding terminology of having *pushouts* and *epis*.

reductions by $\twoheadrightarrow$. They can be identified [20, Def. 8.2.10] with formal compositions $(\cdot)$ of steps (whose targets, sources match) modulo the monoid *identities*. Orienting these into the *rules* (4)–(6) of Tab. 1 gives a complete 2-rewrite system[2] so unique representatives of such reductions.

Proposition 1. *Any residual system on $\to$ extends to a residual system on $\twoheadrightarrow$, defining residuation by normalisation w.r.t. the 2-rewrite system with rules (4)–(8) of Tab. 1.*

$\phi/1$	$\overset{(1)}{=}$	ϕ	$(\gamma\cdot\delta)\cdot\epsilon$	$\overset{(4)}{\Rightarrow}$	$\gamma\cdot(\delta\cdot\epsilon)$	$\gamma/(\delta\cdot\epsilon)$	$\overset{(7)}{\Rightarrow}$	$(\gamma/\delta)/\epsilon$
ϕ/ϕ	$\overset{(2)}{=}$	1	$\gamma\cdot 1$	$\overset{(5)}{\Rightarrow}$	γ	$(\delta\cdot\epsilon)/\gamma$	$\overset{(8)}{\Rightarrow}$	$(\delta/\gamma)\cdot(\epsilon/(\gamma/\delta))$
$(\phi/\psi)/(\chi/\psi)$	$\overset{(3)}{=}$	$(\phi/\chi)/(\psi/\chi)$	$1\cdot\gamma$	$\overset{(6)}{\Rightarrow}$	γ			

Table 1: Residual identities, monoid rules, and residual rules for formal composition

Example 2. *The classical example of a term rewrite system is Combinatory Logic (CL) having the three rules, in applicative notation, $\iota(x):Ix\to x$, $\kappa(x,y):Kxy\to x$, and $\varsigma(x,y,z):Sxyz\to xz(yz)$. We call a term over the signature extended with the so-called* rule *symbols [20, Ch. 8] ι,κ,ς (having as arities the number of variables in the respective rules) a* multistep*, as it can be assigned a source/target by mapping all such rule symbols in it to their lhs/rhs. This naturally induces a residual system on multisteps [20, Prop. 8.7.7], which by the above extends to one on reductions (of multisteps). For example, $\gamma := \varsigma(K,y,Iz)\cdot\kappa(Iz,y(Iz))$ and $\delta := SKI\iota(z)$ are co-initial reductions from $SKy(Iz)$ to Iz respectively $SKyz$. Both these targets are reduced to z by the respective residual reductions: $\delta/\gamma := \iota(z)$ and $\gamma/\delta := \varsigma(K,y,z)\cdot\kappa(z,y(z))$.*

Remark 1. *We introduced the idea of multisteps as terms over the signature extended with rule symbols in [20, Ch. 8] as a generic tool in* structured *rewrite systems, like* string *[6, p. 226],* higher-order term *[2, p. 127], and* graph *[20, Rem. 9.4.30] rewrite systems.*

Then $\twoheadrightarrow$ is a residual system *with composition* [20, Def. 8.7.38], $\simeq$ is a congruence for $/$ and $\cdot$ and quotienting $\simeq$ out yields a residual system whose projection order is a partial order [20, Lem. 8.7.41]. Projection equivalence [20] can alternatively be defined as the homotopy *generated* by the diamond property. This will allow us below to relate the former to local homotopy [13].

Definition 2. Square homotopy *equivalence $\equiv$ on reductions having the same sources/targets, is generated by closing $\phi\sqcup\psi\equiv\psi\sqcup\phi$ for local peaks ϕ,ψ under composition: if $\gamma\equiv\gamma'$ then $\delta\cdot\gamma\cdot\epsilon\equiv\delta\cdot\gamma'\cdot\epsilon$. Here $\phi\sqcup\psi := \phi\cdot(\psi/\phi)$. Correspondingly, we define $\gamma\sqsubseteq\delta$ if $\gamma\cdot\epsilon\equiv\delta$ for some ϵ.*

Lemma 1. $\simeq\,=\,\equiv$ *and* $\lesssim\,=\,\sqsubseteq$.

Example 3. *For γ,δ in Ex. 2 we have $\gamma\cdot(\delta/\gamma)\equiv\varsigma(K,y,Iz)\cdot K(\iota(z))(y\iota(z))\cdot\kappa(z,yz)\equiv\delta\cdot(\gamma/\delta)$.*

Theorem 1. *$\twoheadrightarrow$ up to square homotopy has lubs (δ',γ' is an* upper bound *of γ,δ if $\gamma\cdot\delta'\equiv\delta\cdot\gamma'$;* least *if $\delta'\sqsubseteq\delta''$, $\gamma'\sqsubseteq\gamma''$ for all upper bounds δ'',γ'') and left-cancellation (if $\gamma\cdot\delta\equiv\gamma\cdot\epsilon$ then $\delta\equiv\epsilon$).*

[2] What we refer to as 2-rewrite systems have formal expressions of compositions (and residuations) as objects. Their rules transform such expressions into *reductions* of an ordinary (1-)rewrite system $\to$, i.e. into formal compositions in normal form with respect to the monoid rules. This set-up generalises the 2-rewrite systems as found in the literature by not giving special status to composition, not *assuming* rules to operate on reductions only but on formal expressions. Working *modulo* the monoid identities yields proper 2-rewrite systems.

Proof. By Lem. 1 it follows from the same for projection equivalence $\simeq$ instead of square homotopy $\equiv$, which holds by virtue of $\twoheadrightarrow$ being a residual system with composition [20, Ex. 8.7.52]. We do that exercise: Left-cancellation follows from (see also the proof of Prop. 1):

$$(\gamma\cdot\delta)/(\gamma\cdot\epsilon) \Rightarrow ((\gamma\cdot\delta)/\gamma)/\epsilon \Rightarrow ((\gamma/\gamma)\cdot(\delta/(\gamma/\gamma)))/\epsilon \twoheadrightarrow (1\cdot(\delta/1))/\epsilon \twoheadrightarrow (1\cdot\delta)/\epsilon \Rightarrow \delta/\epsilon$$

That $\delta/\gamma, \gamma/\delta$ is an upper bound up to $\simeq$ of γ,δ, holds by $\twoheadrightarrow$ being a residual system. To see it is least consider any δ'',γ'' such that $\gamma\cdot\delta'' \simeq \delta\cdot\gamma''$. Then $(\gamma\cdot\delta'')/(\delta\cdot\gamma'') \Rightarrow (\gamma/(\delta\cdot\gamma''))\cdot(\delta''/((\delta\cdot\gamma'')/\gamma)) = 1$. Therefore [20, Ex. 8.7.40(iii)] both components must be 1 in particular the 1st $\gamma/(\delta\cdot\gamma'') \Rightarrow (\gamma/\delta)/\gamma'' = 1$. By symmetry $(\delta/\gamma)/\delta'' = 1$ and we conclude.[3] □

2 Multi-redexes and multi-treks

In [13] rewrite systems are equipped with a notion of residuation inducing a notion of *local* homotopy on reductions, based on the four axiomatic properties (SD), (F), (FD), and (PERM). The properties guarantee that multi-redexes/treks can be *developed* into reductions, that such developments have the diamond property, that all developments are locally homotopic, and finally (the main result) that reductions have lubs and left-cancellation up to local homotopy (Thm. 2). In fact *two* sets of four axioms are given in [13], the first one for *multi-redexes* and the second more general one for *multi-treks*. We show that in both cases the main results of [13] follow by known residual theory for a naturally associated residual system (in the sense of Sect. 1) on so-called developments, in particular from Thm. 1. We first develop enough notation to formally *express* the properties required of a rewrite system $\to$ for *multi-redexes* [13, Section 2], which informally read:

(*self-destruction*, SD) no step has a residual after itself;

(*finiteness*, F) every redex has finitely many residuals after a step;

(*finite developments*, FD) developments of multi-redexes are finite; and

(*permutation*, PERM) every peak ϕ,ψ of steps can be completed by a valley of complete developments of the residuals of ψ after ϕ, respectively the residuals of ϕ after ψ, such that both legs of the resulting local confluence diagram induce the same redex-trace relation.

We then show that these properties induce a residual system (Def. 1) on developments whose square homotopy corresponds to local homotopy on reductions, i.e. that Thm. 1 entails Thm. 2:

Theorem 2 (SD,FD,PERM; [13])**.** *$\twoheadrightarrow$ up to local homotopy has lubs and left-cancellation.*

Here *local* homotopy is generated (Def. 5) from the *local* confluence diagrams given by (PERM), instead of the *square* diamonds generating *square* homotopy (Def. 2). As in the statement of this main theorem, we qualify (intermediate) results throughout with the properties used, to enable illustrating that properties are sufficient but not necessary. In [13] residuation is captured by *tracing* a *redex* along a step to its *residuals*.

Definition 3. *A* redex-*trace relation is a function* $[\![\cdot\rangle\!\rangle$ *mapping each step* $\phi : a \to b$ *to a relation* $[\![\phi\rangle\!\rangle$ *between the redexes of a and b, where (*multi-*)*redexes *are reified (sets of) steps.*

[3] That gives a *pushout* as witnessed by $\epsilon := \delta''/(\delta/\gamma)$: On the one hand, $(\delta/\gamma)\cdot\epsilon \simeq \delta'\cdot((\delta/\gamma)/\delta'') \simeq \delta''$ follows from having a residual system and $\delta/\gamma \lesssim \delta'$. On the other hand, $(\gamma/\delta)\cdot\epsilon \simeq \epsilon''$ follows by left-cancellation from $\delta\cdot(\gamma/\delta)\cdot\epsilon \simeq \gamma\cdot(\delta/\gamma)\cdot\epsilon \simeq \gamma\cdot\delta'' \simeq \delta\cdot\epsilon''$ where the 2nd equivalence holds by the above and the others by assumption.

(SD) is formalised as $(\phi\ [\![\phi\rangle\!\rangle) = \varnothing$ and (F) as $(\psi\ [\![\phi\rangle\!\rangle)$ is finite, for any step ϕ and redex ψ. Here we use *section* notation for partial application of relations. The *left* section of a binary relation for an object a is $(a\ R) := \{b \mid a\ R\ b\}$. Similarly, the *right* section is $(R\ a) := \{b \mid b\ R\ a\}$. This is lifted pointwise to sets by $(A\ R) := \bigcup_{a\in A}(a\ R)$ and $(R\ A) := \bigcup_{a\in A}(R\ a)$. Trace relations naturally extend to reductions and conversions since relations constitute an involutive (typed) monoid with respect to composition, the identity relation, and converse, so we may e.g. write $\langle\!\langle\leftarrow]\!]$ for the trace relation of $\leftarrow$. We proceed with reifying tracing, labelling objects of the rewrite system with sets of redexes, which allows to recover the notion of *development* of [13].

Definition 4. *Consider the labelled rewrite system [20, Def. 8.4.5] having for each set Φ of redexes of a the object a^{Φ}, and for each step $\phi{:}a \to b$ the step ϕ^{Φ} from a^{Φ} to $b^{(\Phi[\![\phi\rangle\!\rangle)}$. A reduction γ from an object a is a* development *of Φ if it lifts to a $[\![\to\rangle\!\rangle$-reduction γ^{Φ} from a^{Φ}, where $[\![\to\rangle\!\rangle$ is the restriction of the labelled rewrite system to steps ϕ^{Φ} such that $\phi\in\Phi$. We say γ is a* complete development *of Φ if its lifting ends in a $\varnothing$-labelled object.*

(FD) is formalised by all developments are finite,[4] and (PERM) by every local peak ϕ,ψ is completed by some valley γ,δ of complete developments of $(\psi\ [\![\phi\rangle\!\rangle), (\phi\ [\![\psi\rangle\!\rangle)$ with $[\![\phi\cdot\gamma\rangle\!\rangle = [\![\psi\cdot\delta\rangle\!\rangle$.

Remark 2. *The lifting γ^{Φ} of the reduction γ in Def. 4 is unique. Formally, this is a consequence of the labelling given being a* rewrite *labelling in the sense of [20, Def. 8.4.5].*

Lemma 2 (FD,PERM)**.** *$\langle \multimap\!\!\to, 1, /\rangle$ is a residual system with binary joins/diagonals, for $\multimap\!\!\to$ the rewrite system having as objects the objects of $\to$, and as steps a* multi-redex *$a^{\Phi} : a \multimap\!\!\to b$ if there is a complete development of Φ from a to b; 1_a defined as $\varnothing$; residual Φ/Ψ defined as $(\Phi\ [\![\Psi\rangle\!\rangle)$, and the binary join/diagonal given by $\Phi\cup\Psi$ (cf. [20, Def. 8.7.28]).*

Denoting a multi-redex a^{Φ} by just Φ in the lemma, is justified by that a is the source common to all steps in Φ, and that all complete developments of Φ have the same target. The join being a step from the source to the target of a residual diamond, justifies calling it a diagonal.

Remark 3. *Parallel rewriting $\twoheadrightarrow\!\!\!\!\shortmid\!\shortmid$ [7] does constitute a residual system for orthogonal TRSs, so does give rise to good residual theory [20], but $\twoheadrightarrow\!\!\!\!\shortmid\!\shortmid$ does not have joins, e.g. the join of the single/parallel steps $\iota(Ix)$ and $I\iota(x)$* should *be $\iota(\iota(x))$ but although that is a* multi*step it is not a* parallel *step as it nests ι in itself. Hence, by Lem. 2 it cannot be obtained* via *multi-redexes; a first indication that the properties in [13] are too strong.*[5]

Definition 5 ([13])**.** Local homotopy $\equiv_l$ *on reductions with the same sources/targets, is the equivalence generated by closing $\phi\cdot\gamma \equiv_l \psi\cdot\delta$ for peaks ϕ,ψ and valleys γ,δ given*[6] *by (PERM) under composition: if $\gamma \equiv_l \gamma'$ then $\delta'\cdot\gamma\cdot\epsilon' \equiv_l \delta'\cdot\gamma'\cdot\epsilon'$. We define $\gamma \sqsubseteq_l \delta$ if $\gamma\cdot\epsilon \equiv_l \delta$ for some ϵ.*

We show *local* homotopy $\equiv_l$ on finite $\to$-reductions is the same as *square* homotopy $\equiv$ on finite $\multimap\!\!\to$-reductions. Observe we may embed $\to\ \subseteq\ \multimap\!\!\to$ by mapping a step $\phi : \phi \to \psi$ to $\phi^{\{\phi\}} : a \multimap\!\!\to b$ assuming (SD), and vice versa $\multimap\!\!\to\ \subseteq\ \twoheadrightarrow$ by mapping each multi-redex a^{Φ} to an arbitrary but fixed complete development of Φ from a. Below the corresponding coercions (and their stepwise

[4] Since in [13] only *finite* reductions are defined, (FD) is (must be) circumscribed there as the absence of infinite *sequences* of steps all of whose *prefixes* are developments of the given set.

[5] Following the rewrite approach, residual systems do *not* assume that steps are closed under composition. Indeed, parallel steps are not, but *reductions* of parallel steps do have compositions and therefore also joins as follows from Proposition 1. In our example, both reductions $\iota(Ix)\cdot\iota(x)$ and $I\iota(x)\cdot\iota(x)$ along the two legs of the diamond are (equivalent) joins of $\iota(Ix)$ and $I\iota(x)$.

[6] For a peak, the choice of valley witnessing (PERM) may be non-deterministic. Essentially this follows since FD makes Newman's Lemma apply 'locally' to developments, allowing to show that independently of the choice the induced redex-trace relation is the same; see the proof of Lem. 3 and cf. [15, Prop. 2.4.16] and [17, Thm. 2].

extensions to $\rightarrow$-reductions respectively $\mathrel{-\!\!\circ\!\!\rightarrow}$-reductions) are denoted by overlining respectively underlining, but we omit them as much as possible. Note $\gamma = \underline{(\overline{\gamma})}$ for any γ.

Remark 4. *(FD) entails the equivalence closures of $\rightarrow, \mathrel{-\!\!\circ\!\!\rightarrow}$ are the same, but their reflexive–transitive closures may differ if (SD) does not hold: for steps $\phi : a \rightarrow b$ and $\phi' : b \rightarrow c$ with only $\phi \, [\![\phi\rangle\!\rangle \, \phi'$ non-empty, we have $a \rightarrow b$ and indeed also $a \mathrel{-\!\!\circ\!\!\rightarrow} c \mathrel{\leftarrow\!\!\circ\!\!-} b$, but not $a \mathrel{-\!\!\circ\!\!\twoheadrightarrow} b$.*

Lemma 3 (SD,FD,PERM)**.** *$\equiv \, = \, \equiv_l$ and $\sqsubseteq \, = \, \sqsubseteq_l$ (after embedding; in both directions).*

The main result on multi-redexes of [13] is now a matter of chaining the above results:

Proof of Thm. 2. Lem. 2 for $\rightarrow$ induces a residual system on $\mathrel{-\!\!\circ\!\!\rightarrow}$. By Prop. 1 that induces a residual system with composition on $\mathrel{-\!\!\circ\!\!\twoheadrightarrow}$, which by Thm. 1 has lubs and left-cancellation up to *square* homotopy. Hence $\twoheadrightarrow$ has lubs and left-cancellation up to *local* homotopy by Lem. 3. $\square$

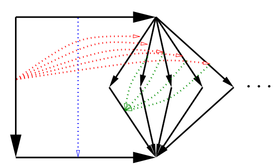

Figure 1: Rewrite system satisfying (SD), (FD) and (PERM) but not (F)

Fig. 1 illustrates the result for a system for which (F) does not hold, a second indication the properties in [13] are too strong. To recover Thms. 1 and 2 of [13] *exactly*, using (F), it suffices to observe that the above can be relativised to a collection $\mathcal{R}$ of sets of redexes such that $\varnothing, \{\phi\} \in \mathcal{R}$ for all redexes ϕ, and $\Phi \cup \Psi, (\Phi \, [\![\Psi\rangle\!\rangle) \in \mathcal{R}$ for all co-initial $\Phi, \Psi \in \mathcal{R}$, and note that the *finite* sets of co-initial steps constitute such a collection. The example in Fig. 1 is rather artificially infinite, but note that although the notion of multi-redex extends naturally (under some provisos) and are at the basis of *infinitary* confluence [20, Ch. 12], (FD) fails for them, a third indication the properties in [13] are too strong.

We generalise *redexes* to *treks* [13], employing $\mathfrak{t}, \mathfrak{s}, \ldots$ ($\mathfrak{T}, \mathfrak{S}, \ldots$) to range over (sets of) them.

Definition 6 ([13])**.** *A* trek*-trace relation maps each step $\phi : a \rightarrow b$ to a relation $[\![\phi\rangle\!\rangle$ between the treks of a and b, where (*multi-*)*treks *of a are elements (subsets) of a set $T(a)$ quasi-ordered by $\leq_a$ having the redexes of a as its minimal elements, and such that $\geq_a \cdot [\![\phi\rangle\!\rangle \subseteq [\![\phi\rangle\!\rangle \cdot \geq_b$.*

Intuitively, a trek is a representation of a reduction and $\leq$ a *causal* order on the redexes contracted; the condition $\geq_a \cdot [\![\phi\rangle\!\rangle \subseteq [\![\phi\rangle\!\rangle \cdot \geq_b$ then captures that if a redex has a residual so do the redexes it causes. Accordingly, we restrict $\phi^{\mathfrak{T}}$ in $[\![\rightarrow\rangle\!\rangle$ (Def. 4) to steps ϕ in the $\leq$-*downward closure* of $\mathfrak{T}$. After these changes and replacing *redex* by *trek* everywhere[7], *everything* above carries over verbatim, in particular Def. 4, Rem. 2, Lem. 2, Rem. 3, Def. 5, Rem. 4, Lem. 3, the main result Thm. 2, and their proofs, using the following remark in the proof of Lem. 2:

Remark 5. *The properties of $\leq$ make $[\![\rightarrow\rangle\!\rangle$ a labelling of itself: if $\phi^{\mathfrak{T}}$ is a $[\![\rightarrow\rangle\!\rangle$-step and $\mathfrak{T} \leq \mathfrak{T}'$, i.e. $\mathfrak{t} \leq \mathfrak{T}'$ for all $\mathfrak{t} \in \mathfrak{T}'$, then $\phi^{\mathfrak{T}'}$ is a $[\![\rightarrow\rangle\!\rangle$-step by transitivity of $\leq$ (if only $\mathfrak{T} \subseteq \mathfrak{T}'$ then transitivity of $\leq$ is not needed) and $(\mathfrak{T} \, [\![\phi\rangle\!\rangle) \leq (\mathfrak{T}' \, [\![\phi\rangle\!\rangle)$ by $\geq \cdot [\![\cdot\rangle\!\rangle \subseteq [\![\cdot\rangle\!\rangle \cdot \geq$ and $[\![\cdot\rangle\!\rangle$ being defined pointwise.*

[7] And also $\in$ into $\leq$ when appropriate, and references to [13, Sect. 2] into corresponding ones to [13, Sect. 3].

Thus we have shown that the axiomatisation of [13] is sufficient but not necessary for obtaining a good residual theory. Although one often *may* factor residual theory through these axioms, there usually is no need to do so, and residual systems can be constructed directly and inductively [8, 20]. We conclude with two remarks on the (FD) axiom:

(FD) was not included among the axioms of residual systems [20] as we did not see a motivation for it. More generally, it is an open question whether finiteness or termination axioms have a place in analysing causality, cf. [21]. Of course, since they give rise to induction measures, they may be practically useful, and we are indeed happy to use them if and when available. For instance, in [15] we showed (FD) to be a consequence of *termination* of the so-called *substitution calculus* (SC) [19] underlying a rewrite format. But for infinitary rewrite systems termination of the SC and hence (FD) are surely too strong, despite that infinitary confluence of orthogonal systems is still based on causality/multi-redexes (up to some provisos).

(FD) may be hard to attain. The application of multi-treks to deal with Lévy's extraction theory for the $\lambda\beta$-calculus in [13, Section 6] is beautiful,[8] but in that application (FD) boils down [13, p. 46] to finiteness of family developments (FFD), cf. [16]. (FFD) is a key result in term rewriting at the basis of standardisation, (hyper-)normalisation of strategies, the theory of optimality, and more, but it also is subtle: It was formulated for the $\lambda\beta$-calculus by Lévy, forming the basis of his beautiful theory of optimality [10], but he resorted [5] to asking the Dutch, van Daalen (whose proof is employed in [10, Sect. II.1.5]) and de Vrijer [4, Stellingen], to prove it.[9] Melliès showed [12, Section 6.2.2] the result [9, Thm. 6.2.4] underlying the proof of (FFD) for Klop's combinatory reduction systems (CRSs) to be incorrect, leaving it and its consequences such as standardisation in limbo. We proved (FFD) for HRSs, hence CRSs, by adapting van Daalen's nifty proof [16], cf. [3].[10]

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[8] It seems worthwhile to adapt it to structured rewrite systems such as TRSs, HRSs, and GRSs.

[9] For first-order term rewrite systems (FFD) is due to Maranget [11]; then it is a simple consequence of RPO.

[10] In view of the subtleties it seems of interest to formalise a proof of (FFD) for HRSs in some proof assistant.

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